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# Option pricing under the double exponential jump-diffusion model by using the Laplace transform

Application to the Nordic market

Master's Thesis in Financial Mathematics

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# Preface

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Halmstad, May 2010  
Natalia Nadratowska  
Damian Prochna



## **Abstract**

In this thesis the double exponential jump-diffusion model is considered and the Laplace transform is used as a method for pricing both plain vanilla and path-dependent options. The evolution of the underlying stock prices are assumed to follow a double exponential jump-diffusion model. To invert the Laplace transform, the Euler algorithm is used. The thesis includes the programme code for European options and the application to the real data. The results show how the Kou model performs on the NASDAQ OMX Stockholm Market in the case of the SEB stock.

**Keywords:** Double exponential jump-diffusion model, Kou model, Laplace transform, Laplace transform inversion, Euler algorithm, stylized facts.





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# Chapter 1

## Introduction

The publication in 1973 of the work of Fischer Black and Myron Scholes "The Pricing of Options and Corporate Liabilities" has been a starting point for the revolution in the finance and option pricing. Their idea was to develop a model based on the assumption that the asset prices follow a geometric Brownian motion. Since then, this model is commonly used by dint of its simplicity and ease of implementation.

Despite the fact that today's knowledge of the behaviour of prices is far more clarified, many references to the Black-Scholes model are still made. Results achieved by this model quite well reflect the real prices, but still the model has certain shortcomings. The real prices show properties which contradict the assumptions of the Black-Scholes model.

As a consequence of a market data analysis, the existence of some phenomena have been noticed, such as a leptokurtic feature, a volatility smile and a volatility clustering effect. It means that the prices do not follow strictly a Brownian motion (Kou, et al., 2003).

This conclusion was the main reason for developing more complicated models. Among the most popular it is worth to mention the Heston model (Heston, 1993), the Generalized Hyperbolic models (Prause, 1999), as well as models based on Lévy processes (Papapantoleon, 2008), Stochastic Volatility models (Hull & White, 1987), time-changed Brownian models (Veraart & Winkel, 2010) and jump-diffusion models. Each of them has some advantages, but there does not exist any model, which could meet all the aspects related to the price movements.

Depending on a problem, types of data and available tools, we have to choose the most suitable model. What is easily visible, is that in financial time series data exist jumps (Maekawa, et al., 2008), caused by various economical, political and social factors. In this context we decide to focus on the jump-diffusion models, among which we should stress on the Merton model

(Merton, 1976) and the Kou model (Kou, 2008). These models are important especially for researches in fields like path-dependent options, pricing short maturity options, interest rate derivatives, but also in pricing plain vanilla options.

Considering jump-diffusion issues, we have chosen the Kou model, because of its assumption of the asymmetric double exponential distribution of jumps. The double exponential distribution has interesting properties that are crucial for the model. Furthermore, the Kou model incorporates two of the empirical stylized facts mentioned before: a leptokurtic feature and a volatility smile.

To price options under the chosen model we use the Laplace transform. This method is popular in pricing both European and path-dependent options. The prices of options are obtained due to the numerical inversion of Laplace transforms, because explicit formulas for the inverted Laplace transform in this case do not exist. For this purpose we use the Euler algorithm because of its fast convergence due to which just a limited number of terms are necessary (Petrella, 2003).

The problem of the existence of jumps was picked up by many authors. The jump-diffusion model was described in several surveys, e.g. Pham (1997), Gukhal (2001), Kijama (2002), Cont & Tankov (2004). This thesis is based mainly on the papers of Steven Kou and Giovanni Petrella from Columbia University, published between 2002 and 2008, see "Jump Diffusion Models for Asset Pricing in Financial Engineering" and "An extension of the Euler Laplace transform inversion algorithm with applications in option pricing". Our objective is to explore the theoretical issues related to the Kou model and the Laplace transform as a method of pricing under this model. Afterwards we transfer the theoretical results to the practical problem. Our goal is also to apply the theory to valuation of European options under the double exponential jump-diffusion model via the Laplace transform. In the literature there is no programme code available for implementation this problem, so a part of our work is to write our own programme. Then we use the results for the analysis of the data derived from the NASDAQ OMX Stockholm market. Our data concerns a stock, the Skandinaviska Enskilda Banken (SEB), which is a part of the index OMX Stockholm 30 (OMXS30) consisting of the 30 most-traded stock classes.

