San Francisco State University
ECON 715

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Final Exam

December 18, 2018

Time: 120 minutes

Name: ______________________________

Instructions:

1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (5 points). Suppose that GDP as a function of time t is modelled according to the Cobb-Douglas production function:

$$Y(t) = A(t)\, K(t)^{\theta}\, L(t)^{1-\theta}, \quad 0 < \theta < 1$$

The inputs are capital K and labor L, and A is productivity parameter. All variables are differentiable functions of time. Prove that the growth rate of GDP, is given by:

$$\hat{Y} = \hat{A} + \theta \hat{K} + (1-\theta)\, \hat{L}$$

where the "hat" on top of a variable indicates the growth rate of that variable.

The growth rate of Y is:

$$\begin{aligned}
\hat{Y} &= \frac{d}{dt} \ln Y(t) \\
&= \frac{d}{dt} \left[\ln A(t) + \theta \ln K(t) + (1-\theta) \ln L(t)\right] \\
&= \frac{d}{dt} \ln A(t) + \theta \frac{d}{dt} \ln K(t) + (1-\theta) \frac{d}{dt} \ln L(t) \\
&= \hat{A} + \theta \hat{K} + (1-\theta)\, \hat{L}
\end{aligned}$$

2. (10 points). Suppose the demand curve for cigarettes is $Q = 110 - P$, and the supply is $Q = -10 + P$, and the government imposes a tax of $\$T$ per unit (pack).

   (a) (4 points). Calculate the equilibrium quantity sold as a function of tax amount: $Q(T)$.

   The price that buyers pay is given by the inverse demand curve $P_B = 110 - Q$, and the price that sellers receive is given by the inverse supply curve $P_S = 10 + Q$. Without tax, these two prices are the same. With the tax is $\$T$, we have $P_B = P_S + T$. Thus, equilibrium quantity with tax is:

$$\begin{aligned} 110 - Q &= 10 + Q + T \\ &\Rightarrow Q(T) = \frac{100 - T}{2} \end{aligned}$$

(b) (6 points). Find the tax amount per unit, T^*, that maximizes the government tax revenues from this market $R(T)$. In your calculations clearly demonstrate the first order necessary condition and the second order sufficient condition for **unique global maximum**.

The government tax revenue is

$$R(T) = Q(T) \cdot T = \left(\frac{100 - T}{2}\right) T = 50T - \frac{1}{2}T^2$$

The government solves

$$\max_{T} R(T) = 50T - \frac{1}{2}T^2$$

The first order necessary condition for maximum:

$$\begin{aligned} R'(T) &= 50 - T = 0 \\ T^* &= 50 \end{aligned}$$

The second order sufficient condition for maximum: $R''(T^*) < 0$

$$R''(50) = -1 < 0$$

The second order sufficient condition is satisfied for any T, and therefore $R(T)$ is strictly concave, and $T^* = 50$ is the unique global maximizer.

3. (25 points). Suppose that the market value of an asset, at time t, is given by $V(t)$ - a twice differentiable function of time. Suppose that the interest rate per period (year) is r.

   (a) The owner of the asset wishes to maximize the present value of the asset, by selling it at the right time. Write the optimization problem of the owner, and derive the first order necessary condition for optimal asset holding.

   Optimization problem is:

$$\max_{t} PV(t) = V(t) e^{-rt}$$

   First order necessary condition for maximum is:

$$\begin{aligned} \frac{d}{dt} PV(t) &= V'(t) e^{-rt} - rV(t) e^{-rt} = 0 \\ &\Rightarrow V'(t) - rV(t) = 0 \end{aligned}$$

   (b) Provide economic intuition of the first order necessary condition from the previous section.

   The first order condition can be written as

$$\frac{V'(t)}{V(t)} = r$$

   This means that the owner should keep the asset until the growth rate of its value equalizes to the interest rate.

(c) Suppose the value of the asset evolves according to $V(t) = Ke^{f(t)}$, where $f(t) = 0.22\ln(1+t)$, where t is time in years and the interest rate is $r = 2\%$. Find the optimal holding time, t^*, of the asset.

$$\begin{aligned}
\frac{V'(t)}{V(t)} &= r \\
\frac{Ke^{f(t)} \cdot f'(t)}{Ke^{f(t)}} &= r \\
f'(t) &= \frac{0.22}{1+t^*} = r \\
t^* &= \frac{0.22}{0.02} - 1 = 11 - 1 = 10 \text{ years}
\end{aligned}$$

(d) Suppose that $f(t)$ is unknown, but it is given that f is increasing and concave ($f'(t) > 0\ \forall t$, and $f''(t) < 0\ \forall t$). Prove that the optimal holding time, t^*, is decreasing in interest rate r.

