

## Midterm Exam 2

October 30, 2018

Time: 1 hour, 30 minutes

Name: \_\_\_\_\_

**Instructions:**

1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (5 points). Find the limit

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$$

This is a limit of the form  $\frac{0}{0}$ . Using L'Hôpital's rule,

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{1} = 1$$

2. (10 points). Prove that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r$$

Remark: This is an important result in finance. When periodic interest rate is  $r$ , and the interest is compounded  $n$  times per period, the amount after one period is:  $(1 + r/n)^n$ . For example,  $r$  is usually annual percentage rate (APR), and  $n = 12$  if the interest compounds monthly, and  $n = 360$  if interest compounds daily, etc. If interest compounds every instant, the amount can be approximated by  $e^r$  - continuously compounded formula.

We can rewrite the required function in a way that allows the use of substitution rule and L'Hôpital's rule:

$$\left(1 + \frac{r}{n}\right)^n = \exp\left(n \ln\left(1 + \frac{r}{n}\right)\right) = \exp\left(r \frac{\ln\left(1 + \frac{r}{n}\right)}{r/n}\right) = \exp\left(r \frac{\ln(1+x)}{x}\right)$$

where  $x = r/n$ . Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} x &= 0 \\ \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} &= 1 \quad [\text{We proved previously, using L'Hôpital's rule}] \end{aligned}$$

Thus, the required limit can be written as

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = \lim_{x \rightarrow 0} \exp\left(r \frac{\ln(1+x)}{x}\right) = \lim_{y \rightarrow 1} \exp(ry) = e^r$$

3. (10 points). Given the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0$$

(a) Prove that

$$\lim_{\sigma \rightarrow 1} u(c) = \ln(c)$$

Since the above limit has  $\frac{0}{0}$  form (i.e. the numerator and the denominator have limit of 0 as  $\sigma \rightarrow 1$ ), we can apply L'Hôpital's rule:

$$\lim_{\sigma \rightarrow 1} \frac{c^{1-\sigma} - 1}{1 - \sigma} = \lim_{\sigma \rightarrow 1} \frac{-\ln(c) c^{1-\sigma}}{-1} = \ln(c)$$

Here we used the rule of derivatives  $(c^x)' = \ln(c) c^x$ , together with the chain rule.

(b) Calculate the Arrow-Pratt measure of Relative Risk Aversion ( $RRA$ ).

$$RRA = -\frac{u''(c)}{u'(c)}c = -\frac{-\sigma c^{-\sigma-1}}{c^{-\sigma}}c = \sigma$$

4. (10 points). Suppose  $u(x, y)$  is a utility function, and  $v(x, y) = f(u(x, y))$  is another utility function generated by composition  $f \circ u$ , where  $f$  is differentiable function. Using the implicit function theorem, prove that the indifference curves of  $u$  and  $v$  have the same slope at every point.

Indifference curves of  $u$  and  $v$ , in the form of implicit function:

$$\begin{aligned}u(x, y) &= \bar{u} \\v(x, y) &= f(u(x, y)) = \bar{v}\end{aligned}$$

Using the Implicit Function Theorem, the slopes of indifference curves are:

$$\begin{aligned}[IC_u] &: \frac{dy}{dx} = -\frac{u_x(x, y)}{u_y(x, y)} \\[IC_v] &: \frac{dy}{dx} = -\frac{v_x(x, y)}{v_y(x, y)} = -\frac{f'(u(x, y)) u_x(x, y)}{f'(u(x, y)) u_y(x, y)} = -\frac{u_x(x, y)}{u_y(x, y)}\end{aligned}$$

The last step is valid as long as  $f'(u)$  is not zero. Thus, the slope of indifference curves of  $u$  is the same as the slope of indifference curves of  $v$ , as long as  $f'(u) \neq 0$ .

