ALGORITHMICALLY FINDING THE IDENTITY OF THE BTW SANDPILE GROUP

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Abstract. Algorithms to find the identity of the group of recurrent BTW sandpiles are described. These algorithms are used to provide experimental data, from which conjectures about the structure of the identity are tested.

1. Introduction

The BTW sandpile model was introduced by Bak, Tang and Wiesenfeld in 1988 [1] to study the 1/f noise. In their paper, they describe one, two and three dimensional sandpiles as grids of heights (indexed by d-tuples, where d is the dimension), with each site toppling sand onto its $2d$ neighbors should it ever reach a height of $2d$. Since its introduction, the BTW sandpile model has been studied extensively by physicists for its demonstration of *self-organized criticality*, the tendency to approach a critical state from many starting positions. Many modifications of the BTW model have been created, but the most important is the generalization by Dhar in [5], [4], and [6] of the BTW model to the Abelian Sandpile Model (ASM). The ASM is defined for arbitrary directed graphs, represented as toppling matrices, Δ, which describe the rules for when to topple at a site and which sites (neighbors) to topple to. The algebra of the ASM can be studied to give general mathematical results that may then be applied to special cases, such as the BTW model. A good introduction to the general ASM, upon which the simplified model here is based, is given in [8]. Most relevant to the topic is the finding that the recurrent elements of the ASM have a group structure whose identity has been studied, with some results, in [2] and [7].

Outline. The remainder of this article is organized as follows. Section 2 precisely defines the specific BTW model studied and gives account of relevant previous results. Section 3 describes the derivation of the algorithms used to find the identity of the group of recurrent states. Section 4 analyses the identities produced to give conjectures about their structure and fractal nature. Finally, Section 5 introduces some further questions warranting future study.

2. BTW Sandpile Model on a Square Lattice

2.1. **Sandpiles.** We will be discussing the BTW sandpile model on the $n \times n$ grid, so we shall define the set of indices of this grid. Let N_n be the set of all integers from 1 to n. $N_n^2 = N_n \times N_n$ is thus our set of indices. A sandpile on the $n \times n$ grid is a height function from the indices of the grid to the natural numbers (starting with 0). Let S_n be the set of all $n \times n$ sandpiles. For an arbitrary sandpile, $\eta \in S_n$ and

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a position $x = (i, j) \in N_n^2$, $\eta(x) = \eta(i, j) = z$, where $z \in \mathbb{N}$ is the height (number of sand grains) of the sandpile at x. A stable sandpile is one which has maximum height 3. Let Ω_n be the set of all stable sandpiles, which is clearly a subset of S_n. Let Q_n and Z_n be sandpile sets defined similarly to S_n, with the sandpiles returning rational numbers and integers respectively, instead of natural numbers. We assign the names $\alpha_n$and $\omega_n$to the minimum and maximum stable sandpiles that return 0 and 3 respectively.

$$
\begin{aligned}
N_n &= \{k \in \mathbb{Z}^+ \mid k \leq n\} = \{1, 2, \ldots, n\} \\
N_n^2 &= N_n \times N_n = \{(1, 1), \ldots, (n, n)\} \\
S_n &= \{\eta \mid \eta : N_n^2 \to \mathbb{N}\} \\
\Omega_n &= \{\eta \mid \eta : N_n^2 \to \{0, 1, 2, 3\}\} \\
Q_n &= \{\eta \mid \eta : N_n^2 \to \mathbb{Q}\} \\
Z_n &= \{\eta \mid \eta : N_n^2 \to \mathbb{Z}\} \\
&\forall x \in N_n^2 : \\
&\quad \alpha_n(x) = 0 \\
&\quad \omega_n(x) = 3
\end{aligned} \tag{1}
$$

We will now define sandpile equality and partial ordering

Definition 2.1 (Sandpile Comparison)**.** For all $\eta, \zeta \in Q_n$:
$\eta = \zeta$ if and only if $\eta(x) = \zeta(x)$ for all $x \in N_n^2$
The comparisons $\leq$, $\geq$, $<$, and $>$ are defined similarly.
$\max(\eta) = \eta(x_{max})$, where $\eta(x_{max}) \geq \eta(x)$, for all $x \in N_n^2$
$\min(\eta)$ is defined similarly.

For convenience, we will define the standard matrix representations of an arbitrary sandpile to be the matrix of all values it returns.

Definition 2.2 (Matrix Form)**.** For $\eta \in Q_n$ and $\{x_1, x_2, \ldots, x_{n^2}\} = N_n^2$:

$$
\mathrm{mat}(\eta) = \begin{pmatrix} \eta(1,1) & \ldots & \eta(1,n) \\ \vdots & \ddots & \vdots \\ \eta(n,1) & \ldots & \eta(n,n) \end{pmatrix} \tag{2}
$$

2.2. **Toppling.** For sandpiles on the grid, we are interested in the *von Neumann neighborhood*, $\mathrm{N}(x)$, of each point $x \in N_n^2$. This neighborhood contains all the points directly above, below, and to the left or right of x. The number of neighbors of x, $|\,\mathrm{N}(x)|$, is thus four in the center of the grid, three on the edges, and two in the corners. Also note that $y \in \mathrm{N}(x)$ if and only if $x \in \mathrm{N}(y)$.

Definition 2.3 (von Neumann Neighborhood)**.** For all For all $x = (x_1, x_2) \in N_n^2$ and $y = (y_1, y_2) \in N_n^2$, let $\mathrm{d}_p = \sqrt[p]{|x_1 - y_1|^p + |x_2 - y_2|^p}$. Thus, $\mathrm{d}_1 = |x_1 - y_1| + |x_2 - y_2|$

$$
\mathrm{N}(x) = \{y \in N_n^2 \mid \mathrm{d}\,1xy = 1\}
$$

The model will be represented using the $n^2 \times n^2$ lattice laplacian toppling matrix, Δ^n, which completely describes the relationships between each lattice point. A point relates to itself with the number of its neighbors should the boundary be removed (i.e.: all 4s), and to its neighbors with -1. Notice that as neighbor relations are symmetric (the graph is undirected), so is Δ^n

Definition 2.4 (Lattice Laplacian Toppling Matrix)**.** For all indices $x \in N_n^2$, and $y \in N_n^2$:

$$\Delta^n_{x,y} = \begin{cases} 4 & \text{if } x = y \\ -1 & \text{if } y \in \mathrm{N}(x) \\ 0 & \text{otherwise} \end{cases}$$

With this toppling matrix, we may describe a types of operation on sandpiles called firing and toppling rules. A firing rule acts on single site, x, by removing four grains of sand from it and adding one to each of its neighbors, or in other words, it fires x. A toppling rule fires only unstable sites.

