Parameterized Complexity of Weak Odd Domination Problems

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Abstract. Given a graph $G = (V, E)$, a subset $B \subseteq V$ of vertices is a *weak odd dominated* (WOD) set if there exists $D \subseteq V \setminus B$ such that every vertex in B has an odd number of neighbours in D. $\kappa(G)$ denotes the size of the largest WOD set, and $\kappa'(G)$ the size of the smallest non-WOD set. The maximum of $\kappa(G)$ and $|V| - \kappa'(G)$, denoted $\kappa_Q(G)$, plays a crucial role in quantum cryptography. In particular deciding, given a graph G and $k > 0$, whether $\kappa_Q(G) \leq k$ is of practical interest in the design of graph-based quantum secret sharing schemes. The decision problems associated with the quantities κ, κ' and κ_Q are known to be NP-Complete. In this paper, we consider the approximation of these quantities and the parameterized complexity of the corresponding problems. We mainly prove the fixed-parameter intractability (W[1]-hardness) of these problems. Regarding the approximation, we show that κ_Q, κ and κ' admit a constant factor approximation algorithm, and that κ and κ' have no polynomial approximation scheme unless P=NP.

1 Introduction

The odd domination problem is a domination type problem on a graph $G = (V, E)$ that consists of finding a set D of vertices such that each vertex has an odd number of neighbours in D, equivalently each vertex is in the close odd neighbourhood of D defined as $Odd[D] = \{u \in V, |N[u] \cap D| = 1 \bmod 2\}$. The odd domination falls into the general framework of (σ, ρ)-domination [5,9]. The parameterized complexity of these problems has been studied, in particular in the parity cases [4]. Weak odd domination is a variation of odd domination, which does not fall into the general framework of (σ, ρ)-domination. Given a graph $G = (V, E)$, a *Weak Odd Dominated* (WOD) set is a set $B \subseteq V$ such that there exists $D \subseteq V \setminus B$ with $B \subseteq Odd(D) := \{v \in V \setminus D, |N(v) \cap D| = 1 \bmod 2\}$, in other words, every vertex in B has an odd number of neighbours in D. The Lemma 1 in [3] gives a good characterization of non-WOD sets: $B \subseteq V$ is not WOD if and only if $\exists C \subseteq B$ such that $|C| = 1 \bmod 2$ and $Odd(C) \subseteq B$. Since a subset of a WOD set is WOD and a superset of a non-WOD is non-WOD, we focus on the largest WOD set and the smallest non-WOD set:

Definition 1 *Given a graph $G = (V, E)$,*

$$\kappa(G) = \max_{B \text{ WOD}} |B| = \max_{D \subseteq V} |Odd(D)| \qquad \kappa'(G) = \min_{B\ \neg\text{WOD}} |B| = \min_{C \subseteq V, |C| = 1 \bmod 2} |C \cup Odd(C)|$$

Weak odd dominated sets have a simple interpretation in a variant of the sigma-game with *fragile* bulbs: given a graph G, to each vertex is attached a bulb which has three possible states: 'on', 'off', and 'broken'; when one plays on a bulb, it makes this bulb 'broken' and flips the states 'on'/'off' of its neighbours. In the initial configuration all bulbs are 'off'. The size $\kappa(G)$ of the largest WOD set corresponds to the largest number of (unbroken) 'on' bulbs one can obtain. Indeed, when one plays a set D of bulbs, $Odd(D)$ is the set of 'on' bulbs.

Weak odd domination is strongly related to graph-based quantum secret sharing protocols, defined in [8]. These protocols are represented by graphs in which every vertex represents a player. It has been proved in [3], that for a quantum secret sharing protocol based on a graph G of order n, $\kappa_Q(G)$ defined as $\max(\kappa(G), n{-}\kappa'(G))$ is the minimal threshold such that any set of more than $\kappa_Q(G)$ players can recover the secret. Graphs with a small quantum threshold (i.e. $\kappa_Q(G){\leq}0.811n$ for a graph G of order n) have been proved to exist using non constructive methods [3]. In fact, a random graph has a small κ_Q with high probability (see [3] for details). Thus, deciding whether a graph has a small threshold is crucial for the generation of good graph-based quantum secret sharing protocols. Unfortunately this problem has been proved to be NP-complete [3].

Since the decision problem associated with κ_Q, as well as those associated with κ and κ' are NP-complete [3], we consider two approaches to tackle the hardness of these problems: parameterized algorithms and approximations.

Parameterized Complexity. Several NP-Complete problems, like deciding whether a graph of order n has a vertex cover of size at most k, have been proved to be fixed parameter tractable, i.e. they can be solved in time $f(k).n^{O(1)}$ for some computable function f. The parameterized complexity hierarchy [1]: FPT $\subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq XP$ is, roughly speaking, a way to distinguish the problems which are fixed parameter tractable (FPT) from those which are not.

Approximation. An optimization problem belongs to APX if it admits a constant factor approximation algorithm. It admits a polynomial approximation scheme if for any $\epsilon > 0$ it admits a $(1+\epsilon)$-approximation [11].

In this paper, the approximation and parameterized complexity of weak odd domination problems are explored. Section 2 is dedicated to bounds on the weak odd domination. We prove a lower bound on κ and an upper bound on κ' using probabilistic methods. We also prove a strong duality property between κ and κ'. In section 3 we define the parameterization of the weak odd domination problems, then we prove using cyclic reductions that WOD Set Of Size At Least $n{-}k$, Non-WOD Set Of Size At Most k and Quantum Threshold At Least $n{-}k$ are equivalent to Oddset, which has been proved to be hard for W[1] and to belong to W[2] in [2], even for bipartite graphs, thus they have no FPT-algorithms unless the parameterized hierarchy collapses with W[1] = FPT. Finally, in section 4 we define the optimisation version of the weak odd domination problems and then prove that Max non-Accessible Set is in APX and that Max WOD Set and Min non-WOD Set are complete for APX, by reduction from Max 3-SAT B which is hard for APX [10], thus Max WOD

Set and Min non-WOD Set have no polynomial approximation scheme unless P=NP.

2 Bounds on WOD sets

In this section we improve the known bounds on the largest WOD and smallest non-WOD sets in a graph. These improved bounds are essential for the choice of the parameterization of the corresponding problems. The largest WOD set of a graph G of order n and degree Δ satisfies $\Delta \leq \kappa(G) \leq \frac{n\Delta}{\Delta+1}$ [3]. The bound $\kappa(G) \geq \Delta$ is coming from the simple fact that any vertex is oddly dominating its neighbourhood. We improve this bound using probabilistic methods.