**Remark.** The above is an important result in consumer choice theory. If  $f$  is also *monotone increasing* function, then consumer with utility function  $v(x, y) = f(u(x, y))$  will have the same demand as consumer with utility function  $u(x, y)$ . This result is known as *invariance of utility functions under monotone increasing transformation*. In other words, the same preferences can be represented with infinitely many utility functions, as long as they are monotone increasing transformations of each other.

5. (15 points). Suppose that the demand function for some good is given by  $x = Ap_x^{-0.5}p_y^{0.8}I^{-0.3}$ , where  $p_x$  is the price of  $x$ ,  $p_y$  is the price of some other good  $y$ , and  $I$  is buyers' income.

- (a) Based on the given information,  $x$  is a normal/ good. Circle the correct answer and provide a brief explanation.

The demand for  $x$  is decreasing in income  $I$ , which is the definition of inferior good.

- (b) Calculate the price elasticity of demand for  $x$ , the cross price elasticity of demand for  $x$  with respect to  $p_y$ , and income elasticity of demand for  $x$ . Denote these elasticities with  $\eta_{x,p_x}$ ,  $\eta_{x,p_y}$  and  $\eta_{x,I}$ .

Taking logs of the demand function

$$\ln x = \ln A - 0.5 \ln p_x + 0.8 \ln p_y - 0.3 \ln I$$

The required elasticities are:

$$\eta_{x,p_x} = \frac{\partial \ln x}{\partial \ln p_x} = -0.5$$

$$\eta_{x,p_y} = \frac{\partial \ln x}{\partial \ln p_y} = 0.8$$

$$\eta_{x,I} = \frac{\partial \ln x}{\partial \ln I} = -0.3$$

- (c) Calculate the approximate % change in the demand for  $x$  as a result of a 1% simultaneous increase in  $p_x$ ,  $p_y$  and  $I$ .

$$\begin{aligned} \% \Delta x &= \eta_{x,p_x} \cdot \% \Delta p_x + \eta_{x,p_y} \cdot \% \Delta p_y + \eta_{x,I} \cdot \% \Delta I \\ &= -0.5 \cdot 1\% + 0.8 \cdot 1\% - 0.3 \cdot 1\% = 0 \end{aligned}$$

**Remark.** This has to be the case with any demand function. When prices and income increase by the same proportion, the budget constraint does not change, and the demand does not change as well. We therefore say that demand (for any good) is always *homogeneous of degree zero*.

6. (10 points). Suppose a firm's profit, as a function of output  $q$  and advertising  $a$  is given by:

$$\pi(q, a) = 1000q - q^2 + 500a - a^2$$

Suppose that the firm is producing  $q = 450$  units of output and spends  $a = 275$  on advertising. Using total differential, what is the approximate change in profit if the firm increases output by 1 unit and decreased advertising by 1 unit (simultaneously)?

Writing the total differential of  $\pi$  with increments  $dq = 1$  and  $da = -1$ , and substituting the initial values ( $q = 450, a = 275$ ), gives:

$$\begin{aligned}d\pi &= \pi_1(q, a) dq + \pi_2(q, a) da \\&= (1000 - 2q) dq + (500 - 2a) da \\&= (1000 - 2 \cdot 450) dq + (500 - 2 \cdot 275) da \\&= 100 \cdot dq - 50 \cdot da \\&= 100 \cdot 1 - 50 \cdot (-1) = 150\end{aligned}$$

**Remark.** The exact change in profit is

$$\begin{aligned}\Delta\pi &= \pi(q + 1, a - 1) - \pi(q, a) \\&= \pi(451, 274) - \pi(450, 275) = 148\end{aligned}$$

7. (15 points). Consider the following utility function from consumption in two periods:

$$u(c_1, c_2) = \frac{c_1^{1-\sigma} - 1}{1 - \sigma} + \beta \frac{c_2^{1-\sigma} - 1}{1 - \sigma}, \quad \beta, \sigma > 0$$

(a) Write the equation of a generic indifference curve with utility level  $\bar{u}$ .

