Learning Mathematics: Findings from the National (U.S.) Mathematics Advisory Panel

David C. Geary
University of Missouri

1

Learning Mathematics

Contributing Members: Learning Processes Task-Group

- Dave Geary
- Dan Berch
- Wade Boykin
- Susan Embretson
- Valerie Reyna
- Bob Siegler
- Jennifer Graban, staff

2

Learning Mathematics: Areas Reviewed

- General principles: from cognitive processes to learning outcomes
- Social, motivational, and affective influences on learning
- Mathematical knowledge children bring to school
- Mathematical learning and cognition in:
  - Whole number arithmetic
  - Fractions, decimals, and ratios
  - Estimation
  - Geometry
  - Algebra
- Individual differences
- Brain Science and Mathematics Learning

3

Learning Mathematics: General Principles

- There is considerable scientific knowledge on learning and cognition that could be but is not applied to improve student achievement.
- Examples:
  - Working memory is the ability to hold multiple pieces of information in mind, while manipulating one or several of them.
    - Working memory capacity is inherently limited.
    - Demands that exceed this limit will result in poor performance or learning.
    - Practice can offset this limitation by achieving automaticity–the fast, effortless retrieval of facts or procedures–which frees up working memory resources.
  - Testing aides in learning by prompting retrieval of content knowledge

4

Learning Mathematics: General Principles

- The learning of facts, algorithms, and concepts are inter-related.
  - Conceptual knowledge aids in the choice of algorithms;
  - practice of algorithms can provide a context for making inferences about concepts;
  - committing facts to long-term memory reduces working memory demands and allows attention to be focused on more complex problem features.
- Conceptual understanding promotes transfer of learning to new problems and better long-term retention.
- Learning is most effective when practice using algorithms is combined with instruction on related concepts.

5

Learning Mathematics: Social, Motivation, and Affective Contributions

- ***Vygotsky's Socio-Cultural Perspective***
  - The socio-cultural perspective of Vygotsky has been influential in education.
    - It treats learning as a social induction process through which learners become increasingly able to function independently through the tutelage of more knowledgeable peers and adults.
  - Due to a shortage of controlled experiments, however, the usefulness of this approach for improving mathematics learning has not been established.

6

Learning Mathematics: Social, Motivation, and Affective Contributions

- Among the factors that can influence mathematical performance above and beyond mathematical competence are:

Self-Regulation

- The ability to set goals, plan, monitor, and evaluate progress is correlated with mathematics achievement.

Mathematics Anxiety

- Doubts about one's competence can lead to intrusions into working memory (e.g., "I don't understand this…) that in turn reduce the capacity that can be used in problem solving.
- Interventions can significantly reduce anxiety and improve test scores..

7

Learning Mathematics: Social, Motivation, and Affective Contributions

Intrinsic and Extrinsic Motivation

- Young children's intrinsic motivation to learn (desire to learn for its own sake) is positively correlated with academic outcomes in mathematics.
- However, intrinsic motivation declines across grades, especially in mathematics and the sciences as material becomes increasingly complex and abstract.
- The educational environment can influence students' intrinsic motivation, but this may not be sufficient for mastery; grades, proficiency tests, and so forth may be needed for some students as sources of external motivation

8

Learning Mathematics: Social, Motivation, and Affective Contributions

Beliefs about Effort versus Ability

- Children who believe that learning mathematics is associated with ability, rather than effort, are likely to lack persistence when the material becomes difficult.
- Experimental studies have demonstrated that children's beliefs about the relative importance of effort and ability can be changed, and that increased emphasis on the importance of effort is related to greater persistence and improved mathematics grades.

9

Learning Mathematics: What Children Bring to School

- Most children start school with a fair amount of numerical knowledge;
  - Many have an implicit sense of numbers, counting, and the effects of adding and subtracting on set size
- The mathematical knowledge that children bring to school influences their learning and mathematics achievement for many years thereafter.
- The numerical knowledge of children from low-income backgrounds lags behind even before they start school and places them at significant risk for achieving below their potential through high school.
- Promising instructional programs exist for increasing low income preschoolers' numerical knowledge.

10

Learning Mathematics: Whole Number Arithmetic

- Many children do not master whole number arithmetic.
- Sources:

  They do not know basic arithmetic facts, prime numbers, laws of exponents – this results in unnecessary errors and use of procedures (e.g., counting) that consume working memory resources that could otherwise be used for other problem features

  They frequently make errors when using standard algorithms; error patterns suggest poor conceptual knowledge (e.g., of base-10).

