Math 340, Midterm 1 Name: ______________________

April 1, 2018

Instructions.

- You are allowed one side of handwritten notes.
- No calculators.
- Leave your answers unsimplified. For example, it is preferable for you to write $13C_{4,4}C_{48,1}$ rather than 624.
- There are 7 problems on 5 pages. Make sure your exam is complete.

Page	Points	Score
1	10	
2	14	
3	6	
4	6	
5	14	
Total:	50	

(Poisson thinning) If $X = \text{Poisson}(\lambda)$ and $(X_1, X_2, \dots, X_n)$ are the groupings of X after multinomial thinning with $p_1, p_2, \dots, p_n$ then the X_i are independent and $X_i = \text{Poisson}(p_i\lambda)$.

(Union bound) For any events $A_1, \dots, A_n$ it holds that $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$.

(Bonferroni inequality) For any events $A_1, \dots, A_n$ it holds that $P(\cup_{i=1}^n A_i) \geq \sum_{i=1}^n P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \cap A_j)$.

$P(\text{Poi}(\lambda) = k) = e^{-\lambda}\lambda^k/k!$

$P(\text{Ber}(p) = 1) = p$ and $P(\text{Ber}(p) = 0) = 1 - p$.

$P(\text{Bin}(n,p) = k) = C_{n,k}p^k(1-p)^{n-k}$

$f(x) = \frac{1}{b-a}$ for $x \in [a,b]$ and 0 otherwise if $U = \text{uniform}(a,b)$.

1. In order to create a more exciting Poker experience, a gambler invents a deck of 48 cards with six suits (apples, oranges, lemons, limes, blueberries, and plums). Each suit has cards of eight denominations $A, K, Q, J, 10, 9, 8, 7$. You pick 5 cards out of this deck.

[3 points] (a) What is the probability you get a three of a kind, for example three Kings, one Ace and one 9? (A full house or a four of a kind is not a three of a kind.)

Solution: (a) We pick the value for the three of a kind then suits for the three cards, choose the two values for the other cards and then their suits

$$\frac{8 \cdot C_{6,3} \cdot C_{7,2} \cdot 6^2}{C_{48,5}}$$

[3 points] (b) What is the probability of an anti-flush, i.e., all five cards are from different suits?

Solution: We pick the suits for the five cards and then their values $C_{6,5} \cdot 8^5 / C_{48,5}$.

[4 points] 2. What is the probability that in a room with 30 people there are two people with the same birthday, three people with the same birthday (that is different from the pair of people) and no other birthday collisions?

Solution:

$$C_{30,2} C_{28,3} P_{365,27} / 365^{30} \approx .42.$$

The twins and triplets are distinguishable so we think of having $2 + 25 = 27$ different "people" to assign birthdays to.

3. A waterpark predicts Poi(6000) attendees on sunny Saturdays. They have two main attractions: the **lazy tube float** and the **wavy water zone**. Guests are only allowed to do one of these. Suppose that each guest independently has probability 1/6 of selecting the lazy tube float, 1/2 of selecting the wavy water zone, and 1/3 of doing **neither**.

[2 points] (a) On a sunny Saturday, what is the expected number of guests that do either the lazy tube float or wavy water zone?

Solution: $1000 + 3000 = 4000$

[4 points] (b) The chance of getting heat stroke varies depending on the activity. Suppose that a person gets heat stroke on the lazy tube float with probability 1/2000, in the wavy water zone with probability 1/3000, and doing neither with probability 1/1000. What is the probability ≥ 1 person gets heat stroke?

Solution: We have Poi(1/2), Poi(1) and Poi(2) getting heat stroke. By thinning these are independent. The probability of no heat stroke is

$$e^{-1/2}e^{-1}e^{-2} = e^{-3.5} \approx .03.$$

We subtract this from 1 to get the answer

$$1 - e^{-3.5}s \approx .97.$$

4. A bacteria colony is multiplying. It starts with a single bacterium. Each bacterium takes Ber(p) minutes with $p = 1/9$ to split into three (yes, sometimes it takes 0 minutes). The children then take an independent Ber(p) minutes to split in three and so on.