Lemma 1 *For any graph G of order n and minimal degree $\delta > 0$,*

$$\kappa(G) \geq \left(\frac{1}{2} - \frac{1+\log(2\delta)}{4\delta}\right) n$$

Moreover, $\kappa(G) \geq \frac{n}{4}$ when $\delta \geq 1$, and $\kappa(G) \geq \frac{8n}{27}$ when $\delta \geq 2$.

Proof. The proof consists in evaluating the expected size of the odd neighbourhood of a randomly chosen set of vertices. Given $q \in [0.5, 1]$, let D be a subset of vertices created by choosing each $v \in V(G)$ independently with probability $1-q$. The expected size of D is $(1-q)n$. For every v, the probability that $v \in Even(D) := \{u \in V \setminus D, |N(u) \cap D| = 0 \bmod 2\}$, is $P_0(v) = q.\sum_{k=0}^{\delta(v)/2} \binom{\delta(v)}{2k}(1-q)^{2k} q^{\delta(v)-2k}$, and the probability that $v \in Odd(D)$ is $P_1(v) = q.\sum_{k=0}^{\delta(v)/2} \binom{\delta(v)}{2k+1}(1-q)^{2k+1} q^{\delta(v)-2k-1}$. Notice that v is not in D with probability $P_0(v)+P_1(v) = q$, moreover $P_0(v)-P_1(v) = q\sum_{k=0}^{\delta(v)} \binom{\delta(v)}{k}(q-1)^k q^{\delta(v)-k} = q(2q-1)^{\delta(v)}$. As a consequence, $P_1(v) = \frac{1}{2}(q-q(2q-1)^{\delta(v)})$, and the expected size of $Odd(D)$ is $E[|Odd(D)|] = \sum_{v\in V(G)} \frac{1}{2}(q-q(2q-1)^{\delta(v)})$. Let $x = 2q-1$ and δ the minimal degree of G, $E[|Odd(D)|] \geq \frac{n}{4}(x+1)(1-x^\delta)$, which is maximal for $x = 0$ when $\delta = 1$, so $E[|Odd(D)|] \geq \frac{n}{4}$. Thus there exists $D \subseteq V(G)$ such that $|Odd(D)| \geq \frac{n}{4}$ so $\kappa(G) \geq \frac{n}{4}$. When $\delta = 2$, $\frac{n}{4}(x+1)(1-x^2)$ is maximal for $x = \frac{1}{3}$, so $E[|Odd(D)|] \geq \frac{8n}{27}$. In the general case, $E[|Odd(D)|] \geq \frac{n}{4}(1+x-2x^\delta)$, which is maximal for $x = (2\delta)^{-\frac{1}{\delta-1}}$. So $E[|Odd(D)|] \geq \frac{n}{4}(1+(2\delta)^{-\frac{1}{\delta-1}}-2(2\delta)^{-\frac{\delta}{\delta-1}}) = \frac{n}{4}(1+\frac{\delta-1}{\delta}e^{-\frac{\log(2\delta)}{\delta-1}}) \geq \frac{n}{4}(1+\frac{\delta-1}{\delta}(1-\frac{\log(2\delta)}{\delta-1})) = n(\frac{1}{2}-\frac{1+\log(2\delta)}{4\delta})$. □

The bound of lemma 1 is not known to be tight. For the graph C_5^k of order $n = 5k$ composed of the disjoint union of k C_5, $\kappa(C_5^k) = 2n/5$. Regarding connected graphs, the largest WOD set of a comb graph of order $2k$ (a path P_k with a pending vertex on every vertex of the path) is of size k. We conjecture that for any connected graph G, $\kappa(G) \geq \lfloor n/2 \rfloor$.

Most of the graphs of order n have no WOD set larger than $0.811n$. Indeed, theorem 8 in [3] implies that a random graph $G(n, 1/2)$ (graph of order n where every possible pair of vertices has an edge with probability 1/2), $Pr(\kappa(G(n, 1/2)) \leq 0.811n) \geq 1 - \frac{1}{n}$.

Similarly to the largest WOD set, the smallest non-WOD set of a graph G of order n and minimal degree δ satisfies $\frac{n}{\delta+1} \leq \kappa'(G) \leq \delta+1$ [3]. The bound

$\kappa'(G) \leq \delta{+}1$ is coming from the fact that any vertex together with its neighbourhood is not a WOD set. Notice that a similar probabilistic technique as of the proof of Lemma 1 fails to improve this bound: the expected size of $D \cup Odd(D)$ for a randomly chosen subset D does not produce an upper bound on the smallest non-WOD set because of the additional constraint that D must be of odd size. Instead, we improve the upper bound for the smallest non-WOD set by strengthening the duality property $\kappa'(G){+}\kappa(\overline{G}){\geq}n$ proved in [3] as follows:

Lemma 2 *For any graph G of order n, $n - \kappa(\overline{G}) \leq \kappa'(G) \leq n - \frac{\kappa(\overline{G})}{2}$.*

Proof. The proof consists in showing that for any graph G, $\kappa'(\overline{G}){\leq}n{-}\frac{\kappa(G)}{2}$. To this end, first we show that $\exists D{\subseteq}V(G)$ s.t. $|D|{=}1 \bmod 2$ and $|Odd(D)|{\geq}\frac{\kappa(G)}{2}$. Indeed let $D{\subseteq}V(G)$ be a non empty set s.t. $|Odd(D)|{=}\kappa(G)$. If $|D|{=}1 \bmod 2$ then we obviously have $|Odd(D)|{\geq}\frac{\kappa(G)}{2}$. Othewise, if D is of even size then $\forall v{\in}D$, $|N(v)|{+}|Odd(D{\setminus}\{v\})|{\geq}|Odd(D)|{=}\kappa(G)$. So either $\{v\}$ or $D{\setminus}\{v\}$, which are both of odd size, has an odd neighbourhood larger than $\frac{\kappa(G)}{2}$. Thus, $\exists C{\subseteq}V(G)$ s.t. $|C|{=}1 \bmod 2$ and $|Odd(C)|{\geq}\frac{\kappa(G)}{2}$. Since $|C|{=}1 \bmod 2$ implies that $\forall v \notin C$, $v{\in}Odd(C) \Leftrightarrow v{\notin}Odd_{\overline{G}}(C)$ (where $Odd_{\overline{G}}(C)$ is the odd neighbourhood of C in $\overline{G}$), $|C{\cup}Odd_{\overline{G}}(C)|{\leq}n{-}\frac{\kappa(G)}{2}$. Thus $\kappa'(\overline{G}){\leq}|C{\cup}Odd_{\overline{G}}(C)|{\leq}n{-}\frac{\kappa(G)}{2}$. □