Indifference curve:

$$\frac{c_1^{1-\sigma} - 1}{1 - \sigma} + \beta \frac{c_2^{1-\sigma} - 1}{1 - \sigma} = \bar{u}$$

(b) Find the marginal rate of substitution between  $c_1$  and  $c_2$  ( $MRS_{c_1, c_2}$ ).

Note, the indifference curve is implicit function. Using the implicit function theorem:

$$\frac{dc_2}{dc_1} u(c_1, c_2) = -\frac{u_1(c_1, c_2)}{u_2(c_1, c_2)} = -\frac{c_1^{-\sigma}}{\beta c_2^{-\sigma}} = -\frac{c_2^\sigma}{\beta c_1^\sigma}$$

The marginal rate of substitution between  $c_1$  and  $c_2$  ( $MRS_{c_1, c_2}$ ) is the absolute value of the slope of indifference curves, i.e.

$$MRS_{c_1, c_2} = \frac{1}{\beta} (c_2/c_1)^\sigma$$



(c) Find the elasticity of substitution between goods  $c_1$  and  $c_2$  ( $ES_{c_1, c_2}$ ).

$$\begin{aligned}\frac{1}{ES_{c_1, c_2}} &= \frac{\partial MRS_{c_1, c_2}}{\partial (c_2/c_1)} \cdot \frac{c_2/c_1}{MRS_{c_1, c_2}} \\ &= \frac{\partial \frac{1}{\beta} (c_2/c_1)^\sigma}{\partial (c_2/c_1)} \cdot \frac{c_2/c_1}{\frac{1}{\beta} (c_2/c_1)^\sigma} \\ &= \frac{1}{\beta} \sigma (c_2/c_1)^{\sigma-1} \frac{c_2/c_1}{\frac{1}{\beta} (c_2/c_1)^\sigma} = \sigma\end{aligned}$$

Thus,

$$ES_{c_1, c_2} = \frac{1}{\sigma}$$

8. (15 points). Consider the following system of nonlinear equations:

$$\begin{aligned}\ln(x) + y + xa &= 5 \\ x + xy^2 - ya &= -1\end{aligned}$$

The endogenous variables are  $x$  and  $y$ , while  $a$  is an exogenous variables (or parameter).

- (a) Suppose that initially,  $(x, y, a) = (1, 2, 3)$ . Verify that this point is a solution to the given system of equations.

$$\begin{aligned}\ln(1) + 2 + 1 \cdot 3 &= 5 \\ 1 + 1 \cdot 2^2 - 2 \cdot 3 &= -1\end{aligned}$$

- (b) Derive the Jacobian matrix, and calculate the Jacobian determinant at the initial point.

The given system can be written as:

$$\begin{aligned}F_1(x, y, a) &= 5 \\ F_2(x, y, a) &= -1\end{aligned}$$

The Jacobian with respect to the endogenous variables  $(x, y)$ :

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{x} + a & 1 \\ 1 + y^2 & 2xy - a \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix}$$

Jacobian determinant:

$$|J| = 4 \cdot 1 - 5 \cdot 12 = -1 \neq 0$$

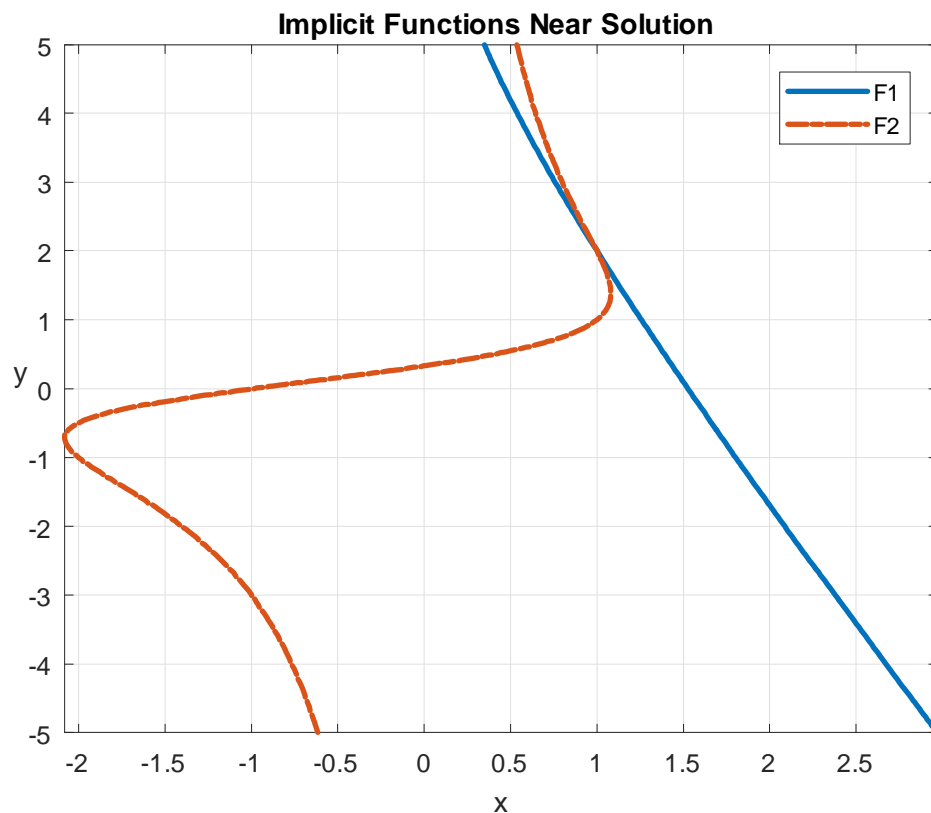
- (c) Suppose the exogenous parameter changes:  $da = 0.01$ . Calculate the approximate changes in endogenous variables:  $dx$ ,  $dy$ .

$$\begin{aligned} \begin{bmatrix} dx \\ dy \end{bmatrix} &= -J^{-1} \begin{bmatrix} \frac{\partial F_1}{\partial a} \\ \frac{\partial F_2}{\partial a} \end{bmatrix} da = - \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ -y \end{bmatrix} da \\ &= - \frac{\begin{bmatrix} 1 & -1 \\ -5 & 4 \end{bmatrix}}{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} da = \begin{bmatrix} 3 \\ -13 \end{bmatrix} \cdot 0.01 = \begin{bmatrix} 0.03 \\ -0.13 \end{bmatrix} \end{aligned}$$

Remark: when solving the system numerically with  $a = 3.01$  (using Matlab), the precise changes in endogenous variables are

$$\begin{aligned} \Delta x &= 0.02716650371840980420472563150556... \\ \Delta y &= -0.11857522030587645995421198469615... \end{aligned}$$

demonstrating that the approximation above ok, but not great. It turns out that the given system is very sensitive to changes in  $a$  in the vicinity of the given solution. The next figure shows the plot of the two implicit functions at the original solution.



9. (10 points). Briefly explain what the following Matlab commands are doing.

(a) `z = linspace(-5,5,101)`

Creating evenly spaced grid (named `z`) of 101 points between -5 and 5.

(b) `syms x y`

Declaring symbolic variables `x` and `y`.

(c) `limit(x^x,x,0)`

Finding the limit

$$\lim_{x \rightarrow 0} x^x$$

(d) `diff(x^a, a)`

Calculating the partial derivative

$$\frac{\partial}{\partial a} x^a$$

(e) `J = jacobian([x^2+x*y; y^2 - x/y])`

Calculating the Jacobian (matrix of second derivatives) of the system of functions:

$$\begin{aligned} F_1(x, y) &= x^2 + xy \\ F_2(x, y) &= y^2 + \frac{x}{y} \end{aligned}$$

That is,

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$