Proof 1. Taking differential of both sides of the optimality condition $f'(t) = r$:

$$\begin{aligned} f''(t)\, dt &= dr \\ &\Rightarrow \frac{dt}{dr} = \frac{1}{f''(t)} < 0 \end{aligned}$$

The last inequality follows from the given that $f''(t) < 0$.

Proof 2. Write the F.O.N.C. as implicit function:

$$F(t, r) = f'(t) - r = 0$$

Then, by the implicit function theorem:

$$\frac{dt}{dr} = -\frac{F_r}{F_t} = -\frac{-1}{f''(t)} = \frac{1}{f''(t)} < 0$$

(e) Provide economic intuition for the result in the previous section.

The interest rate is the opportunity cost of holding the asset for additional time. The owner can sell the asset, and invest the money at interest rate r. Thus, higher interest rate, means that the opportunity cost of holding the asset is higher, and the seller would like to sell it earlier.

4. (25 points). Consider a monopoly that sells a single product to n segmented markets. The revenue function in market i, as a function of quantity of product sold in that market, is $R_i(Q_i)$, with $R_i'(Q_i) > 0$ and $R_i''(Q_i) < 0$. Assume that the total cost function, $C(Q) = C\left(\sum_{i=1}^{n} Q_i\right)$, is increasing and strictly convex.

   (a) Write the optimization problem of the monopoly, and derive the first order necessary condition.

   Profit maximization problem:

$$\max_{Q_1,...,Q_n} \pi(Q_1, ..., Q_n) = \sum_{i=1}^{n} R_i(Q_i) - C(Q)$$

   First order necessary condition:

$$\frac{\partial}{\partial Q_i}\pi(Q_1, ..., Q_n) = R_i'(Q_i) - C'(Q) = 0 \quad \forall i = 1, ..., n$$

   (b) Provide economic interpretation of the first order necessary condition from the previous section.

   The above condition states that the Marginal Revenue (MR) in all markets must be equal to the common marginal cost.

(c) Show that the monopoly's selling price in market i is equal to the markup rate μ_i above the marginal cost, i.e.

$$p_i = [1 + \mu_i]\, MC$$

where the markup rate is $\mu_i = 1/\left(|\eta_i| - 1\right)$, and η_i is the price elasticity of demand in market i.

The revenue in any market is $R(Q) = P(Q)\,Q$ (here we omit the market subscript i for ease of notation). Thus, the Marginal Revenue can be expressed as a function of price elasticity of demand:

$$MR = \frac{d}{dQ}R(Q) = \frac{dP}{dQ}\cdot Q + P = P\left[1 + \frac{dP}{dQ}\frac{Q}{P}\right] = P\left[1 + \frac{1}{\eta}\right] = P\left[1 - \frac{1}{|\eta|}\right]$$

Using the $MR = MC$ profit maximization condition:

$$\begin{aligned} P\left[1 - \frac{1}{|\eta|}\right] &= MC \\ P &= \left[\frac{|\eta|}{|\eta| - 1}\right] MC = \left[1 + \frac{|\eta|}{|\eta| - 1} - 1\right] MC \end{aligned}$$

Thus, the selling price in market i is

$$P_i = \left[1 + \frac{1}{|\eta_i| - 1}\right] MC = [1 + \mu_i]\, MC$$

(d) Suppose that Genentech (a pharmaceutical company) sells its medicine to India and U.S., and price elasticities of demand in the two countries are $\eta_{India} = -11$, $\eta_{US} = -1.1$. Find the markup rate (in % above the marginal cost) that Genentech will charge in the two countries.

$$\begin{aligned} \mu_{India} &= \frac{1}{|\eta_{India}| - 1} = \frac{1}{11 - 1} = 0.1 = 10\% \\ \mu_{US} &= \frac{1}{|\eta_{US}| - 1} = \frac{1}{1.1 - 1} = 10 = 1000\% \end{aligned}$$

Thus, the company will charge 1000% markup in the U.S. and 10% markup in India.

(e) Prove that the critical value of the profit function is a unique global maximum. Clearly state the theorems used in your proof.

Proof. It is given that $R_i(Q_i)$ is strictly concave $\forall i$, and therefore $\sum_{i=1}^{n} R_i(Q_i)$ is strictly concave (sum of strictly concave functions is str. concave). The cost function is given to be strictly convex, so $-C(Q)$ is strictly concave (f is concave if and only is $-f$ is convex, strict or not). Thus, the profit function is a sum of strictly concave functions and therefore strictly concave. Consequently, the critical point of a strictly concave function is automatically a unique global maximum.