Definition 2.5 (Toppling Rules)**.** A firing rule is an operation $\mathrm{F}_x : Z_n \to Z_n$
For all $\eta \in Z_n$ and $x, y \in N_n^2$:

$$\mathrm{F}_x(\eta)(y) = \eta(y) - \Delta^n_{x,y}$$

A toppling rule is an operation $\mathrm{T}_x : S_n \to S_n$ such that for all $\eta \in S_n$:

$$\mathrm{T}_x(\eta) = \begin{cases} \mathrm{F}_x(\eta) & \text{if } \eta(x) \geq 4 \\ \eta & \text{otherwise} \end{cases}$$

By composing a finite number of toppling rules, we may obtain a *toppling function* that relaxes an arbitrary sandpile, $\eta \in S_n$ to a unique stable sandpile, called the relaxation of η.

Definition 2.6 (Toppling Function)**.** The toppling function, $\mathcal{T}_{\Delta^n} : S_n \to \Omega_n$, is defined for some minimal sequence $(x_1, x_2, \ldots, x_N)$, where $x_i \in N_n^2$, to be:

$$\mathcal{T}_{\Delta^n} = \prod_{i=1}^{N} \mathrm{T}\, x_i = \mathrm{T}\, x_i \circ \mathrm{T}\, x_2 \circ \ldots \circ \mathrm{T}\, x_N$$

$(x_1, x_2, \ldots, x_N)$ is minimal in the sense that for $\eta \in (S_n \oplus \Omega_n)$ (i.e.: η is unstable).

$$\mathcal{T}_{\Delta^n}(\eta) \neq \left(\prod_{i=1}^{N-1} \mathrm{T}\, x_i \right) (\eta) \notin \Omega_n$$

For convenience, we will denote the relaxation of $\eta \in S_n$ as $\lfloor \eta \rfloor = \mathcal{T}_{\Delta^n}(\eta)$.
Also, $\lfloor \eta \rfloor = \zeta$ may be denoted $\eta \to \zeta$.

It is proven in section 2.3 of [8] that $\mathcal{T}_{\Delta^n}$ is well defined by Definition 2.6, using the fact the toppling rules are commutative (i.e.: $\mathrm{T}_x \circ \mathrm{T}_y = \mathrm{T}_y \circ \mathrm{T}_x$). It is also worthwhile to note that $\mathrm{T}_x \circ \mathcal{T}_{\Delta^n} = \mathcal{T}_{\Delta^n}$, for all $x \in N_n^2$, as toppling rules only act when a site is unstable. Following from this, we can see that $\mathcal{T}^k_{\Delta^n} = \mathcal{T}_{\Delta^n}$ for all $k \geq 1$.

2.3. **Addition.** Let the operation of adding a single grain of sand at the site $x \in N_n^2$ be denoted $\mathrm{p}_x : Q_n \to Q_n$. Let $\mathrm{a}_x : S_n \to \Omega_n$, be the operation of adding a grain of sand at x and allowing the sandpile to collapse. $\mathrm{a}_x = \mathcal{T}_{\Delta^n} \circ \mathrm{p}_x$. Thus, for all $\eta \in S_n$, $\mathrm{p}_x(\eta) \to \mathrm{a}_x(\eta)$. Note that p_x and a_x are both associative and commutative, and that:

$$\mathrm{a}_x^{\Delta^n_{x,x}} = \prod_{y \in \mathrm{N}(x)} \mathrm{a}_y^{-\Delta^n_{x,x}} \tag{3}$$
$$\mathrm{a}_x^4 = \prod_{y \in \mathrm{N}(x)} \mathrm{a}_y$$

The sum of two sandpiles is simply the sum of their heights at all grid points. The sum with relaxation is the summation, followed by relaxation. Clearly, both operations are associative and commutative.

Definition 2.7. Let us extend p_x, so that p_x^k corresponds to adding $k \in \mathbb{Q}$ grains of sand when k in nonnegative, and subtracting it is negative. For sandpiles $\eta \in S_n$ and $\zeta \in S_n$, the summation of η and ζ is the result of summing the heights at all points $x \in N_n^2$.:

$$(\eta + \zeta)(x) = \eta(x) + \zeta(x) = \zeta(x) + \eta(x)$$
$$\eta + \zeta = \left(\prod_{x \in N_n^2} \mathrm{p}_x^{\zeta(x)} \right)(\eta) \in S_n.$$

Scalar multiplication for sandpiles is defined similarly to scalar multiplication for vectors. For $k \in \mathbb{Q}$:

$$(k \cdot \eta)(x) = k \cdot \eta(x)$$
$$k \cdot \eta = \left(\prod_{x \in N_n^2} \mathrm{p}_x^{k \cdot \zeta(x)} \right)(\eta) \in Z_n.$$

Subtraction of $\eta \in Q_n$ and $\zeta in Q_n$ is defined as:

$$\eta - \zeta = \eta + (-1) \cdot \zeta$$

Definition 2.8. For sandpiles $\eta \in S_n$ and $\zeta \in S_n$, the sum with relaxation is $\eta \oplus \zeta = \lfloor \eta + \zeta \rfloor$.

For convenience, let us also define the scalar multiplication by $k \in \mathbb{Z}^+$

$$k \otimes \eta = \overbrace{\eta \oplus \ldots \oplus \eta}^{k\ \eta\text{'s}} = \lfloor k \cdot \eta \rfloor$$

Parentheses may be omitted because $\oplus$ is associative.

The following translations from operations to sandpile sums can prove useful:

$$\forall \eta \in Q_n \ \ \mathrm{p}_x \, \eta = (\mathrm{p}_x \, \alpha_n) + \eta \tag{4}$$
$$\forall \eta \in S_n \ \ \mathrm{a}_x \, \eta = (\mathrm{a}_x \, \alpha_n) \oplus \eta \tag{5}$$

Theorem 2.9. *For all $\eta \in S_n$, and $\zeta \in S_n$:*

$$\eta \oplus \zeta = \left(\prod_{x \in N_n^2} \mathrm{a}_x^{\zeta(x)} \right) (\eta)$$

Proof.

$$\eta \oplus \zeta = \lfloor \eta + \zeta \rfloor = \left(\mathcal{T}_{\Delta^n} \prod_{x \in N_n^2} \mathrm{p}_x^{\zeta(x)} \right) (\eta)$$

We can now use the property that $\mathcal{T}_{\Delta^n} = \mathcal{T}_{\Delta^n}^k$ for all $k \geq 1$ to obtain:

$$\eta \oplus \zeta = \left(\mathcal{T}_{\Delta^n}^{n^2} \prod_{x \in N_n^2} \mathrm{p}_x^{\zeta(x)} \right) (\eta) = \left(\prod_{x \in N_n^2} \mathcal{T}_{\Delta^n} \circ \mathrm{p}_x^{\zeta(x)} \right) (\eta) = \left(\prod_{x \in N_n^2} \mathrm{a}_x^{\zeta(x)} \right) (\eta).$$