A biologist is interested in studying all of the bacteria in G_n, the nth generation of this colony (i.e. the bacteria formed after n splits). So there are 3^n of them, each with an ancestry of n splits back to the original.

[2 points] (a) Let $T_1, \dots, T_{3^n}$ be the time it took for each bacterium in G_n to be born starting from the first bacterium. Explain why each T_i is a Bin(n, p) random variable.

Solution: The age is a sum of n independent Bernoulli(p) random variables.

[3 points] (b) Are the T_i independent? Briefly explain.

Solution: No. For example, siblings have the same lineage back to the original, so these are highly dependent.

[3 points] (c) Let $E_{i,n}$ for $i = 1, 2, \dots, 3^n$ be the event that the ith bacteria from G_n took n minutes to be born. What is $P(E_{i,n})$?

Solution: $P(E_{i,n}) = p^n$

[3 points] (d) Let E_n be the event that there exists a bacterium from G_n that is n minutes old. Use a union bound to upper bound $P(E_n)$.

Solution: $P(E_n) \leq \sum_1^{3^n} P(E_{i,n}) = 3^n p^n$.

[3 points] (e) Let E be the event that there exists an n such that a bacterium from G_n is n minutes old. Write E in terms of the E_n and apply a union bound to upper bound $P(E)$. Simplify by using the formula $\sum_{n=1}^{\infty} a^n = \frac{a}{1-a}$.

Solution: $P(E) = P(\cup_1^{\infty} E_n) \leq \sum_1^{\infty} 3^n p^n = \sum_1^{\infty} 3^{-n} = 1/2$.

5. In 2017 Stephen Curry made 47 free throws in a row. Let's say a typical starter takes 100 free throws in a season, and that Stephen Curry makes free throws independent of one another with probability p. Let B be the event that he makes 47 free throws in a row sometime during a season.

[4 points] (a) Let B_0 be that he makes his first 47 free throws, and let B_i for $i = 1, \ldots, 53$ be the event that he misses his ith shot, and then makes his $i+1, \ldots, i+47$ shots. Use these to give an upper bound on $P(B)$ in terms of p.

> **Solution:** $P(B) \leq \sum_0^{53} P(B_i) p^{47} + 53(1-p)p^{47}$

[2 points] (b) Let $A_0 = B_0$ and A_i be that he makes the $i+1, i+2, \ldots, i+47$ shots. Without doing any major calculations, explain why the upper bound you found in (a) is more accurate than if you had instead used a union bound with the A_i.

> **Solution:** The A_i have much more overlap, so we are double counting more outcomes. For example, A_1 and A_2 require making the same 46 shots.

[4 points] 6. Say you get up to six rolls of five dice and are trying to end with as many 6's as possible. At each turn you reroll any dice that are not 6's. What is the probability you get at least four? You can leave your answer as an unsimplified sum of exactly two numbers.

Solution: The number of sixes is $N = \text{Bin}(5, 1-(5/6)^6)$. So the probability of at least 4 sixes is $P(N \geq 4) = C_{5,4}p^4(1-p) + C_{5,5}p^5$ where $p = 1-(5/6)^6$.

7. Let U and U' be independent uniform$(0,1)$ random variables, and $Y = \max\{U, U'\}$.

[2 points] (a) What is the distribution function $F(x) = P(Y \leq x)$?

Solution: $F(x) = x^2$.

[2 points] (b) What is the density function of Y?

Solution: $f(x) = 2x$.

[3 points] (c) What is the density function of $W = e^Y$? Specify where it is 0 and nonzero.

Solution: The inverse of $s(y) = e^y$ is $\log y$. So we have a density $f_W(w) = f_Y(\log w)(1/w) = 2\log y/y$. Since f_Y is nonzero in $[0,1]$ we have f_W is nonzero in $[1,e]$.

[3 points] (d) Explain why the density function $f_{W,Y}(w,y)$ is not equal to $f_W(w)f_Y(y)$ and provide a value (w,y) for which equality fails.

Solution: No because they are dependent. Consider $f_{W,Y}(e,1/2) = 0$ while $f_W(e) = P(Y=1)f_Y(1) = 2$ and $f_Y(1/2) = 1$.