The data are collected by using the *SIX EDGE<sup>TM</sup>* program, which is a professional software designed for this purpose.

The reason for investigating of the NASDAQ OMX market is the fact, that the Laplace transform method has to our best knowledge not been implemented there. We compare our numerical results for the double exponential jump-diffusion model via Laplace transform with real market prices, together

with the prices obtained by the Black-Scholes model.

Our thesis consists of seven parts and is organized as follows: in Chapter 2 we describe empirical phenomena of stock returns which are examined by the double exponential jump-diffusion model; in Chapter 3 we present the most popular financial models for option pricing; Chapter 4 contains the precise description of the Kou model; the programme manual and practical results for NASDAQ OMX Stockholm market are included in Chapter 5; summary and conclusions for our research are included in Chapter 6; all derivations and complete programme code for the European-style option pricing are attached in the Appendix.



# Chapter 2

## Stylized facts

In this chapter we introduce some basic concepts and the properties of the jump-diffusion model.

### 2.1 A leptokurtic distribution

**Definition 2.1.1** (*Wuensch, 2005*)

The **skewness**  $S$  is a measure of asymmetry of the distribution of a random variable  $X$ ,

$$S = E\left[\left(\frac{X - \mu}{\sigma}\right)^3\right]. \quad (2.1)$$

The value  $S < 0$  implies that left tail is longer;

$S = 0$  implies that distribution is symmetric;

$S > 0$  implies that right tail is longer.

An estimation of the  $S$  is given by

$$\hat{S} = \frac{1}{(n-1)\hat{\sigma}^3} \sum_{i=1}^n (X_i - \bar{X})^3, \quad (2.2)$$

where  $\hat{\sigma}$  is the standard deviation of the sample.

**Definition 2.1.2** (*Wuensch, 2005*)

The **kurtosis**  $K$  is a measure of whether a data have sharp or flat peak in comparison to the normal distribution.

$$K = E\left[\left(\frac{X - \mu}{\sigma}\right)^4\right] - 3. \quad (2.3)$$

If  $K < 0$  then the distribution is platykurtic (lower and wider peak around the mean). If  $K = 0$  then the distribution is mesokurtic.

If  $K > 0$  then the distribution is leptokurtic (higher and sharper peak around the mean and heavier tails than those of the normal distribution).

By similar derivation we can obtain (2.3) from the estimation as

$$\hat{K} = \frac{1}{(n-1)\hat{\sigma}^4} \sum_{i=1}^n (X_i - \bar{X})^4. \quad (2.4)$$

The discussion of the distribution of the asset returns and weights of its tails is the base of the study of a jump-diffusion model. The double exponential density function is an example of a leptokurtic distribution, as it has higher peak and tails heavier than the normal distribution.

The classical models simply ignore the leptokurtic feature, while the empirical evidences of this feature are commonly known. As an example it is worth to mention the Black-Scholes Brownian motion model, in which the price is modelled as a geometric Brownian motion

$$S(t) = S(0)e^{\mu t + \sigma W(t)}, \quad (2.5)$$

where  $W(t) \sim N(0, t)$ ,  $\mu$  is the drift and  $\sigma$  is the volatility.

In the Brownian motion model the continuously compounded return  $r(t) = \ln \frac{S(t)}{S(0)}$  has a normal distribution. This is not in line with the leptokurtic feature. The jump-diffusion model is developed to overcome this kind of drawbacks.

## 2.2 Types of tails

It is a fact (Kou, 2008) that the asset returns have the leptokurtic distribution due to their higher peaks and tails heavier than those of the normal distribution. A problem related to this, is how heavy the tail distributions are.

In the literature (Kou, 2008) there are two basic classes of tail distributions

- 1) the power-type tail distributions,
- 2) the exponential-type tail distributions.

In the first case we describe the left and the right tail of a random variable  $X$  as

$$P(X < -x) \approx Cx^{-a}, \quad (2.6)$$

$$P(X > x) \approx Cx^{-a}, \quad (2.7)$$



respectively, for  $x > 0$  and where  $x \rightarrow \infty$ .