□

Theorem 2.10. *For $\eta \in S_n$ and $\zeta \in S_n$:*

$$\eta \oplus \zeta = \eta \oplus \lfloor \zeta \rfloor = \lfloor \eta \rfloor \oplus \lfloor \zeta \rfloor$$

Proof. By Theorem 2.9:

$$\eta \oplus \zeta = \left(\prod_{x \in N_n^2} \mathrm{a}_x^{\zeta(x)} \right) (\eta)$$

Notice that in this form, Equation 3 is the same as applying T_x to ζ, where $\zeta(x) \geq 4$. Thus, we may apply any number of T_x's to ζ and still obtain the same result. Thus:

$$\eta \oplus \zeta = \left(\prod_{x \in N_n^2} \mathrm{a}_x^{\mathcal{T}_{\Delta^n}(\zeta)(x)} \right) (\eta) = \left(\prod_{x \in N_n^2} \mathrm{a}_x^{\lfloor \zeta \rfloor (x)} \right) (\eta) = \eta \oplus \lfloor \zeta \rfloor$$

We may now reverse the order order of η and ζ and apply the same reduction to obtain:

$$\eta \oplus \zeta = \eta \oplus \lfloor \zeta \rfloor = \lfloor \eta \rfloor \oplus \lfloor \zeta \rfloor$$

□

Corollary 2.11. *From Theorem 2.10, we can see that in a relaxed sum of sandpiles, only the outer relaxation must remain, with relaxation of the addends being optional. Thus,* $\lfloor \eta + \zeta + \theta \rfloor = \lfloor \lfloor \eta + \zeta \rfloor + \theta \rfloor = \lfloor \eta + \lfloor \zeta + \theta \rfloor \rfloor = \lfloor \lfloor \eta \rfloor + \lfloor \zeta + \theta \rfloor \rfloor = \ldots$

2.4. **Firing Sandpiles.** The set Q_n is an n^2 dimensional vectorspace over the rational numbers. Similarly $Z_n \subset Q_n$ is an n^2 dimensional vectorspace over the integers.

Let $\delta_x^n \in Z_n$ be the firing sandpile at $x \in N_n^2$, corresponding to the xth row of Δ^n.

$$\forall x, y \in N_n^2 \; \delta_x^n(y) = \Delta_{x,y}^n \tag{6}$$

Let φ be the complete toppling operation on a sandpile that transforms it with toppling matrix, Δ^n.

Definition 2.12. $\varphi : Q_n \to Q_n$. For all $\eta \in Q_n$:

$$\varphi(\eta) = (\Delta^n)^{\mathrm{T}}\eta = \sum_{x\in N_n^2} \eta(x)\cdot\delta_x^n = \left(\prod_{x\in N_n^2} \mathrm{F}_x\right)(\eta) \in R_n.$$

Since Δ^n is invertible, φ is a non-degenerate linear transformation of Q_n and $\forall \eta \in Q_n$ $\varphi^{-1}(\eta) = (\Delta^n)^{-1}\eta$.

For any $\eta \in S_n$, $\varphi^{-1}(\eta)$ gives the number of firings necessary at each point to obtain the minimum sandpile.

$$\left(\prod_{x\in N_n^2} \mathrm{F}_x^{\varphi^{-1}(\eta)(x)}\right)(\eta) = \alpha_n \tag{7}$$

Lemma 2.13. *$\eta \in Q_n^+$ if and only if $\varphi^{-1}(\eta)(x) \in Q_n^+$.*

Proof. $\varphi^{-1}(\eta)(x) \in Q_n^+$ implies $\eta \in Q_n^+$ as:

$$\eta = \sum_{x\in N_n^2} \varphi^{-1}(\eta)(x)\cdot\delta_x^n \geq \sum_{x\in N_n^2} \delta_x^n \geq 0$$

Now, we will prove $\varphi^{-1}(\eta)(x) \in Q_n^+$. Assume to the contrary that $m = \min(\varphi^{-1}(\eta)) < 0$. Let $m_a = \{x \in N_n^2 \mid \varphi^{-1}(\eta)(x) = m\}$. If $x \in m_a$ then

$$\begin{aligned}\eta(x) &= \varphi(\varphi^{-1}(\eta))(x) = 4\,\varphi^{-1}(\eta)(x) - \sum_{y\in \mathrm{N}(x)} \varphi^{-1}(\eta)(y)\\ &= 4m - \sum_{y\in\mathrm{N}(x)} \varphi^{-1}(\eta)(y) \leq (4 - |\,\mathrm{N}(x)|)\cdot m \leq 0\end{aligned}$$

Thus, $\eta(x) \leq 0$, with equality if and only if $|\,\mathrm{N}(x)| = 4$ and $\varphi^{-1}(\eta)(y) = m$ for all $y \in \mathrm{N}(x)$. Since $\eta(x) \geq 0$ we have $\mathrm{N}(x) \subset m_a$ whenever $x \in m_a$. Thus, $m_a = N_n^2$, meaning for all $y \in N_n^2$, $\eta(y) = m$ and $|\,\mathrm{N}(y)| = 4$. Since there exist points on the edges with fewer than 4 neighbors, this is a contradiction. Thus, $m \geq 0$. □

Now that we have the operator φ^{-1}, how we can be represent sandpiles as combinations of the rows of Δ^n is of interest. Thus, we define $D_n \subset Z_n$ to be all integer combinations of the rows of Δ^n, i.e.: the rowspace of Δ^n over $\mathbb{Z}$. Let $D_n^+ = D_n \cap S_n$.

Lemma 2.14. *$\eta \in D_n$ if and only if $\varphi^{-1}(\eta) \in Z_n$.*

Proof. $\eta \in D_n$ means there exists some $\zeta \in Z_n$ such that $(\Delta^n)^{\mathrm{T}}\zeta = \eta$. Thus, $\varphi(\zeta) = \eta$ and $\eta = \varphi^{-1}(\zeta)$. □

Following directly from Lemmas 2.13 and 2.14:

Corollary 2.15. *$\varphi^{-1}(\eta) \in S_n$ if and only if $\eta \in D_n^+$*

Lemma 2.16. *If $\eta \in \Omega_n$ and $\zeta \in D_n^+$ then $\varphi^{-1}(\eta) \leq \varphi^{-1}(\eta \oplus \zeta)$.*

Proof. For some firing sequence $(x_1, x_2, \ldots, x_N)$, Let $\zeta_0 = \zeta$ and $\zeta_i = \zeta_{i-1} - \delta_{x_i}^n$ and let $\eta_0 = \eta + \zeta_0$ and $\eta_i = \eta + \zeta_i$ for all $0 \leq i \leq N$:

$$\eta_0 = \eta + \zeta, \eta_1 = \eta + \zeta - \delta_{x_1}^n, \ldots, \eta_N = \eta + \zeta - \sum_{i=1}^{N} \delta_{x_i}^n = \eta \oplus \zeta$$

$\zeta_i \in D_n$ for all $0 \leq i \leq N$ by the following induction: $\varphi^{-1}(\zeta_0) = \varphi^{-1}(\zeta) \in D_n^+$. Assuming $\varphi^{-1}(\zeta_k) \in D_n$, then $\varphi^{-1}(\zeta_{k+1}) = \mathrm{p}_{x_k}^{-1}\,\varphi^{-1}(\zeta_k) \in D_n$.