Corollary 1 *For any graph G of order n and degree $\Delta < n-1$,*

$$\kappa'(G) \leq \frac{7n}{8} \qquad \text{and} \qquad \kappa'(G) \leq \left(\frac{3}{4} + \frac{1+\log(2(n-\Delta-1)}{8(n-\Delta-1)}\right) n$$

The restriction $\Delta < n{-}1$ in Corollary 1 is crucial since for any n, $\kappa'(K_n) = n$, where K_n is the complete graph of order n. Similarly, the condition $\delta > 0$ in lemma 1 is necessary since $\kappa(\overline{K_n}) = 0$. We consider the class of graphs which do not satisfy these conditions, i.e. graphs having a universal vertex or an isolated vertex, and show that such graphs satisfy a stronger duality property:

Lemma 3 *For any graph G of order n with a universal or isolated vertex,*

$$\kappa'(G) + \kappa(\overline{G}) = n$$

Proof. If G has an isolated vertex then $\kappa'(G){=}1$ and $\kappa(\overline{G}){=}n{-}1$. Otherwise, let u be a universal vertex in G. Let $D{\subseteq}V(G)$ s.t. $|Odd_{\overline{G}}(D)|{=}\kappa(\overline{G})$. Notice that u is isolated in $\overline{G}$, so $|Odd_{\overline{G}}(D{\oplus}\{u\})|{=}|Odd_{\overline{G}}(D)|{=}\kappa(\overline{G})$, where $\oplus$ denotes the symmetric difference. Since either D or $D{\oplus}\{u\}$ is of odd size, $\exists C{\in}\{D, D{\oplus}\{u\}\}$ s.t. $|C|{=}1 \bmod 2$ and $|Odd_{\overline{G}}(C)|{=}\kappa(\overline{G})$. Moreover $|Odd_G(C)|{=}n{-}\kappa(\overline{G})$, so $\kappa'(G) \leq n{-}\kappa(\overline{G})$, which implies $\kappa'(G){=}n{-}\kappa(\overline{G})$ since $\kappa'(G){\geq}n{-}\kappa(\overline{G})$ for any graph. □

The strong duality property gives a way to get ride of the universal and isolated vertices in the computation of $\kappa(G)$ and $\kappa(\overline{G})$. For instance, if G has a universal vertex u, $\kappa'(G) = n - \kappa(\overline{G}) = n - \kappa(\overline{G} \setminus u)$, since u is isolated in $\overline{G}$.

3 Parameterized Complexity

3.1 Parameterization of weak odd domination problems

The results of the previous section imply that when parameterised by the size of the WOD set, the largest WOD set problem is in FPT. More precisely, given a graph G and a parameter k, deciding whether $\kappa(G) \geq k$ is fixed parameter tractable using the following algorithm `Kappa`(G, k):
• Remove all isolated vertices. Let n be the order of the resulting graph G'
• If $k \leq \frac{n}{4}$ then 'true'
• Else if $\forall D \subseteq V(G')$, $|Odd(D)| < k$ then 'false' else 'true'.

The first step of the algorithm is correct since for any isolated vertex u, $\kappa(G) = \kappa(G \setminus \{u\})$. The complexity of the algorithm is $O^*(2^n)$ and since in the worst case $k = \frac{n}{4}$ the complexity in fonction of k is $O^*(2^{4k})$, so the problem WOD of size at least k is FPT.

Regarding the smallest non-WOD set, given a graph G of order n and a parameter k, deciding whether $\kappa'(G) \leq n{-}k$ is fixed parameter tractable using the following algorithm `Kappa'`(G, k):
• If G has a universal vertex u then `Kappa`$(\overline{G} \setminus u, k)$
• Else if $k \leq n/8$ then 'true'
• Else if $\forall D$ of odd size $|D \cup Odd(D)| > n - k$ then 'false' else 'true'.

The correctness of the first step is based on Lemma 3: if G has a universal vertex u, $\kappa'(G) = n - \kappa(\overline{G}) = n - \kappa(\overline{G} \setminus u)$.

The fixed parameter tractability of the problems WOD of size at least k and non-WOD of size at most $n{-}k$ is not relevant because only based on the existence of bounds for $\kappa(G)$ and $\kappa'(G)$. As a consequence, we focus in the rest of this paper on the dual parameterization of these problems:

WOD Set Of Size At Least $n{-}k$	non-WOD Set Of Size At Most k
input: A graph G of order n	input: A graph G of order n
parameter: An integer k	parameter: An integer k
question: Is $\kappa(G) \geq n{-}k$?	question: Is $\kappa'(G) \leq k$?

Concerning the quantum threshold problem, given a graph G of order n and a parameter k deciding whether $\kappa_Q(G) = \max(\kappa(G), n{-}\kappa'(G)) \geq k$ is in FPT since $\kappa_Q(G) \geq 0.506n$ [6]. As a consequence we consider its dual parameterization:

Quantum Threshold At Least $n{-}k$
input: A graph G of order n
parameter: An integer k
question: Is $\kappa_Q(G) \geq n{-}k$?

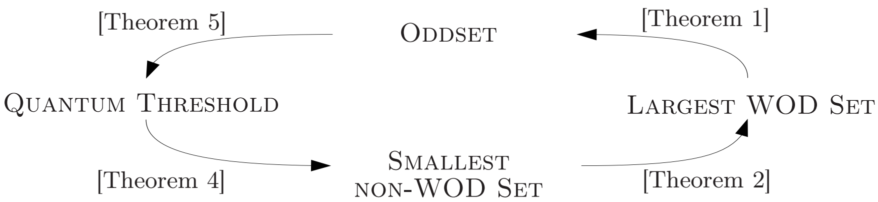
3.2 Fixed parameter intractability

In this section we show that the three problems WOD Set Of Size At Least $n{-}k$, non-WOD Set Of Size At Most k, and Quantum Threshold At Least $n{-}k$ are all hard for W[1] and belong to the class W[2]. W[1]-hardness implies the fixed parameter intractability of these problems (unless FPT = W[1]).

W[2] membership and W[1]-hardness of these problem are proved using reductions from the Oddset problem:

Oddset of size at most k
input: A bipartite graph $G = (R \cup B, E)$
parameter: An integer k
question: Is there a subset $R' \subseteq R$, $|R'| \leq k$ such that $B = Odd(R')$?

This problem is known [2] to be W[1]-hard and W[2]. We prove the following circular reductions, where $B \to A$ stands for A is FPT-reducible to B:



As a consequence, all these problems are FPT-equivalent to Oddset. Moreover we show that they remain FPT-equivalent to Oddset when restricted to bipartite graphs.