Similarly, we can define the exponential type of tails as

$$P(X < -x) \approx Ce^{-ax}, \quad (2.8)$$

$$P(X > x) \approx Ce^{-ax}, \quad (2.9)$$

respectively, for  $x > 0$  and where  $x \rightarrow \infty$ .

Not in every model we can use distributions defined above. The problem occurs when the right power-type tail is used in models with continuous compounding. If the return distribution  $X$  has this type of tail, then the price tomorrow has an infinite expectation. As explained in Kou (2008) the price of a call option can also be infinitely large.

Generally, this problem stands for any  $t$ -distribution with any degrees of freedom, but only when a continuous compounding is considered, as long as the right power-type distribution is used. It means that only discretely compounded models are appropriate for this kind of tail, but unfortunately analytical solutions are extremely hard to obtain for them.

## 2.3 Implied volatility

**Definition 2.3.1** (Derman, 1998)

The **implied volatility** is a parameter such that after its substitution to the Black-Scholes formula we get a quoted price (market price)  $C$  of the call option, or  $P$  of the put option, assuming that all parameters  $t, T, r, S, E$ , are known. Because the price of a call or put option under the Black-Scholes formula is a monotone increasing function of a volatility  $\sigma$ , then there exists a one-to-one correspondence between  $C$  and  $\sigma$  (respectively between  $P$  and  $\sigma$ ).

**Definition 2.3.2** (Hull, 2005)

Consider a call option and a put option, both written for the same strike  $K$  and expiry at the same date  $T$ , on some stock, which pays no dividend. The no arbitrage argument requires that the relationship between the call and put options must satisfy

$$P(t) + S(t) = C(t) + K \cdot B(T), \quad (2.10)$$

where  $P(t)$  is the price of the put option,  $C(t)$  is the price of the call option,  $K$  is the strike price (the same for both options),  $S(t)$  is the value of the share and  $B(T)$  is a zero coupon bond.

The relationship (2.10) is known as the **put-call parity**.  
In the case of dividend payments we obtain the formula

$$P(t) + S(t) = C(t) + K \cdot B(T) + D(t), \quad (2.11)$$

where  $D(t)$  is the present value of the dividends to be paid out before the expiration of the option.

Considering the put-call parity, the immediate supposition is that for the same maturity time  $T$  and same strike price  $K$ , the implied volatility for a call option  $\sigma_C(T, K)$  and put option  $\sigma_P(T, K)$  should have exactly the same value. The theoretical verification of this matter is not complicated. By using (2.10) for options priced with the Black-Scholes formula ( $C_{BS}(S, K)$  and  $P_{BS}(S, K)$ , call and put respectively) and for the market price of the corresponding call and put options ( $C_M(S, K)$  and  $P_M(S, K)$ ), we obtain

$$C_{BS}(S, K) - C_M(S, K) = P_{BS}(S, K) - P_M(S, K). \quad (2.12)$$

Let us denote the implied volatilities derived from the quoted prices in the market for the call option as  $\sigma_C(T, K)$  and for the put option as  $\sigma_P(T, K)$ . By definition of the implied volatility we immediately get from (2.12), that  $\sigma_C(T, K) = \sigma_P(T, K) = \sigma(T, K)$ .

As a result, we have that if we refer to the relationship between the implied volatility and the strike price/maturity time, it is not necessary to precise whether we talk about call or put options. Moreover, the calculations for the options on the same underlying asset, but with different strike price  $K$  and different time to maturity  $T$ , should give the same implied volatility, as we assume that the Brownian motion hypothesis is in power.

According to the fact that theory is not fully consistent with the real world, it is commonly known (Ahoniemi, Lanne 2007), that for real data the implied volatility for call and put options are not equal. As in-the-money and out-of-the-money options have higher implied volatilities than at-the-money options, the curve of implied volatilities against a strike price is a convex function. Due to its U-shape, this feature is known as the **volatility smile**.