We show that $\varphi^{-1}(\zeta_i) \in S_n$ by induction.

From Corollary 2.15, $\varphi^{-1}(\zeta_0) \in S_n$.

Let us assume $\varphi^{-1}(\zeta_k) \in S_n$ for some $0 \leq k < N$.

As $\varphi^{-1}(\zeta_{k+1}) = \mathrm{p}_{x_{k+1}}^{(|\mathrm{N}(x_{k+1})|-4)} \varphi^{-1}(\zeta_k)$, $\varphi^{-1}(\zeta_{k+1}) \notin S_n$ if and only if $\varphi^{-1}(\zeta_k)(x_{k+1}) < 4 - |\mathrm{N}(x_{k+1})|)$ and $\zeta_k(x_{k+1}) \geq 4$, and may thus be fired.

$$\begin{aligned}
\zeta_k(x_{k+1}) &= \left(\sum_{y \in N_n^2} \varphi^{-1}(\zeta_k)(y)\delta_y^n \right)(x_{k+1}) \\
&= 4 \cdot \varphi^{-1}(\zeta_k)(x_{k+1}) - \left(\sum_{z \in \mathrm{N}(x_{k+1})} \varphi^{-1}(\zeta_k)(z)\delta_z^n \right)(x_{k+1}) \\
&\leq 4 \cdot \varphi^{-1}(\zeta_k)(x_{k+1}) - \varphi^{-1}(\zeta_k)|\mathrm{N}(x_{k+1})| \\
&\leq 4 \cdot (4 - |\mathrm{N}(x_{k+1}|) - (4 - |\mathrm{N}(x_{k+1}|) \cdot |\mathrm{N}(x_{k+1})| - 1 \\
&= 16 - 8|\mathrm{N}(x_{k+1})| + |\mathrm{N}(x_{k+1})|^2 - 1 \leq 16 - (8) * 2 + (2)^2 - 1 = 3 < 4
\end{aligned}$$

Thus, if a firing at x_{k+1} would make $\varphi^{-1}(\zeta_k) \notin S_n$, that site is already stable, and thus no firing will take place. Thus $\varphi^{-1}\,\zeta_{k+1} \in S_n$. And thus, by induction, $\varphi^{-1}\,\zeta_N \in S_n$.

As $\varphi^{-1}(\zeta_N) \geq \alpha_n$,

$$\varphi^{-1}(\eta \oplus \zeta) = \varphi^{-1}(\eta + \zeta_N) \geq \varphi^{-1}(\eta)$$

□

2.5. **Group Properties.** We will now define the reachability of a sandpile from another sandpile. This coincides with the intuitive notion of a "larger" sandpile, up to relaxation.

Definition 2.17 (Reachability)**.** A sandpile, $\eta \in \Omega_n$ is reachable from $\zeta \in S_n$ if and only if there there exists a sandpile $\theta \in \Omega_n$ (By Theorem 2.10, this is equivalent to $\theta \in S_n$), such that $\eta = \zeta \oplus \theta$. This is denoted $\zeta \hookrightarrow \eta$.

ζ and η are said to communicate ($\zeta \sim \eta$) if and only if $\zeta \hookrightarrow \eta$ and $\eta \hookrightarrow \zeta$.

Rechability may be used to define a class of stable sandpiles, known as reccurent sandpiles, that are reachable from all stable sandpiles.

Definition 2.18 (Reccurent States)**.** A recurrent sandpile, $\eta \in \Omega_n$, is one that is reachable from all $\zeta \in \Omega_n$. The set of all recurrent sandpiles is thus

$$\mathcal{R}_n = \{\eta \in \Omega_n \mid \forall \zeta \in \Omega_n\ \zeta \hookrightarrow \eta\}$$

Note that for all $\eta \in \mathcal{R}_n$ and $\zeta \in \mathcal{R}_n$, $\eta \sim \zeta$.

Two sandpiles are called equivalent if and only if there exists some sequence of firings sandpile $\varphi^{-1}(\eta - \zeta)$ that transform between the $\eta \in Q_n$ and $\zeta \in Q_n$ are equivalent if and only if $\eta - \zeta \in D_n$, denoted $\eta \simeq \zeta$. Note that if $\zeta \in \Omega_n$, $\eta \to \zeta$ implies $\eta \simeq \zeta$, as some integer number of firings can performed on η to give ζ.

Corollary 2.19. *If $\eta \in \Omega_n$, $\zeta \in \mathcal{R}_n$, and $\eta \simeq \zeta$, then $\varphi^{-1}(\eta) \leq \varphi^{-1}(\zeta)$. If $\eta \in \mathcal{R}_n$, then $\eta = \zeta$.*

Proof. For some $\theta \in D_n^+$, $\zeta = \eta \oplus \theta$. Thus, $\varphi^{-1}(\zeta) = \varphi^{-1}(\eta \oplus \theta)$. By Lemma 2.16, $\varphi^{-1}(\eta) \leq \varphi^{-1}(\zeta)$

If $\eta \in \mathcal{R}_n$, then by the symmetry of $\simeq$, $\varphi^{-1}(\zeta) \leq \varphi^{-1}(\eta)$. Thus, $\eta = \zeta$. □

Corollary 2.20. *If $\eta, \zeta \in \Omega_n$, $\eta \simeq \zeta$, and $\varphi^{-1}(\eta) \leq \varphi^{-1}(\zeta)$, then $\eta \notin \mathcal{R}_n$.*

Proof. Assume to the contrary that $\eta \in \mathcal{R}_n$. Thus, by Corollary 2.19, $\varphi^{-1}(\zeta)(x) \leq \varphi^{-1}(\eta)$, which is a contradiction. □

Theorem 2.21. *$\mathcal{R}_n$ with the operation $\oplus$ forms an abelian group.*

Proof. Let $\eta \in \mathcal{R}_n$ and $\zeta \in \mathcal{R}_n$

(1) $\oplus$ must be associative.
By Corollary 2.11:

$$a \oplus (b \oplus c) = \lfloor a + \lfloor b + c \rfloor \rfloor = \lfloor \lfloor a + b \rfloor + c \rfloor = (a \oplus b) \oplus c$$

Thus $\oplus$ is associative

(2) $\oplus$ must be commutative.
$\eta \oplus \zeta = \lfloor \eta + \zeta \rfloor = \lfloor \zeta + \eta \rfloor = \zeta \oplus \eta$, thus $\oplus$ is commutative,

(3) $\mathcal{R}_n$ must be closed under addition:
By the definition of a recurrent sandpile: For any $\beta_1, \beta_2 \in \Omega_n$, there exists $\theta_1, \theta_2 \in \Omega_n$ such that:

$$\eta = \beta_1 \oplus \theta_1$$

$$\zeta = \beta_2 \oplus \theta_2$$

Thus, $\eta \oplus \zeta = (\beta_1 \oplus \beta_2) \oplus (\theta_1 \oplus \theta_2)$, and $(\beta_1 \oplus \beta_2) \hookrightarrow (\eta \oplus \zeta)$.
As β_2 is varying over all values in Ω_n, we may take it to be α_n, and thus get: $(\beta_1 \oplus \alpha_n) \hookrightarrow (\eta \oplus \zeta)$.
Thus, for all $\beta_1 \in \Omega_n$, $\beta_1 \hookrightarrow (\eta \oplus \zeta)$. Thus $(\eta \oplus \zeta) \in \mathcal{R}_n$.