  Poor understanding of core concepts, such as associativity, commutativity, distributive property, and the inverse relation between division and multiplication.

11

Learning Mathematics: Whole Number Arithmetic

- Learning of facts, concepts, and procedures is interrelated, but may require somewhat different experiences to master.
  - Mastery – fast and effortless retrieval of facts or execution of procedures – requires practice extended across time.
  - Overlearning aids in long-term retention; practice after automatic recall is achieved.

12

Learning Mathematics: Whole Number Arithmetic

- Mixing procedures during practice can facilitate learning, by requiring discrimination of potentially confusable procedures.
  - E.g., practicing problems in the form of $y^a \times y^b$ with $(y^a)^b$ helps students learn and discriminate the different rules

Concepts may not require as extensive practice but their application to different types of problems and contrast with related concepts is needed.

13

Learning Mathematics: Fractions

- Fractions are formally introduced in elementary school, and yet remain difficult for many adults:
  - 27% of U.S. 8th graders could not correctly shade 1/3 of a rectangle (NAEP, 2005)
  - 45% could not solve a word problem involving dividing fractions (NAEP, 2004).
- Preschoolers show an intuitive awareness of fractions based on part-whole relations and sharing:
- But, the relation between this knowledge and a formal mathematical understanding of fractions is not well understood.

14

Learning Mathematics: Fractions

- Absence of a coherent and empirically supported theory of how children learn and understand fractions is a major stumbling block to developing practical interventions.
- What we do know:
  - Conceptual and procedural knowledge of fractions reinforce and bootstrap one another and influence such varied tasks as estimation, word problems, and computations.
  - A key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a physical, and ultimately mental, number line.

15

Learning Mathematics: Fractions

- What we do know:
  - On-task time, motivation, working memory, well-learned basic arithmetic skills and reading ability also determine performance on fraction problems.
  - Instruction focusing on conceptual knowledge of fractions is likely to have the broadest and largest impact on problem-solving performance provided that it is aimed at accurate solution of specific problems.

16

Learning Mathematics: Estimation

- Numerical estimation is an important part of mathematical cognition, because it is frequently used in everyday life and in scientific, mathematical, and technological professions, and because it is closely related to overall math achievement.
- Poor estimation performance often reveals underlying difficulties in understanding of mathematics in general.
- Extensive use of rounding as the only way to estimate can result in children not understanding that the purpose of estimation is to approximate the correct value.
- Children’s estimation of the magnitudes of fractions is especially poor.

17

Learning Mathematics: Geometry

- The component of geometry most directly relevant for the early learning of algebra is that of *similar triangles.*
  - But, it is difficult to draw firm, scientifically-based conclusions from the empirical research on how students’ acquire knowledge of similarity and related concepts.
- Piaget theorized about children’s understanding of geometry. However, the mathematical inaccuracies of his hypotheses along with the mounting negative empirical evidence suggest that it should no longer inform the design of instructional approaches in geometry.
- Despite the widespread use of mathematical manipulatives such as geoboards, dynamic software, and so forth, evidence regarding their usefulness in helping children learn geometry is tenuous at best.

18

Learning Mathematics: Geometry

- Students must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations. The crucial steps in making such transitions are not clearly understood at present.
- One of the challenges to effective learning in geometry is the persistence of misconceptions and their resistance to instruction:
  - E.g., "illusion of linearity" where students incorrectly believe that if the perimeter of a geometric figure is enlarged *k* times, its area (and/or volume) is enlarged *k* times as well.

19

Learning Mathematics: Algebra

- Cognitive studies of algebra have focused on linear equations and word problems and reveal that many students in high-school algebra classes are woefully unprepared for learning the basics of algebra.
- The errors students make when solving algebraic equations reveal many do not have a firm understanding of basic principles of arithmetic (e.g., commutativity, distributivity, laws of exponents), and many do not understand the concept of mathematical equality.