## 2.4 The volatility clustering effect

**Definition 2.4.1** (Jakubowski & Sztencel, 2002)

For a random variable  $X$  defined on a probability space  $(\Omega, \mathcal{F}, P)$  the **expected value**  $E(X)$  is given by

$$E(X) = \int_{\Omega} X dP = \int_{-\infty}^{+\infty} x f(x) dx, \quad (2.13)$$

where  $f(x)$  is a probability density function.

When  $X$  is a discrete random variable with the probability mass function  $p(x)$ , then the expected value is defined as

$$E(X) = \sum_i x_i p(x_i). \quad (2.14)$$

**Definition 2.4.2** (Jakubowski & Sztencel, 2002)

The **variance** of the random variables  $X$  with the expected value  $\mu = E(X)$  is given by

$$\text{Var}(X) = E[(X - \mu)^2] = E[(X - E(X))^2] = E(X^2) - [E(X)]^2. \quad (2.15)$$

**Definition 2.4.3** (Jakubowski & Sztencel, 2002)

The **covariance** between two real-valued random variables  $X$  and  $Y$  with finite second moments is

$$\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = E(X \cdot Y) - E(X) \cdot E(Y). \quad (2.16)$$

**Definition 2.4.4** (Jakubowski & Sztencel, 2002)

The **correlation**  $\rho$  between two real-valued random variables  $X$  and  $Y$  is given by

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}}. \quad (2.17)$$

For a stock return discrete time series  $r_t$  we define the autocovariance  $\gamma_k$

$$\gamma_k = \text{Cov}(r_t, r_{t-k}) = \text{Cov}(r_t, r_{t+k}) \quad (2.18)$$

and the autocorrelation  $\rho_k$

$$\rho_k = \frac{\text{Cov}(r_t, r_{t-k})}{\sqrt{\text{Var}(r_t) \cdot \text{Var}(r_{t-k})}}. \quad (2.19)$$

The value  $\rho_k$  can be estimated by

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^T (r_t - \bar{r})(r_{t-k} - \bar{r})}{\sum_{t=1}^T (r_t - \bar{r})^2}. \quad (2.20)$$

Considering the stock returns, or more precisely the squared returns or absolute values of returns, there is quite a notable correlation  $\text{Cor}(|r_t|, |r_{t+k}|) > 0$ ,

where  $k$  spreads from few minutes to several weeks. Surprisingly, the returns themselves seem to have approximately no correlation. These two empirical circumstances give the phenomenon called the **volatility clustering effect**. The effect cannot be incorporated to any financial model, which bases on assumption that stock returns have independent increments (such as Lévy processes). As the jump-diffusion model is a special case of a Lévy process, it cannot deal with the volatility clustering effect straightway. In this case only the combination of the jump-diffusion process with other processes can deal with the problem of the volatility clustering effect.

# Chapter 3

## Some financial models

Since a long time ago the asset price movements were on the field of concerns of investors, who wanted to predict future price levels to earn as much as possible. It led to the expansion of the branch of finance called theory of pricing. During the spread over forty years many models of price movements were invented, firstly very simple and later on more complicated. In this chapter we present the Black-Scholes model and its modifications.

### 3.1 The Black-Scholes model

The Black-Scholes model is a model invented in 1973 by Fischer Black and Myron Scholes. The first description of the model was published in the paper "The Pricing of Options and Corporate Liabilities". Theoretical expansion of the model was made by Robert C. Merton. For their work Merton and Scholes were rewarded with the Nobel Prize in Economics in 1997.

Due to its simplicity, the Black-Scholes model is a base for other models, which were developed to overcome its shortcomings.

**Model assumptions** (Black & Scholes, 1973)

- Investors can borrow and lend money for a constant risk-free rate.
- There are no transaction costs and taxes.
- Asset prices follow a geometric Brownian motion with a constant drift and volatility.
- Stocks pay no dividend.
- There are no arbitrage possibilities.

- All securities are perfectly divisible.
- A short selling is permitted.
- Liquidity in a trading of assets.
- Options can be exercised only in maturity.

In this model the asset price is described as follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (3.1)$$

where  $W_t$  is a Brownian motion,  $\sigma$  denotes the volatility,  $\mu$  is the drift and  $S_t$  is the stock price.