(4) There exists a unique $-\eta \in \mathcal{R}_n$ such that $\eta \oplus -\eta \oplus \zeta = \eta \ominus \eta \oplus \zeta = \zeta$ As proven by Creutz in [3], $-\eta = (|\Delta^n| - 1) \otimes \eta$, where $|\Delta^n|$ is the determinant of the toppling matrix and the number of recursive states.

(5) There exists an identity $e_n \in \mathcal{R}_n$, with the property that $e_n \oplus \zeta = \zeta$.
From the previous, we see that $e_n = \eta \ominus \eta$.

□

We can now extend our notation of scalar multiplication for recurrent states.

Definition 2.22 (Scalar Multiplication for $\mathcal{R}_n$)**.** For all $\eta \in \mathcal{R}_n$ and $k \in \mathbb{Z}^+$:

$$0 \otimes \eta = e_n$$
$$(-k) \otimes \eta = k \otimes -\eta \tag{8}$$

3. Algorithms to Find the Identity

Definition 3.1. $j_n \in D_n$ is the sandpile such that $j_n(x) = 4 - |\mathrm{N}(x)|$ for all $x \in N_n^2$. This definition implies that $\varphi^{-1}(j_n)(x) = 1$ for all $x \in N_n^2$.

$$\mathrm{mat}(j_n) = \begin{pmatrix} 2 & 1 & \cdots & 1 & 2 \\ 1 & 0 & & 0 & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & 0 & & 0 & 1 \\ 2 & 1 & \cdots & 1 & 2 \end{pmatrix}$$

Lemma 3.2. *For all $\eta \in \mathcal{R}_n$, $j_n \oplus \eta = \eta$.*

Proof. This follows from Corollary 2.19 as $j_n \oplus \eta$ is a recurrent state and it is equivalent to η. □

Lemma 3.3. *$k \otimes j_n \in \mathcal{R}_n$ for some $k \in \mathbb{N}$.*

Proof. Let $\eta_i = i \otimes j_n$ for all $i \in \mathbb{N}$. As $\eta_i = \eta_{i-1} \oplus j_n$, the sequence $\varphi^{-1}(\eta_i)$ is monotonically increasing by Lemma 2.16. Since $\eta_i \in \Omega_n$ and $|\Omega_n| = 4^{n^2} < \infty$ the sequence $\varphi^{-1}(\eta_i)$ must eventually become constant □

Theorem 3.4. *$k \otimes j_n = e_n$ for all $k \geq N$. $k, N \in \mathbb{N}$.*

Proof. By the previous lemma for some N we have $a = k \otimes j_n$ a recurrent state. Since $a \oplus a = 2k \otimes j_n = a$ therefor $a = e_n$ and the Theorem follows. □

Theorem 3.4 leads directly to an algorithm (Algorithm 1) for finding the identity: simply start with j_n and keep adding j_n until no changes are made, which indicates by Lemma 3.2 that a reccurent state has been reached. A similar algorithm (Algorithm 2), starts with j_n, and doubles (with toppling) until no changes are made.

Let $\chi_n \in S_n$ be defined such that $\chi_n(x) = 4$ for all $x \in N_n^2$.

Theorem 3.5. *$e_n = 4 \otimes (\chi_n - \lfloor \chi_n \rfloor)$*

Proof. Let $a = \chi_n - \lfloor \chi_n \rfloor \in T_n$. For any $x \in N_n^2$ the height $a(x) = \chi_n(x) - \lfloor \chi_n \rfloor(x) = 4 - \lfloor \chi_n \rfloor(x) \geq 4 - 3 = 1$. Therefore $4 \cdot a$ has height 4 at every position of the grid and $\lfloor 4 \cdot a \rfloor$ is a recurrent state. Since it is also in T_n it is the unit e_n. □

It is useful to analyse the identity as its toppling matrix, $\varphi^{-1}(e_n)$. From experimental results, this has a convex paraboloid-like structure, especially in the center, but it is much lower on the sides. Also of interest is the relationship between $\varphi^{-1}(e_n)$ and $\varphi^{-1}(e_{n+2})$. Let us define the maximum height on the side, $s_{max} = max(\varphi^{-1}(e_n)(1, j))$, $j \in \mathbb{N}$. $\varphi^{-1}(e_{n+2})(i+1, j+1) \approx \varphi^{-1}(e_n) + s_{max}(i, j)$, for $1 \leq i, j \leq n$. Let us define τ_1 to be this expansion along the sides of $\varphi^{-1}(e_n)$.

$$\tau_{1;n+2}(i,j) = \begin{cases} \varphi^{-1}(e_n) + s_{max}(i,j) & \text{if } 2 \leq i, j \leq n \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$e_{n+2} = k \cdot j_n \oplus \varphi(\tau_1) \tag{10}$$

This provides a good estimate in the center of the sandpile, but toward the edges it overcompensates and requires much toppling. Let us define a better estimate for the sides of the sandpile, τ_2, that expands along the center.

(11)

$$\tau_{2;n+2}(i,j) = \begin{cases} \varphi^{-1}(e_n)(i,j) & \text{if } 1 \leq i, j \leq n/2 - 1 \\ \varphi^{-1}(e_n)(i+2, j+2) & \text{if } \lfloor n/2 \lfloor + \rfloor 1 n/2 \leq i, j \leq n \; \varphi^{-1}(e_n)(i \pm 1, j) \\ \quad \text{if } |i - (\lfloor n/2 \rfloor + 1)| < 1 & \\ \varphi^{-1}(e_n)(i, j \pm 1) & \text{if } |j - (\lfloor n/2 \rfloor + 1)| < 1 \end{cases}$$

(12)

$$e_{n+2} = k \cdot j_n \oplus \varphi(\tau_2)$$

4. Results From Computational Model

As can be seen in Appendix B, the identities approach a stable, fractal state, with much symmetry. Of particular note is how much 2's and 3's (green and red) dominate. In general, there is a large square of 2's at the center, with four triangular patterns of 3's radiating outward to the edges. e_{2n+1} is related to e_{2n} in the following way: after seperating e_{2n} into it's four main symmetrical regions (top left, top right, bottom left, bottom right), create a single empty column and a single empty row, both in the center. Place 0 in the very center, with a single column and row cross of 1's inside of the square of 2's, and 2's filling up the rest.