20

Learning Mathematics: Algebra

- Students have difficulty grasping the syntax or structure of algebraic expressions and do not understand procedures for transforming equations (e.g., adding or subtracting the same value from both sides of the equation) or *why* transformations are done the way they are.
    - In the following, mathematicians scan the expressions from top to bottom, whereas students treat the division lines and subtraction sign as a continuous line and scan the top from left to right and then the bottom.

$$\frac{5x - 2}{2y + 7} - \frac{3y^2 - 1}{4}$$

21

Learning Mathematics: Algebra

- Word problems are a persistent source of difficulty:
- Translation of relational statements into algebraic expressions and equations is a core source of error:
    - "There are six times as many students (s) as professors (p) at this university". What is the relation between number of students and professors?
        - Common error is direct translation, 6s = p, rather than 6p = s.
        - There are many other sources of translation error.

22

Learning Mathematics: Algebra

- Learning a problem-solving *schema* aids in translation and problem solving:
  - A long-term memory representation that results in fast and automatic recognition of key elements of a word problem, enables the classification of the problem into a conceptual group (e.g., velocity problems, interest problems), and has a linked system of procedures that can be used to solve the problem
- A common difficulty, however, is recognizing that a problem with an unfamiliar cover story can be solved using already learned schemas
- Performance is aided by
  - Familiarity with common problem types (e.g., interest)
  - Use of worked examples
  - Instruction on spatially representing relations

23

Learning Mathematics: Individual Differences

Between 5% and 10% of students will experience a significant learning disability or learning difficulty in mathematics that is unrelated to cognitive ability.

> The corresponding cognitive deficits include a compromised working memory system, difficulties memorizing basic facts, and a poor sense of number and magnitude. These have been identified as impeding arithmetic learning.
>
> Much less is known about how these difficulties are related to learning fractions, estimation, geometry, and algebra.

Mathematically gifted students have an enhanced ability to remember and process numerical and spatial information.

Acceleration of motivated students is the best approach to enhancing their potential

24

Learning Mathematics: Brain Sciences

Brain sciences research has identified core regions involved in aspects of mathematical problem solving

This research has the potential to contribute unique knowledge about mathematical learning and cognition and for eventually informing educational practice.

At the same time, the application of research in the brain sciences to classroom teaching and student learning in mathematics is premature.

Instructional programs in mathematics that claim to be based on brain sciences research remain to be validated.

25

Learning Mathematics: Conclusions

Cognitive Mechanisms

- Working memory is an inherent constraint on learning, but can be functionally improved with automatic recall of core facts and procedures.
- Achievement of automaticity requires practice extended over time as well as overlearning.
- Testing aides learning by requiring the recall of content relevant information.

26

Learning Mathematics: Conclusions

Social, Affective, and Motivational Mechanisms

- The utility of Vygotsky's social-constructivist approach to teaching has not been demonstrated.

- An emphasis on the importance of effort in learning mathematics will promote persistence in learning when the material becomes difficult.

What Children Bring to School

- There are considerable individual differences in children's mathematical knowledge when they enter school.

- Without interventions that will narrow this gap, children who start behind are at significant risk for staying behind.

27

Learning Mathematics: Conclusions

Content Areas

- The curriculum must simultaneously develop conceptual understanding, computational fluency, and problem-solving skills.
    - Debates regarding the relative importance of these are misguided. These capabilities are mutually supportive, each facilitating learning of the others.
    - Teachers should emphasize these interrelations;
    - Conceptual understanding of mathematical operations, fluent execution of procedures, and fast access to number combinations together support effective and efficient problem-solving.

28

Learning Mathematics: Conclusions

Content Areas

- Fractions are a critical roadblock to learning more complex mathematics
  - One key mechanism linking conceptual and procedural knowledge is the ability to represent fractions on a number line.
  - Instruction focusing on conceptual knowledge of fractions is likely to have the broadest and largest impact on problem-solving that involves use of fractions but only when it is directed toward the accurate solution of specific problems.
- Piaget's hypotheses about children's understanding of geometry have not been supported by research and should not be used to guide instruction.

29

Learning Mathematics: Conclusions

Content Areas

- Without a firm grasp of basic arithmetic, children are seriously disadvantaged when it comes to learning algebra
- Students often do not understand the syntax of algebraic equations, or understand the procedures for transforming equations. Problems are compounded with word problems:
  - Remediation of deficits in basic arithmetic may be needed
  - As with arithmetic, practice of core procedures is needed for long-term retention
  - Use of worked examples and instruction on using spatial representations of key relations will help in solving word problems

30