By using the Itô lemma

$$dV = \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \quad (3.2)$$

and for a delta-hedged portfolio with no arbitrage argument, we obtain the Black-Scholes second order partial differential equation (Wilmott, et al., 1995)

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (3.3)$$

Solving Equation (3.3) enables us to calculate the prices of the European call and put options

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad (3.4)$$

$$P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \quad (3.5)$$

where  $C$  denotes the price of the call option,  $P$  is the price of the put option,  $(T - t)$  is the time to maturity,  $S$  is the spot price of the underlying asset,  $K$  is the strike price,  $r$  denotes the risk-free rate,  $\sigma$  is the volatility and  $N(\cdot)$  is the cumulative distribution function of the standard normal distribution calculated by the formula

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy. \quad (3.6)$$

The parameters  $d_1$  and  $d_2$  are given by

$$d_1 = \frac{\log \frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \quad (3.7)$$

$$\begin{aligned} d_2 &= \frac{\log \frac{S}{E} + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \\ &= d_1 - \sigma\sqrt{T - t}. \end{aligned} \tag{3.8}$$

Because of the idealization of assumptions, the results obtained by the Black-Scholes model differ from real-world prices. To modify the Black-Scholes model and explain some stylized facts (the leptokurtic feature, the volatility clustering effect and the implied volatility smile) many studies have been undertaken. Below we present some of the alternative models to the Black-Scholes one.

### 3.2 The chaos theory and fractal Brownian motions

In this type of models the Brownian motion, which appears in Black-Scholes model, is replaced by a fractal Brownian motion (Kou, 2008).

**Definition 3.2.1** (Cont & Tankov, 2004)

A continuous-time Gaussian process  $X = (X_t)_{t \geq 0}$  starting at zero, with a zero mean and a covariance function

$$Cov(X_s, X_t) = E[X_s X_t] = \frac{1}{2} \left( |s|^{2H} + |t|^{2H} - |t - s|^{2H} \right)$$

is called a **fractal Brownian motion** with the Hurst parameter  $0 < H \leq 1$ . The value of  $H$  determines, what kind of process the fractal Brownian motion is.

- If  $H = \frac{1}{2}$ , the process is a regular Brownian motion.
- If  $H > \frac{1}{2}$ , the increments of the process are positively correlated.
- If  $H < \frac{1}{2}$ , the increments of the process are negatively correlated.

This model allows a long-range dependence between returns on different days but it also allows arbitrage opportunities (Rogers, 1997).

### 3.3 Models based on Lévy processes

In models based on a Lévy processes the asset price  $S(t)$  is represented as follows

$$S_t = S_0 \cdot e^{X_t}, \tag{3.9}$$

where  $X_t$  is the Lévy process.

**Definition 3.3.1** (Pitman, 2003)

The process  $X = (X_t)_{t \geq 0}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $\mathbb{R}^d$  is said to be a **Lévy process**, if

1.  $X$  has independent increments,
2.  $X(0) = 0$  a.s.,
3.  $X$  is stochastically continuous (also called continuous in probability or  $P$ -continuous), if for  $s \geq 0$

$$X(t+s) - X(s) \rightarrow 0, \quad P - a.s. \text{ as } t \rightarrow 0,$$

4.  $X$  is time homogeneous, i.e., for  $t \geq 0$ ,  $\mathcal{L}(X(t+s) - X(s))$  does not depend on  $s \geq 0$ ,
5.  $X$  is cadlag almost surely.

There are many models based on Lévy processes. One of them is the generalized hyperbolic model.

### 3.4 The generalized hyperbolic models

In this type of models assumption on the normal distribution, which appears in the Black-Scholes model, is replaced by some other distribution (Kou, 2008).

**Definition 3.4.1** (Cont & Tankov, 2004)

The one-dimensional **generalized hyperbolic model** is a five-parameter model that is usually defined via its Lebesgue density

$$p(x; \lambda, \alpha, \beta, \delta, \mu) = C(\delta^2 + (x - \mu)^2)^{\frac{\lambda}{2} - \frac{1}{4}} K_{\lambda - \frac{1}{2}}(\alpha \sqrt{\delta^2 + (x - \mu)^2}) e^{\beta(x - \mu)}$$

$$C = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi} \alpha^{\lambda-1/2} \delta^\lambda K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})},$$

where  $K$  is the modified Bessel function of the second kind.