Let us define $C_n(i) = \{x \in S_n \mid e_n(x) = i \in \{0, 1, 2, 3\}\}$. From the previous, $|C_{2n+1}(i)| \geq C_{2n}(i)|$ for all $i \in \{0, 1, 2\}$, and $|C_{2n+1}(3)| \geq C_{2n}(3)|$. Graphs of all $|C_n|$ for $3 < n < 125$ Let $\overline{C_n} = \frac{|C_n|}{n^2}$ be the normalized $|C_n|$, such that $\sum_{i=0}^{3} \overline{C_n} = 1$.

5. Further Questions

Let us define a function $\sigma : [0, 1]^2 \to \{0, 2, 3\}$ (where $[0, 1]^2$ is the unit square) as follows:

Definition 5.1. For $i = 2, 3$, $\sigma(x) := i$ if and only if there exists an $\varepsilon > 0$ and an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $y \in N_n^2$ with the property $\mathrm{d}_2(x, \frac{y}{n}) < \varepsilon$, $e_n(y) = i$.

$$\begin{aligned} &\sigma(x) = i \iff \exists \varepsilon > 0, N \in \mathbb{Z}^+ \forall n \geq N \forall y \in N_n^2 \ \mathrm{d}_2(x, \frac{y}{n}) < \varepsilon \to e_n(y) = i \\ &\sigma(x) = 0 \text{ otherwise.} \end{aligned} \tag{13}$$

Using σ, we may now define subsets of $[0, 1]^2$.

Definition 5.2. Let $A_i = \{x \mid \sigma(x) = i\}$, for $i = 1, 2, 3$ and $x \in [0, 1]^2$. Both sets A_2 and A_3 are open subsets of the unit square by definition. Let $B_2 \subset A_0$ and $B_3 \subset A_0$ be the boundaries of A_2 and A_3, respectively.

Let $B = B_2 \cap B_3 \subset A_0$ be the common boundary of A_2 and A_3.

We conjecture that the Hausdorff dimension of B is greater then one, but smaller then two.

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Appendix A. Algorithms

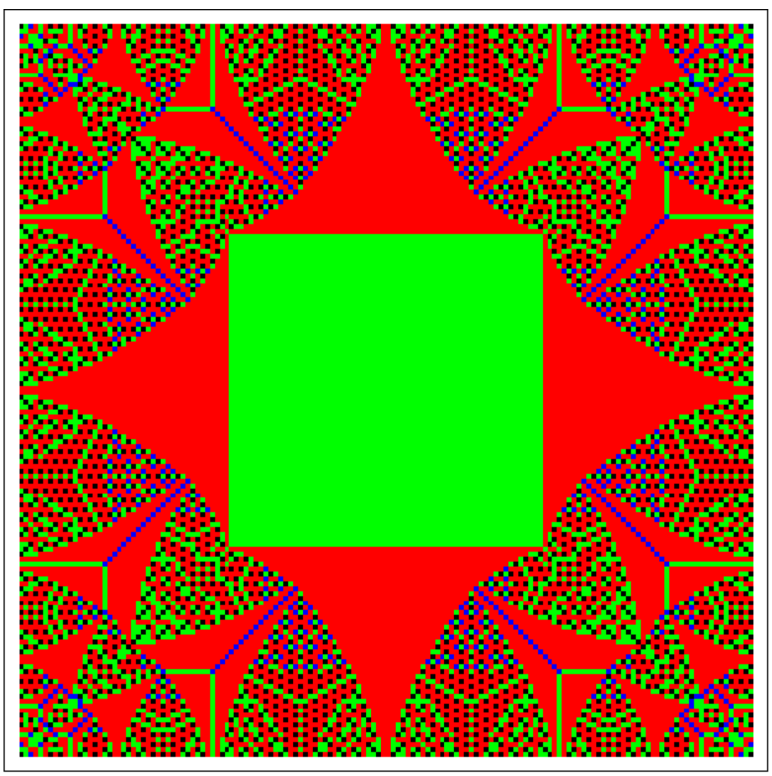
Algorithm 1 Find Identity by adding j_n.

$e_n \leftarrow j_n$
repeat
  $olde_n \leftarrow e_n$
  $e_n \leftarrow e_n \oplus j_n$
until $olde_n = e_n$
return e_n

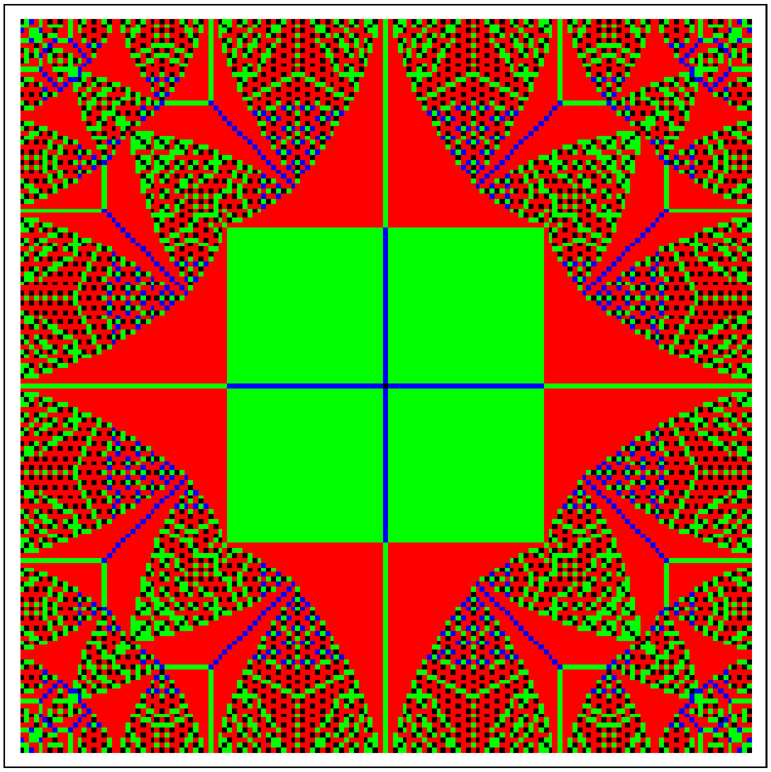
Algorithm 2 Find Identity by starting with j_n and doubling.