3.2.1 Largest WOD set problem

Theorem 1 WOD Set Of Size At Least $n{-}k$ *is harder than* Oddset *by an* FPT-*reduction.*

Proof. Given (G,k) where $G{=}(R{\cup}B,E)$ an instance of Oddset let (G',k') (see Figure 1) be an instance of WOD Set Of Size At Least $n'{-}k'$ such that $G'{=}(A{\cup}D{\cup}F{\cup}c,\ E_1{\cup}E_2{\cup}E_3)$, $n'{=}|R|{+}(k{+}2)|B|{+}(k{+}2){+}1$ and $k'{=}k{+}1$ where:
$A{=}\{a_u, u{\in}R\}$ $\quad E_1{=}\{cf_i, 1 \leq i \leq k{+}2\}$
$D{=}\{d_{u,i}, u{\in}B, 1 \leq i \leq k{+}2\}$ $\quad E_2{=}\{ca_u, u{\in}R\}$
$F{=}\{f_i, 1 \leq i \leq k{+}2\}$ $\quad E_3{=}\{a_u d_{v,i}, uv{\in}E, 1 \leq i \leq k{+}2\}$

If (G',k') is a positive instance of WOD Set Of Size At Least $n'{-}k'$, let $C{\subseteq}V(G')$ be the smallest set such that $|Odd(C)|{\geq}n'{-}k'$. F is an independent set of size $k{+}2{>}k'$ so there exists $f{\in}F{\cap}Odd(C)$. Since c is the unique neighbour of f, c belongs to C. For every $u{\in}B$, the subset $D_u{=}\{d_{u,i}, 1{\leq}i{\leq}k{+}2\}$ is an independent set of size $k{+}2 > k'$ whose neighbourhood is included in A, thus $\forall u{\in}B, D_u{\subseteq}Odd(C{\cap}A)$ so $D{\subseteq}Odd(C{\cap}A)$. Since $c{\in}C$ and $A{\subseteq}Odd(\{c\})$, by minimality of C, $D{\cap}C{=}\emptyset$. Let $R'{=}\{u{\in}R, a_u{\in}C\}$, of size $|R'|{=}|C{\cap}A|{=}|C|{-}1{\leq}k$. Since $\forall u{\in}B, u{\in}Odd(R') \Leftrightarrow D_u{\subseteq}Odd(C)$, $B{\subseteq}Odd(R')$ so (G,k) is a positive instance of Oddset.

If (G,k) is a positive instance of Oddset, there exists $R'{\subseteq}R$, s.t. $|R'|{\leq}k$ and $B{=}Odd(R')$. Let $A'{=}\{a_u, u{\in}R'\}$, since $\forall u{\in}B, u{\in}Odd(R') \Leftrightarrow D_u{\subseteq}Odd(A')$, $D{\subseteq}Odd(A')$ so $D{\subseteq}Odd(A'{\cup}\{c\})$. Since $A{\cup}F$ is an independent set dominated by c, $(F{\cup}A{\cup}D){\setminus}A'{=}V(G'){\setminus}(A'{\cup}\{c\}){\subseteq}Odd(A'{\cup}\{c\})$. Moreover $|Odd(A'{\cup}\{c\}) \geq n'{-}(k{+}1){=}n'{-}k'$, so (G',k') is a positive instance of WOD Set Of Size At Least $n'{-}k'$. □

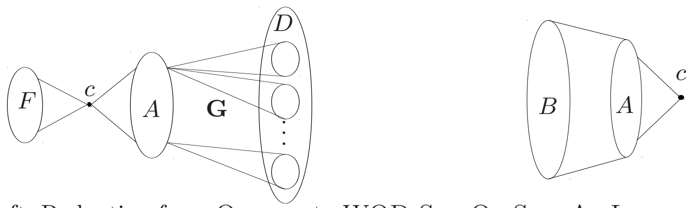


Fig. 1. Left: Reduction from ODDSET to WOD SET OF SIZE AT LEAST $n-k$; Right: Reduction from WOD SET OF SIZE AT LEAST $n-k$ to NON-WOD SET OF SIZE AT MOST k

Since ODDSET is hard for W[1], so is WOD SET OF SIZE AT LEAST $n-k$. Moreover, notice that the graph used in the proof of Theorem 1 is bipartite (see figure 1), as a consequence:

Corollary 2 WOD SET OF SIZE AT LEAST $n-k$ *is hard for* W[1] *even for bipartite graphs.*

3.2.2 Smallest non-WOD set problem

In this section we prove that NON-WOD SET OF SIZE AT MOST k is hard for W[1] even for bipartite graphs.

Theorem 2 NON-WOD SET OF SIZE AT MOST k *is harder than* WOD SET OF SIZE AT LEAST $n-k$ *by an* FPT*-reduction.*

Proof. Given (G, k) where $G = (V, E)$ and $n = |V|$ an instance of WOD SET OF SIZE AT LEAST $n-k$, let (G', k') (see Figure 1) be an instance of NON-WOD SET OF SIZE AT MOST k' such that $G' = (A \cup B \cup \{c\},\ E_1 \cup E_2 \cup E_3)$ and $k' = k + 2$ where:

$A = \{a_i, 1 \leq i \leq k+3\}$ $E_1 = \{b_u b_v, uv \in E\}$
$B = \{b_u, u \in V\}$ $E_2 = \{a_i c, 1 \leq i \leq k+3\}$
$E_3 = \{b_u a_i, u \in V, 1 \leq i \leq k+3\}$

If (G, k) is a positive instance of WOD SET OF SIZE AT LEAST $n-k$, there exists $C \subseteq V$ such that $|Odd(C)| \geq n-k$. There are two cases: if $|C| = 0 \bmod 2$, let $C' = \{b_u, u \in C\} \cup \{a\}$, where a is any vertex in A. Since A is connected by a complete bipartite graph to B, $A \subseteq Even(C')$ and $c \in Odd(C')$. Since $a \in C'$ is connected to all the vertices in B, for every $u \in Odd(C)$ in G, $b_u \notin Odd(C')$ in G', so $|C' \cup Odd(C')| \leq k+2 = k'$. Otherwise, if $|C| = 1 \bmod 2$, let $C' = \{b_u, u \in C\} \cup \{a, c\}$, where a is any vertex in A. A is connected by a complete bipartite graph to B and each vertex in A is connected to c since $|B \cap C'| = 1 \bmod 2$, $A \subseteq Even(C')$. Similarly $|Odd(C') \cup C'| \leq k+2 = k'$. Thus in both cases (G', k') is a positive instance of NON-WOD SET OF SIZE AT MOST k.