To define the modified Bessel function, let us consider the equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + v^2)w = 0. \quad (3.10)$$

for an arbitrary real or complex  $v$ .

**Definition 3.4.2** (Cont & Tankov, 2004)

The **modified Bessel function of the first kind**  $I_v(z)$  for  $z \geq 0$  and  $v \geq 0$  is equal to the solution of Equation (3.10) that is bounded, when  $z \rightarrow 0$ .

The **modified Bessel function of the second kind**  $K_v(z)$  for  $z \geq 0$  and  $v \geq 0$  is equal to the solution of Equation (3.10) that is bounded, when  $z \rightarrow \infty$ .



The characteristic function of the generalized hyperbolic models has the form

$$\phi(u) = e^{i\mu u} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_\lambda(\delta\sqrt{\lambda^2 - (\beta + iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}. \quad (3.11)$$

The generalized hyperbolic models have some disadvantages. First of all they are inappropriate for data on different time scales (Cont & Tankov, 2004). It is difficult to use generalized hyperbolic model to price options with several maturities. This is result of the fact that the sum of two independent GH random variables is not a GH random variable.

### 3.5 The CEV Model

The constant elasticity of variance (CEV) model extends the Black-Scholes model and allows the volatility to change with the stock price. Under the CEV model the stock price is assumed to follow the diffusion process

$$dS(t) = \mu S(t)dt + \sigma S^{\frac{\beta}{2}}(t)dW(t) \quad (3.12)$$

where  $\mu$  is a drift,  $\sigma$  is a volatility,  $\beta$  is an elasticity ( $0 < \beta \leq 2$ ) and  $W(t)$  is a Wiener process.

If  $\beta = 2$  then (3.12) reduces to the geometric Brownian motion. If  $\beta < 2$  then the volatility  $\sigma$  increases, while the stock price  $S$  decreases. This kind of distribution is similar to the observed leptokurtic feature. We can transform the diffusion process into one with a constant volatility. Let us consider the process  $x(t) = S^\alpha(t)/\alpha\sigma$ , with  $\alpha = 1 - \beta/2$ . By applying Itô formula to  $x(t)$  we obtain

$$dx(t) = \frac{S^{\alpha-1}(t)}{\sigma}dS(t) + \frac{(\alpha - 1)\sigma}{2}dt. \quad (3.13)$$

Due to (3.12) and the definition of the process  $x(t)$  we have

$$dx(t) = \left( x(t)\alpha\mu + \frac{(\alpha - 1)\sigma}{2} \right)dt + dW(t). \quad (3.14)$$

We thus obtain a process with constant volatility equal to 1.

### 3.6 An implied binomial tree

Assuming the markets are complete, we can price any security as an expected present value of its future payoffs. To calculate the present value we must

use a risk-neutral (martingale) measure, as well as a risk-free interest rate  $r$ . When we have a required risk-neutral measure, we can build the **implied binomial tree**. Under the Binomial Path Independence assumption, which states that any path that ends at the same point on the tree, has the same risk-neutral probability, we can easily calculate the probability of the path, as we can calculate a number of paths that lead to the certain point. Using a backward induction we can build the entire tree.

**Example** for a discrete binomial  $(B, S)$ -market with a horizon  $N$ . Let the  $(B, S)$ -market be a market described by the following system of stochastic differential equations

$$\begin{cases} dS_t = S_t \mu dt + S_t \sigma dW_t, & S_0 > 0, \\ dB_t = B_t r dt, & B_0 = 1, r > 0, \end{cases} \quad (3.15)$$

where  $W_t$  is a Wiener process (Brownian motion).

Let  $C_n$  be the price of the call option at the moment  $n$ . In this case the terminal condition is  $C_N = (S_N - K)^+$ .