$e_n \leftarrow j_n$
repeat
  $olde_n \leftarrow e_n$
  $e_n \leftarrow 2 \cdot e_n$
until $olde_n = e_n$
return e_n

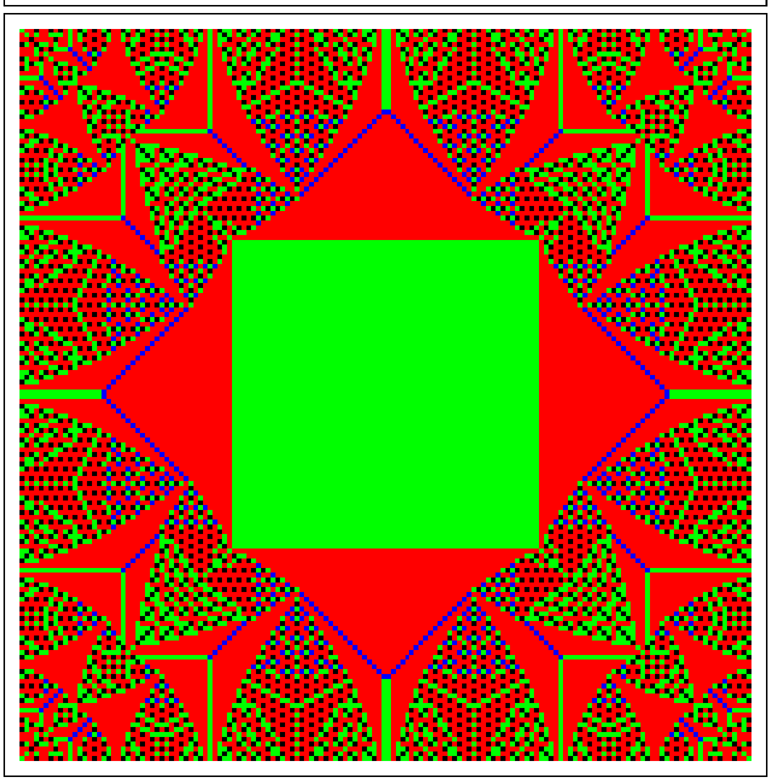
Appendix B. Identities with their toppling vectors and other Data