If (G', k') is a positive instance of NON-WOD SET OF SIZE AT MOST k, there exists $C' \subseteq V(G')$ such that $|C' \cup Odd(C')| \leq k$ and $|C'| = 1 \bmod 2$. A is an independent set of size $k + 3 > k'$, so there exists $a \in A$ such that $a \in Even(C')$. Since A is connected to $V(G') \setminus A$ by a complete bipartite graph and $|C'| = 1 \bmod 2$, $|C' \cap A| = 1 \bmod 2$ then by minimality $|C' \cap A| = 1$, let a be this vertex. Let $C = \{u, b_u \in C'\}$, since a is connected to every vertex in B, $\forall u \in V, b_u \in Odd(C') \Leftrightarrow u \in Even(C)$ so $|Even(C)| < k$ thus (G, k) is a positive instance of WOD SET OF SIZE AT LEAST $n-k$. □

Corollary 3 NON-WOD SET OF SIZE AT MOST k *is hard for* W[1].

The proof of the W[1]-hardness of NON-WOD SET OF SIZE AT MOST k does not respect the bipartition of the graph. However we prove that the problem is W[1]-hard even for bipartite graph by reduction from the general case:

Theorem 3 NON-WOD SET OF SIZE AT MOST k *in bipartite graphs is harder than* NON-WOD SET OF SIZE AT MOST k *by an* FPT*-reduction.*

Proof. Given an instance (G, k) of NON-WOD SET OF SIZE AT MOST k let (G', k') (see Figure) be a bipartite instance of NON-WOD SET OF SIZE AT MOST k with:

$G' = (A \cup B_1 \cup B_2 \cup D \cup F \cup H, E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5)$, $k' = 2k$

$A = \{a_u, u \in V\}$

$B_1 = \{b_{1,u}, u \in V\}$

$B_2 = \{b_{2,u}, u \in V\}$

$D = \{d_{i,u,j}, i \in \{1,2\}, u \in V, 1 \leq j \leq 2k+1$

$F = \{f_{i,u,j,l}, i \in \{1,2\}, u \in V, 1 \leq j, l \leq 2k+1\}$

$H = \{h_i, 1 \leq i \leq 2k+1\}$

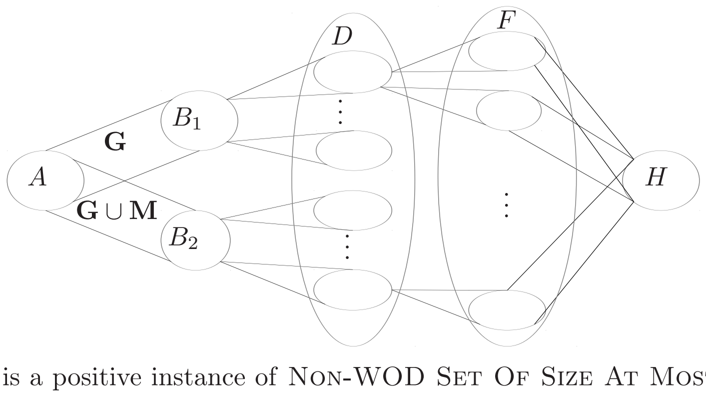
$E_1 = \{a_u b_{i,v}, i \in \{1,2\}, uv \in E\}$

$E_2 = \{a_u b_{2,u}, u \in V\}$

$E_3 = \{b_{i,u} d_{i,u,j}, i \in \{1,2\}, u \in V, 1 \leq j \leq 2k+1\}$

$E_4 = \{d_{i,u,j} f_{i,u,j,l}, i \in \{1,2\},\} u \in V, 1 \leq j, l \leq 2k+1\}$

$E_5 = \{f_{i,u,j,l} h_p, i \in \{1,2\}, u \in V, 1 \leq j, l, p \leq 2k+1\}$



If (G, k) is a positive instance of NON-WOD SET OF SIZE AT MOST k, there exists $C \subseteq V$ such that $|C \cup Odd(C)| \leq k$. Let $C' = \{a_u, u \in C\}$, notice that $|Odd(C') \cap B_1| = |Odd(C)|$ since $\forall u, v \in V, a_u b_{1,v} \in E_1 \Leftrightarrow uv \in E$. Moreover, $|Odd(C') \cap B_2| = |Odd(C) \oplus C|$, since $\forall u, v \in V,\ a_u b_{2,v} \in E_1 \Leftrightarrow uv \in E$ and $\forall u \in V, a_u b_{2,u} \in E_2$. So $|C' \cup Odd(C')| = |C| + |Odd(C)| + |Odd(C) \oplus C| = 2|C \cup Odd(C)| \leq 2k = k'$ thus (G', k') is a positive bipartite instance of NON-WOD SET OF SIZE AT MOST k.

If (G', k') is a positive bipartite instance of NON-WOD SET OF SIZE AT MOST k, there exists $C' \subseteq V'$ such that $|C' \cup Odd(C')| \leq 2k$ and $|C'| = 1 \bmod 2$. Notice that H is an independent set of size $2k+1 > k'$, so there exists $h \in H$ such that $h \in Even(C')$, since H is connected by a complete bipartite graph to F thus $|F \cap C'| = 0 \bmod 2$. F is composed of $(2k+1)2|V| > k'$ independent sets F_i of size $2k+1 > k'$ so $\forall i \in [1, (2k+1)2|V|] \exists f_i \in F_i$ such that $f_i \in Even(C')$. Since each F_i is connected to a vertex $d \in D$ and connected to H by a complete bipartite graph, $|F \cap C'| = 0 \bmod 2$ and $D \cap C' = \emptyset$. By minimality of C' $|F \cap C'| = 0 \bmod 2$ implies $C' \cap F = \emptyset$. D is composed of $2|V| > k'$ independent sets D_i of size $2k+1 > k'$, so there exists $i \in [1, 2|V|]$ such that $D_i \subseteq Even(C')$, since each D_i is connected to a vertex $b \in B_1 \cup B_2$, $C' \cap (B_1 \cup B_2) = \emptyset$. Thus $C' \subseteq A$ and $Odd(C') \subseteq B_1 \cup B_2$, let $C = \{u \in V, a_u \in C'\}$, so $|C' \cup Odd(C')| = |C| + |Odd(C)| + |Odd(C) \oplus C| = 2|C \cup Odd(C)|$. Since $|C' \cup Odd(C')| \leq k'$, $|C \cup Odd(C)| \leq k$ which implies that (G, k) is a positive instance of NON-WOD SET OF SIZE AT MOST k. □

Corollary 4 NON-WOD SET OF SIZE AT MOST k *is hard for* W[1] *even for bipartite graphs.*

3.2.3 Quantum Threshold problem

In this section we consider the quantum threshold problem. The quantum threshold $\kappa_Q(G)$ of a graph G of order n is defined as $\kappa_Q(G) = \max(\kappa(G), \kappa(\overline{G})) = \max(\kappa(G), n - \kappa'(G))$. Like the largest WOD set problem, this problem is proved to be hard for W[1] by a reduction from NON-WOD SET OF SIZE AT MOST k.