As the binomial market is complete and with non-arbitrage opportunities, therefore  $C_0 = B_0 \cdot E_Q \left[ \frac{(C_1 - K)^+}{B_1} \right]$  and  $C_{N-1} = B_{N-1} \cdot E_Q \left[ \frac{C_N}{B_N} \right]$ , where  $Q$  is a martingale measure defined by a parameter  $p^* = \frac{r-a}{b-a}$  describing the probability of going up.

To extend the package of information in the implied binomial tree we can change the binomial path independence assumption and allow paths (that lead to the same ending point) to have different probabilities. It enables us to organize the tree in more realistic way, regarding prices of any derivative.

## 3.7 Time changed Brownian motions and Lévy processes

**Definition 3.7.1** (Veraart & Winkel, 2010).

Let  $X = (X_t)_{t \geq 0}$  be a stochastic process, also called the base process and let  $T = (T_s)_{s \geq 0}$  be a non-negative, non-decreasing stochastic process, independent on  $X$  or not.

The **time-changed process** is then defined as  $Y = (Y_s)_{s \geq 0}$ , where

$$Y_s = X_{T_s}. \quad (3.16)$$

The process  $T_s$  is also referred to as a **business time** or a **stochastic clock**.

Having many different financial models implies having different methods for choosing a time change. The most popular of them we can arrange into two classes, which are

- 1) subordinators,
- 2) absolutely continuous time changes.

In the class of subordinators we find non-decreasing Lévy processes, which can be referred to as a pure jumps with a linear drift. Among the examples it is worth to mention a simple Poisson process and an increasing compound Poisson process.

The second class is a group of a time change of the form  $T_s = \int_0^s \tau_u du$ , where the process  $\tau = (\tau_s)_{s \geq 0}$  is positive and integrable. With such definition,  $T$  is always continuous. In the literature (Veraart & Winkel, 2010) the process  $\tau$  is called instantaneous activity rate process.

**Definition 3.7.2** (Mörters & Peres, 2008).

A real-valued stochastic process  $X(t) : t \geq 0$  is called a linear Brownian motion with start point in  $x \in \mathbb{R}$ , if the following holds

1.  $X(0) = x$ ,
2. the process has independent increments, i.e. for all times  $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$  the increments  $X(t_n) - X(t_{n-1}), X(t_{n-1}) - X(t_{n-2}), \dots, X(t_2) - X(t_1)$  are independent random variables,
3. for all  $t \geq 0$  and  $h > 0$ , the increments  $X(t+h) - X(t)$  are normally distributed with the zero expectation and variance  $h$ ,
4. almost surely, the function  $t \rightarrow X(t)$  is continuous.

We say that  $B(t) : t \geq 0$  is a **standard Brownian motion** if  $x = 0$ . It can be shown that the standard Brownian motion is a Lévy process.

In the time-change Brownian motions or Lévy processes the asset price  $S(t)$  is modelled as

$$S(t) = G(T(t)),$$

where  $G$  is either a geometric Brownian motion or a Lévy process and  $T$  is a business time.

### 3.8 The Merton jump-diffusion model

The jump-diffusion model, introduced in 1976 by Robert Merton, is the model for a stock price behaviour that incorporates small day-to-day diffusive movements together with larger, randomly occurring jumps. Under this model an

asset price is described (Merton, 1976) by the equation

$$S(t) = S(0) \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right\} \prod_{i=1}^{N(t)} e^{Y_i}, \quad (3.17)$$

where  $N(t)$  is a Poisson process.

The value  $Y_i$ ,  $\{i = 1, \dots, N(t)\}$  have a normal distribution with the density function

$$f_{Y_i}(y) = \frac{1}{\sigma' \sqrt{2\pi}} \exp \left\{ -\frac{(y - \mu')^2}{2\sigma'^2} \right\}, \quad (3.18)$$

where  $\mu'$  and  $\sigma'$  are the mean and the standard deviation of  $Y$ .

The Merton model is useful especially to pricing options with a short time to maturity considering the fact (Kou, 2008) that this model does not incorporate the volatility clustering effect.