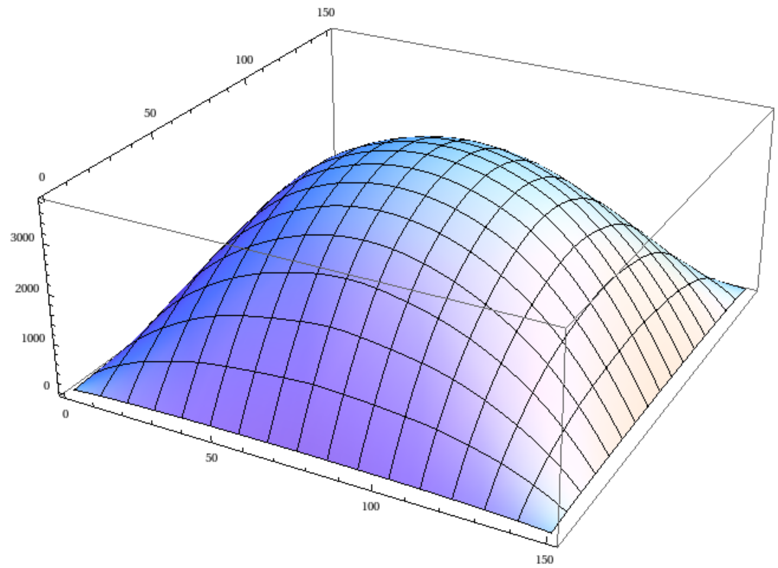
e_{150}



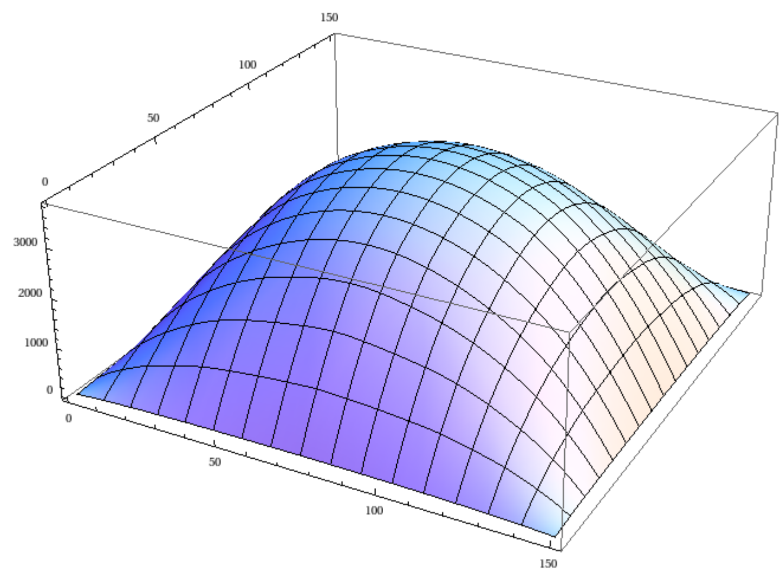
e_{151}



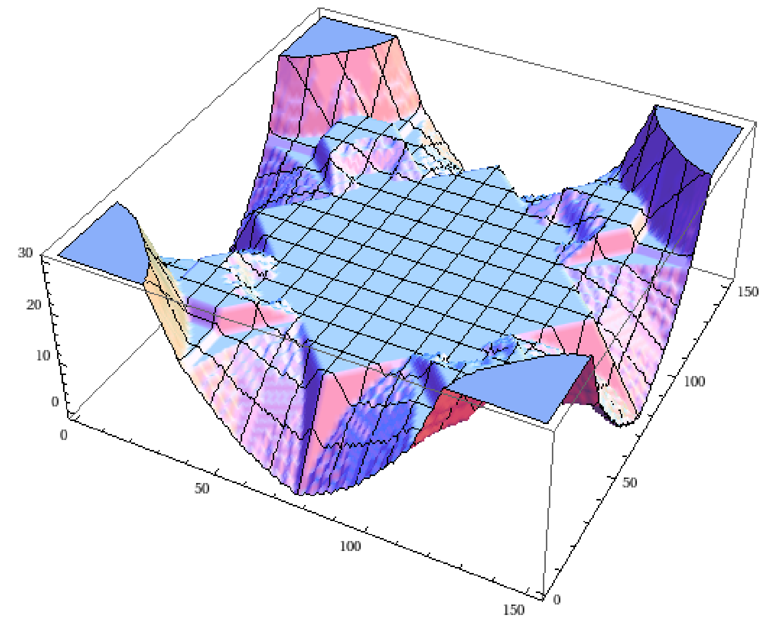
e_{152}



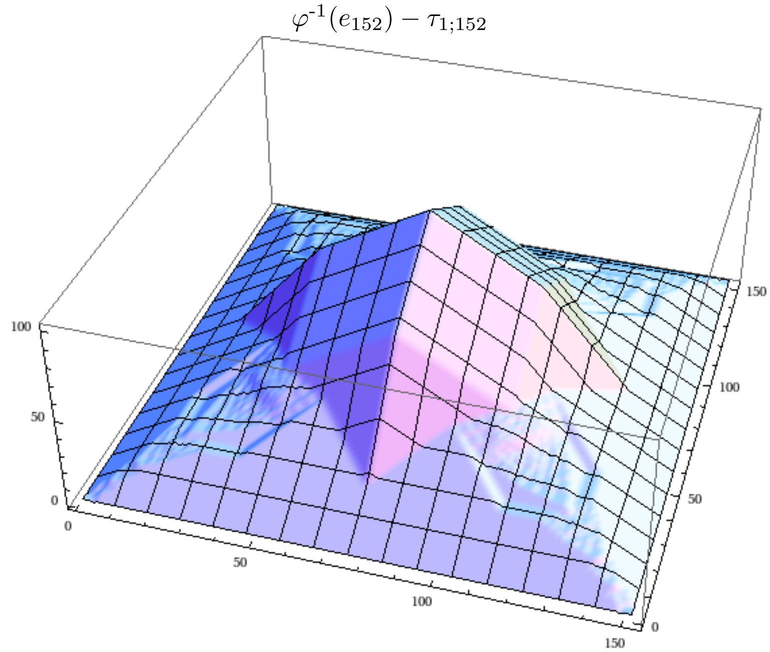
$[\varphi^{-1}(e_{150})]$



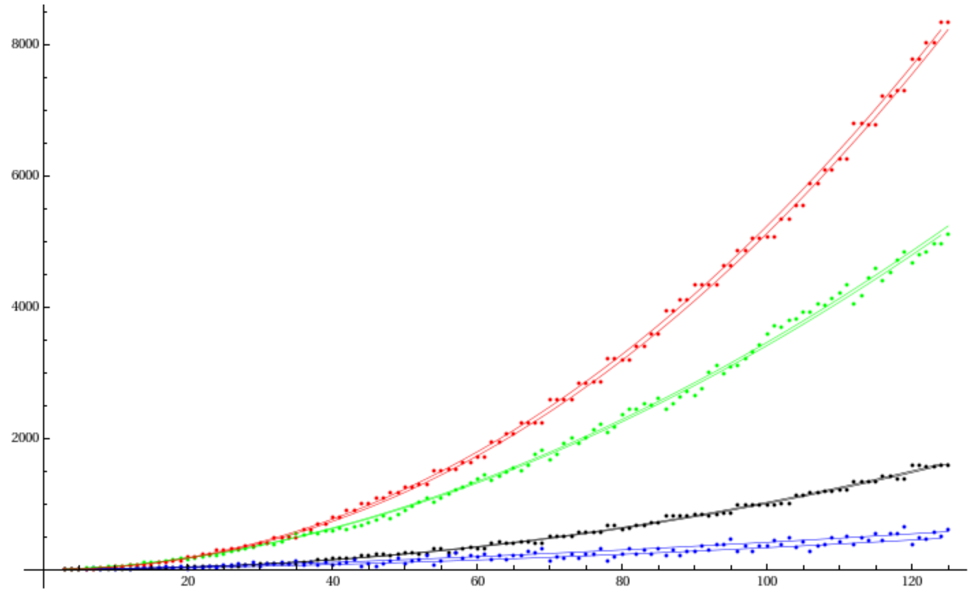
$\varphi^{-1}(e_{150})$



$\varphi^{-1}(e_{152}) - \tau_{1;152}$



$\varphi^{-1}(e_{152}) - \tau_{2;152}$

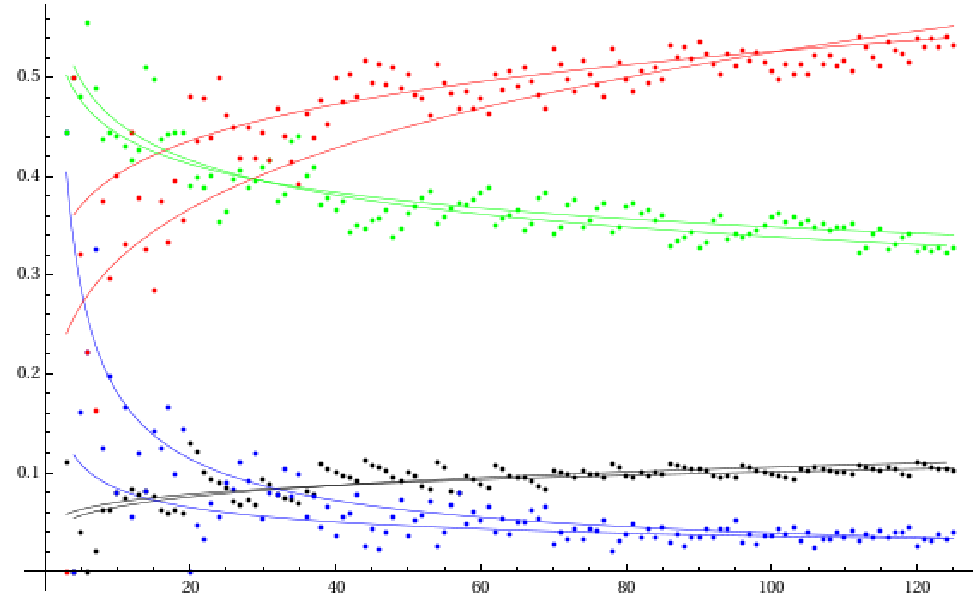


$|C_n|(i) for 3 \leq n \leq 125$ Even n are fit with:

$$|C_n|(0) = 0.0638703x^2.10212$$
$$|C_n|(1) = 0.166812x^1.64857$$
$$|C_n|(2) = 0.642312x^1.86251$$
$$|C_n|(3) = 0.332719x^2.09833$$

Odd n are fit with:

$$|C_n|(0) = 0.0565012x^{2.12427}$$
$$|C_n|(1) = 0.439039x^{1.48653}$$
$$|C_n|(2) = 0.674312x^{1.855}$$
$$|C_n|(3) = 0.289106x^{2.12412}$$



$\overline{C_n}(i)$ for $3 \leq n \leq 125$ Even n are fit with:

$$\overline{C_n}(0) = 0.0403779x^{0.208401}$$

$$\overline{C_n}(1) = 0.197368x^{-0.372473}$$

$$\overline{C_n}(2) = 0.610848x^{-0.128}$$

$$\overline{C_n}(3) = 0.306736x^{0.117251}$$

Odd n are fit with:

$$\overline{C_n}(0) = 0.0485335x^{0.158368}$$

$$\overline{C_n}(1) = 0.840877x^{-0.667482}$$

$$\overline{C_n}(2) = 0.563351x^{-0.104149}$$

$$\overline{C_n}(3) = 0.18834x^{0.22276}$$

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