Theorem 4 QUANTUM THRESHOLD AT LEAST $n-k$ *is harder than* NON-WOD SET OF SIZE AT MOST k *by an* FPT*-reduction.*

Proof. Given (G, k) an instance of NON-WOD SET OF SIZE AT MOST k let (G^{k+1}, k) be an instance of QUANTUM THRESHOLD AT LEAST $n-k$ where G^p is the graph obtained by copying p times G. Since $\kappa_Q(G) = \max(\kappa(G), n - \kappa'(G))$ by Lemma 6 of [3], there are two possibilities for $\kappa_Q(G^{k+1}) \geq (k+1)n-k$, either $\kappa(G^{k+1})$ or $n(k+1) - \kappa'(G^{k+1})$ is maximal:
• If $\kappa(G^{k+1}) \geq (k+1)n-k$, since the $k+1$ copies of G are independent the largest WOD set of G^{k+1} is $k+1$ copies of the largest WOD set of G, so $(k+1)\kappa(G) \geq (k+1)n - k$. Thus $\kappa(G) \geq n - \frac{k}{(k+1)}$, but $\frac{k}{k+1} < 1$ so $\kappa(G) \geq n$. On the other hand, for any graph G of order n we have $\kappa(G) < n$ so there is a contradiction.
• If $(k+1)n - \kappa'(G^{k+1}) \geq (k+1)n-k$. Since the $k+1$ copies are independent the smallest non-WOD set of G^{k+1} is one of the smallest non-WOD set in a copy of G, so $\kappa'(G^{k+1}) \leq k \Leftrightarrow \kappa'(G) \leq k$ which is equivalent to (G, k) is a positive instance of NON-WOD SET OF SIZE AT MOST k. □

Notice that the reduction preserves the bipartition of the graph, so QUANTUM THRESHOLD AT LEAST $n-k$ is W[1]-hard even for bipartite graphs.

Corollary 5 QUANTUM THRESHOLD AT LEAST $n-k$ *is hard for* W[1] *even for bipartite graphs.*

Now we have proved that all the problems related to weak odd domination are hard for W[1], by successive FPT-reductions starting from ODDSET. These problems are not only harder than ODDSET but equivalent to ODDSET and belong to W[2]. Indeed, ODDSET is harder than QUANTUM THRESHOLD OF SIZE AT LEAST $n-k$ by an FPT-reduction:

Theorem 5 Oddset *is harder than* Quantum Threshold At Least $n-k$ *by an* FPT*-reduction.*

Proof. Given an instance (G,k) of QUANTUM THRESHOLD AT LEAST $n-k$, let (G',k') (see Figure) be an instance of ODDSET with:
$G' = (A \cup \{d_1\} \cup \{d_2\} \cup \{c\}, E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup \{d_1c\} \cup \{d_2c\})$

$$A = \bigcup_{1\leq i\leq 2, 1\leq j\leq 5} A_{i,j} \qquad E_1 = \{a_{1,2,u}a_{1,j,v}, j \in \{4,5\}, uv \in E\}$$

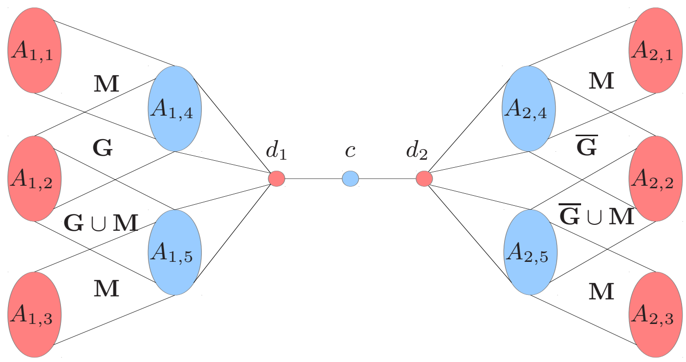
$$A_{i,j} = \{a_{i,j,u}, u \in V\} \qquad E_2 = \{a_{2,2,u}a_{2,j,v}, j \in \{4,5\}, uv \notin E\}$$

$$E_3 = \{a_{i,j,u}a_{i,l,u}, i{\in}\{1,2\}, j{\in}\{1,3\}, l{\in}\{4,5\}, u{\in}V\}$$

$$E_4 = \{d_i a_{i,j,u}, i \in \{1,2\}, j \in \{4,5\}, uv \in E\}$$

$$E_5 = \{a_{i,2,u}a_{i,5,u}, i \in \{1,2\}, u \in V\}$$

And with $B = A_{i,4} \cup A_{i,5} \cup \{c\}$, $R = A_{i,1} \cup A_{i,2} \cup A_{i,3}$ and $k' = 2k+1$.



If (G,k) is a positive instance of QUANTUM THRESHOLD AT LEAST $n-k$, since $\kappa_Q(G){=}\max(\kappa(G), \kappa(\overline{G}))$, either $\kappa(G)$ or $\kappa(\overline{G})$ is greater than $n-k$:

- If $\kappa(G) \geq n-k$, there exists C such that $|Odd(C)| \geq n-k$. let $C' \subseteq R$ be $\{a_{1,2,u}, u \in C\} \cup \{a_{1,1,u}, u \in Even(C)\} \cup \{a_{i,3,u}, u \in C \oplus Even(C)\} \cup \{d_2\}$ where $\oplus$ is the symmetric difference. Since $d_2 \in C'$, then c, $A_{2,4}$ and $A_{2,5}$ are in $Odd(C')$. Since $a_{1,2,u}a_{1,4,v} \in E_1 \Leftrightarrow uv \in E$ and $\{a_{1,2,u}, u \in C\} \subseteq C'$, then $\{a_{1,4,u}, u \in Odd(C)\} \subseteq Odd(C')$. $A_{1,1}$ is connected to $A_{1,4}$ by a matching and $\{a_{1,1,u}, u \in Even(C)\} \subseteq C'$ so $\{a_{1,4,u}, u \in Even(C)\} \subseteq Odd(C')$ thus $A_{1,4} \subseteq Odd(C')$. Since $A_{1,5}$ is connected to $A_{1,2}$ like $A_{1,4}$ plus a matching, $\{a_{1,5,u}, u \in Odd(C) \oplus C\} \subseteq Odd(C' \cap A_{1,2})$ and $\{a_{1,5,u}, u \in Even(c) \oplus C\} \subseteq Odd(C' \cap A_{1,3})$, thus $A_{1,5} \subseteq Odd(C')$. So $B \subseteq Odd(C')$, and $|C'| = |C| + |Even(C)| + |Even(C) \oplus C| + 1 = 2|Even(C) \cup C| + 1 \leq 2k+1 = k'$, thus (G',k') is a positive instance of ODDSET.
- If $\kappa(\overline{G}) {\geq} n-k$ then $d_1 \in C'$ instead of d_2 and using the same process with neighbourhood relations of $\overline{G}$ instead of G we obtain (G',k') is a positive instance of ODDSET.