5. (15 points). Suppose that consumer spends income I on n goods. Let the quantities purchased on the n goods (consumption bundle) be $x = (x_1, ..., x_n)$ and the (given) market prices be $p = (p_1, ..., p_n)$. The budget set is defined as the set of all consumption bundles, which cost no more than consumer's income:

$$B = \{x | p'x \leq I\}$$

(a) Prove that the budget set is convex.

Let x^1, x^2 be two bundles in the budget set, and $\alpha \in (0,1)$. We need to prove that $\alpha x^1 + (1-\alpha) x^2 \in B$. The two bundles satisfy

$$\begin{aligned} p'x^1 &\leq I \\ p'x^2 &\leq I \end{aligned}$$

Multiply the first inequality by α and the second by $1-\alpha$

$$\begin{aligned} \alpha p'x^1 &\leq \alpha I \\ (1-\alpha) p'x^2 &\leq (1-\alpha) I \end{aligned}$$

and add the two:

$$\begin{aligned} \alpha p'x^1 + (1-\alpha) p'x^2 &\leq \alpha I + (1-\alpha) I \\ p' \left[\alpha x^1 + (1-\alpha) x^2\right] &\leq I \end{aligned}$$

Thus, $\alpha x^1 + (1-\alpha) x^2 \in B$.

(b) Suppose consumer derives utility from n goods according to utility function $u(x) = u(x_1, ..., x_n)$. Write the consumer's utility maximization problem, assuming that the consumer must sepnd all his income on cinsumption, and derive the first order necessary conditions for optimal consumption bundle.

$$\max_x u(x)$$
$$s.t.$$
$$p'x = I$$

The Lagrange function:

$$\begin{aligned}
\mathcal{L} &= u(x) - \lambda [p'x - I] \\
\mathcal{L} &= u(x_1, ..., x_n) - \lambda [p_1 x_1 + ... + p_n x_n - I]
\end{aligned}$$

The first order necessary conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} = ... = \frac{\partial \mathcal{L}}{\partial x_n} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

(c) Suppose $x^* = (x_1^*, ..., x_n^*)$ satisifies the first order necessary conditions in the previous section. Write the sufficient condition which will guarantee that x^* is the **unique constrained global maximum**.

We proved in part a that the budget set is convex. If in addition the utility function is **strictly quasiconcave**, then the critical point x^* is the unique constrained global maximum.

6. (10 points). Prove that any Cobb-Douglas function $u(x_1, ...x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$, with $\alpha_i > 0$ $\forall i = 1, ..., n$, is strictly quasiconcave. Clearly state the theorems used in your proof.

The given function can be written as

$$u(x_1, ...x_n) = \exp\left(\sum_{i=1}^{n} \alpha_i \ln x_i\right)$$

The $\ln(\cdot)$ function is strictly concave ($\frac{d}{dx}\ln(x) = x^{-1}$, $\frac{d^2}{dx^2}\ln(x) = -x^{-2} < 0$), the weighted sum of strictly concave functions $\sum_{i=1}^{n} \alpha_i \ln x_i$ is strictly concave. The exponential function $\exp(\cdot)$ is monotone increasing, and since monotone increasing transformation of strictly concave function is strictly quasiconcave, $u(x_1, ...x_n)$ is strictly quasiconcave.

7. (10 points). Consider the Matlab script below:

```
1
2   syms x y p_x p_y I k a
3   u(x,y) = a*log(x) + (1-a)*log(y);
4   L(x,y,k) = u(x,y) - k*(p_x*x + p_y*y - I);
5   [x,y,k] = solve(gradient(L,[x,y,k])==0, [x,y,k]);
6   d = subs([x,y],[a,p_x,p_y,I],[0.5,3,0.75,100]);
```

(a) What is the purpose of the entire program?

The program defines a lagrange function corresponding to a consumer utility maximization problem, and solves for the demand.

(b) What is the role of the variable `k` in the above program?

Lagrange multiplier.

(c) What does the function **`gradient`** in line 5 do?

Differentiates the Lagrange function with respect to `x,y` and `k`, i.e. obtaining the First Order Necessary Conditions for the optimization problem.

(d) What is the purpose of line 6? In particular, what is the purpose of the numerical values in line 6?

The command in line 6 substitutes the numerical values of the parameters into the symbolic solution, to get the numerical demand and lagrange multiplier. Here `a` is the weight on `x` in the utility function, so the utility is $u(x, y) = 0.5 \ln x + 0.5 \ln y$. The values 3,0.75 are the prices p_x, p_y and consumer's income is $I = 100$.