If (G',k') is a positive instance of ODDSET, there exists $C' \subseteq R$ such that $B \subseteq Odd(C')$. c is dominated either by d_1 or d_2. If $d_2 \in C'$, then $A_{2,4}$ and $A_{2,5}$ are in $Odd(C')$. Since $A_{1,4}$ is connected to $A_{1,1}$ by a matching $A_{1,2} \cap C' \neq \emptyset$, let $C = \{u, a_{1,2,u} \in C'\}$ be a set of vertices in V, so similarly $|C'| = 2|Even(C) \cup C| + 1$ which implies that $|C'| = 2|Odd(C)| + 1$, so $\kappa(G) \geq n-k$ thus (G,k) is a positive instance of QUANTUM THRESHOLD OF SIZE AT LEAST $n-k$. If

$d_1 \in C'$, similarly $\kappa(\overline{G}) \geq n{-}k$, so (G, k) is a positive instance of Quantum Threshold At Least $n{-}k$. □

Corollary 6 *All the following problems:* WOD Set Of Size At Least $n{-}k$, Non-WOD Set Of Size At Most k *and* Quantum Threshold At Least $n{-}k$ *even for bipartite graph are* FPT*-equivalent to* Oddset *and thus are all hard for* W[1] *and in* W[2].

4 Approximability

Weak odd domination problems WOD Set Of Size At Least $n{-}k$, non-WOD Set Of Size At Most k, and Quantum Threshold At Least $n{-}k$ are not fixed parameter tractable unless FPT = W[1] (see section 3). In this section we consider the question of the existence of an approximation algorithm for optimisation problems associated with the weak odd domination:

Max WOD Set	Min non-WOD Set
input: A graph $G = (V, E)$	input: A graph $G = (V, E)$
cost function: $D \mapsto \lvert Odd(D)\rvert$	cost function $D \mapsto \lvert D \cup Odd(D)\rvert$
output: $D {\subseteq} V$ s.t. $\lvert Odd(D)\rvert{=}\kappa(G)$	output: $D {\subseteq} V$ s.t. $\lvert D \cup Odd(D)\rvert{=}\kappa'(G)$

Max non-Accessible Set
input: A graph $G = (V, E)$
cost function $D \mapsto \max(|Odd_G(D)|, |Odd_{\overline{G}}(D)|)$
output: $D \subseteq V$ s.t. $\max(|Odd_G(D)|, |Odd_{\overline{G}}(D)|) = \kappa_Q(G)$

Notice that Max non-Accessible Set is in APX as it admits the 2-approximation algorithm which consists in outputting any vertex of the graph. Indeed, given a graph $G{=}(V, E)$ of order n, $\forall v{\in}V$, $\kappa_Q(G){\geq} \max(|N(v)|, |V \setminus N(v)|{-}1) \geq \frac{n-1}{2}$. Since $\kappa_Q(G){\leq}n{-}1$, it produces a 2-approximation.

In the rest of the section, we prove the APX-completeness of Max WOD Set and Min non-WOD Set, which implies the non-existence of polynomial approximation scheme for these problems unless P=NP.

4.1 Maximum WOD Set

Theorem 6 *The* Max WOD Set *problem is complete for* APX.

The proof of the harness consists in a reduction from Max 3-Sat B proved to be hard for APX in [10], for any $B \in \mathbb{N}$,

Max 3-Sat B
input: a 3-CNF formula ψ where every variables occurs at most B times
output: an assignment of variable of ψ s.t. the maximum number of clauses is satisfied

Lemma 4 *For any* $B{\geq}0$ Max WOD Set *is harder than* Max 3-Sat B *by an L-reduction.*

Proof. Given ψ an instance of MAX 3-SAT B with n clauses, let f such that $f(\psi) = G'$ (see Figure) be an instance of MAX WOD SET such that $G' = (C \cup D \cup F, E_1 \cup E_2)$ where:

$C = \{a, \overline{a} | a \text{ is a variable of } \psi\}$

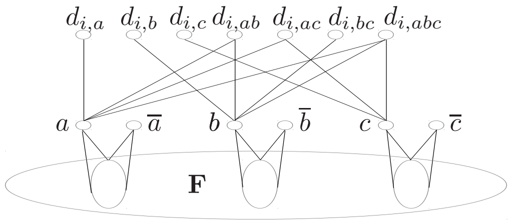
$F = \{f_{a,j} | a \text{ is variable of } \psi, j \in [1, 4B+1]\}$

$D_i = \{d_{i,a}, d_{i,b}, d_{i,c}, d_{i,ab}, d_{i,ac}, d_{i,bc}, d_{i,abc} | a, b, c \text{ are the literals of the } i^{th} \text{ clause of } \psi\}$

$D = \bigcup_{i \in [1,n]} D_i$

$E_1 = \{ad_{i,a}, ad_{i,ab}, ad_{i,ac}, ad_{i,abc} | i \in [1, n]\}$

$E_2 = \{af_{a,j}, \overline{a}f_{a,j} | j \in [1, 4B+1], F_{i,j} \subseteq F\}$



Let g such that given $X \subseteq V(G')$ a set of G', $g(X)$ is an assignment of ψ such that the variable a is true if the vertex $a \in D$ (even if $\overline{a} \in D$) and is false otherwise. g is computable in polynomial time and since G' is polynomial in $|\psi|$, f is computable in polynomial time.

In G', $|C| \leq 6n$ and $|D| = 7n$, and since each variable of ψ is in a most B clauses then $|F| \leq 3n \times 4B + 1$, so the size of G' is linear in the size of ψ. Let $MaxSAT(\psi)$ be the value of the number of true clauses in the optimal solution of MAX 3-SAT B on ψ, since $|G'| = \alpha|\psi|$ and that $MaxSAT(\psi) \geq \frac{|\psi|}{2}$, $\kappa(G') \leq 2\alpha MaxSAT(\psi)$.

Given $C' \subseteq C$, let $(a \vee b \vee c)$ be the i^{th} clause of ψ, $C' \cap \{a, b, c\} \neq \emptyset$ if and only if $|Odd(C') \cap D_i| = 4$ since:

- If $|C'| = 1$, let $a \in C'$ then $d_{i,a}, d_{i,ab}, d_{i,ac}, d_{i,abc} \in Odd(C')$ and $d_{i,b}, d_{i,c}, d_{i,bc} \in Even(C')$ and symmetrically for b and c.
- If $|C'| = 2$, let $a, b \in C'$ then $d_{i,a}, d_{i,b}, d_{i,ac}, d_{i,bc} \in Odd(C')$ and $d_{i,c}, d_{i,ab}, d_{i,abc} \in Even(C')$ and symmetrically for b, c and a, c.
- If $|C'| = 3$, let $a, b, c \in C'$ then $d_{i,a}, d_{i,b}, d_{i,c}, d_{i,abc} \in Odd(C')$ and $d_{i,ab}, d_{i,ac}, d_{i,bc} \in Even(C')$.

For all variable a of ψ, there exists $F_a = \{f_{a,j} | j \in [1, 4B+1]\}$ an independent set of size $4B + 1$ connected only to a and $\overline{a}$. Given $C' \subseteq C$, since $|Odd(C') \cap D_i| \leq 4$ and that a variable a appears in at most B clauses in ψ, if $a, \overline{a} \in C'$, $|Odd(C' \setminus \overline{a})| > |Odd(C')|$ thus there is no $a, \overline{a}$ pairs in the optimal solution of MAX WOD SET.

Let X_{opt} be the optimal solution of MAX WOD SET, since there is no $a, \overline{a}$ pairs in X_{opt}, then $F \subseteq Odd(X_{opt})$ and the number of D_i dominated by X_{opt} is $MaxSAT(\psi)$. Let X be a solution of MAX WOD SET and l the number of satisfied clauses by $g(X)$. $MaxSAT(\psi) - l = k$ implies that $g(X)$ satisfies k minus clauses than $g(X_{opt})$. Thus X oddly dominates at least k minus D_i than X_{opt} or contains $a, \overline{a}$, so X oddly dominates at least $4k$ minus vertices than X_{opt}. This implies that $MaxSAT(\psi) - l \leq \beta(\kappa(G') - |Odd(X)|)$. □

Proof of Theorem 6. For all $B \geq 3$ Max 3-Sat B is hard for APX and there exists an L-reduction from Max 3-Sat B to Max WOD Set, thus Max WOD Set is also hard for APX. The completeness is obtained by applying the conditional probabilities method [13] on the linear bound of $\kappa(G)$ given by Lemma 1 which gives a polynomial 4-approximation of κ.

Corollary 7 *There exists $\epsilon > 0$ such that there is no polynomial time $(1+\epsilon)$-approximation of* Max WOD Set *unless* P=NP.

4.2 Minimum non-WOD Set

Theorem 7 *The* Min non-WOD Set *problem is complete for* APX.

The proof of the hardness consists in a reduction from Max WOD Set proved to be hard for APX in Theorem 6.

Lemma 5 Min non-WOD Set *is harder than* Max WOD Set *by an L-reduction.*

Proof. Given $G = (V, E)$ an instance of Max WOD Set of size n, let f such that $f(G) = G'$ be an instance of Min non-WOD Set such that $G' = (V \cup d, E_1 \cup E_2)$ where $E_1 = \{uv | u \in V, v \in V, uv \notin E\}$ and $E_2 = \{ud | u \in V\}$. Let g such that given $X \subseteq V(G')$ a non-WOD set of G', $g(X)$ is a WOD set of $V(G)$ defined by $g(X) = \{u | u \notin X, u \neq d\}$. Since X is a non-WOD set there exists $D \subseteq X$ such that $|D| = 1 \bmod 2$ and $Odd(D) \subseteq X$, so $\overline{X} \subseteq Even(D)$, thus in $\overline{G'}$ $\overline{X} \subseteq Odd(D)$. Since d is isolated in $\overline{G'}$, $\overline{X} \subseteq Odd(D)$ in G thus $g(X)$ is a WOD set in G and $|g(X)| = n - |X|$. g is computable in polynomial time and since G' is polynomial in $|\psi|$, f is computable in polynomial time. By Lemma 3, $\kappa'(G') + \kappa(\overline{G'}) = n + 1$, since $\overline{G'}$ is G and an isolated vertex and that an isolated vertex does not change the value of κ, then $\kappa'(G') = n + 1 - \kappa(G)$. By Lemma 1, $\kappa(G) \geq \frac{n}{4}$, thus $\kappa(G') \leq \alpha\kappa(G)$. By adding $|g(X)| = n - |X|$ to $\kappa'(G') = n + 1 - \kappa(G)$ we have that for all solution X of Min non-WOD Set, $|\kappa(G) - |g(X)|| \leq \alpha|\kappa'(G') - |X||$. □

Proof of Theorem 7. Max WOD Set is hard for APX and there exists an L-reduction from Max WOD Set to Min non-WOD Set, thus Min non-WOD Set is also hard for APX. Similarly to Max WOD Set the completeness is given by applying the conditional probabilities method [13] on Corollary 1 which gives a polynomial 8-approximation of κ'.

Corollary 8 *The exits $\epsilon > 0$ such that there is no polynomial time $(1+\epsilon)$-approximation of* Min non-WOD Set *unless* P=NP.

5 Conclusion

In this paper we have explored the parameterized complexity and approximation of weak odd domination problems: largest WOD set, smallest non-WOD set, quantum threshold, and their respective related quantities κ, κ' and κ_Q. We have proved bounds on these quantities. These bounds imply that for any of these problems, there is a parameterization which is FPT. However, if one considers a more natural parameterization of these problems, it turns out that all the variants of weak odd domination problems are hard for W[1] and in W[2] even for bipartite graphs. More precisely we show that they are equivalent to ODDSET. Regarding the approximation we have proved that (1) maximum WOD set and minimum non-WOD set are complete for APX, so they have no polynomial approximation scheme unless P=NP; and (2) maximum non-accessible set is in APX with a trivial constant factor approximation. The existence of a polynomial approximation scheme for maximum non-accessible set, which would be of great interest for the design of graph-based quantum secret sharing schemes, remains open. Regarding the parameterized complexity, *tight* bounds on κ, κ' and κ_Q would allow for above guarantee technics to tackle the existence of FPT-algorithms only based on the existence of bounds on the parameter (see [12] details).

Acknowledgements. The authors want to thank Sylvain Gravier, Jérôme Javelle, and Mehdi Mhalla for fruitful discussions and the anonymous reviewer for his useful comments. This work has been funded by the ANR-10-JCJC-0208 CausaQ grant and by Région Rhône-Alpes (ADR Cible R637).

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