### 3.9 The Kou model

The Kou model is very similar to the Merton model. The only difference is the assumption about the distribution of the jumps. Instead of the normal distribution, Kou proposed to use the double exponential distribution of the jumps with the density function as

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} \mathbb{1}_{y \geq 0} + q \cdot \eta_2 e^{\eta_2 y} \mathbb{1}_{y < 0}, \quad (3.19)$$

where  $\eta_1 > 1$ ;  $\eta_2 > 0$ ;  $p, q \geq 0$ ;  $p + q = 1$ ;  $p$  is the probability of the upward jumps and  $q$  is the probability of the downward jumps.

We develop the Kou model in Chapter 4.

# Chapter 4

## The double exponential jump-diffusion model

There are many modifications of the Black-Scholes model. One of them is the jump-diffusion model.

The double exponential jump-diffusion model, called also the Kou model, was proposed by Steven Kou in 2002. We decided to focus on this model due to its many advantages listed below.

### 4.1 Advantages of the double exponential jump-diffusion model

This model overcomes some difficulties connected with the stylized facts (Kou, 2002). The double exponential jump-diffusion model captures two of three empirical phenomena described in Chapter 2. The model can reproduce the leptokurtic feature of the return distribution and the "volatility smile" effect in option prices. Because of this fact the model's behaviour in empirical tests is quite good.

The next attraction of the Kou model is its simplicity. This model retains the analytical tractability of the Black-Scholes model. The computations under the Kou model are quite easy and we can obtain closed-form solutions for the vanilla options, as well as for path-dependent options, while it seems to be difficult for many other models. These closed-form solutions are possible to be obtained because of the special property of the double exponential distribution, called the memoryless property.

Moreover, the double exponential jump-diffusion model is internally self-consistent (Kou, 2008). It means that the model is arbitrage-free and can

be embedded in a rational expectations equilibrium setting, explained in Chapter 4.3.

Another motivation for the Kou model comes from the behavioral finance. Empirical studies show that the markets tend to have overreaction and underreaction to the outside news (good or bad). Due to the high peak of the double exponential distribution, it can be used to model the underreaction to outside news. Because of the heavy tails of the double exponential distribution, it can be used to model the overreaction. The jump part of the model can be interpreted as the market response to outside news. If there are not any outside news, the asset price changes according to a geometric Brownian motion.

## 4.2 The model specification

Let  $S(t)$  be the asset price. Under the probability measure  $P$ ,  $S(t)$  is modeled as (Kou, 2008)

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \quad (4.1)$$

where  $W(t)$  is a standard Brownian motion,  $N(t)$  denotes a Poisson process with parameter  $\lambda$ ,  $\{V_i\}$  is a sequence of independent identically distributed nonnegative random variables,  $S(t-)$  denotes the asset price just before the jump,  $\mu$  is the drift and  $\sigma$  is the volatility.

For the simplicity we assume that the Brownian motion and the jumps are one-dimensional while  $\mu$  and  $\sigma$  are constants. These assumptions are helpful especially if we want to find analytical solutions for complicated cases of the option pricing problems. In addition the model assumes that the Poisson process  $N(t)$ , the Brownian motion  $W(t)$  and the jump sizes are independent.

We can solve the stochastic differential equation (4.1) by using the Itô's formula for the jump-diffusion (Cont & Tankov, 2004). A complete derivation of the solution is presented in the work of Walachowska and Walachowski (2009). This procedure leads to the equation for the dynamics of the underlying asset price

$$S(t) = S(0) \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right\} \prod_{i=1}^{N(t)} V_i. \quad (4.2)$$

In the Kou model  $Y = \log(V)$  has an asymmetric double exponential distribution and its density is given by

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} \mathbb{1}_{y \geq 0} + q \cdot \eta_2 e^{\eta_2 y} \mathbb{1}_{y < 0}, \quad (4.3)$$

where  $\eta_1 > 1$ ;  $\eta_2 > 0$ ;  $p, q \geq 0$ ;  $p + q = 1$ ;  $p$  is the probability of the upward jumps and  $q$  is the probability of the downward jumps.

$Y = \log(V)$  can also be defined in another way (Maekawa, et al., 2008)

$$Y = \log(V) = \begin{cases} \xi^+ & \text{with probability } p, \\ -\xi^- & \text{with probability } q, \end{cases} \quad (4.4)$$

where  $\xi^+$  and  $\xi^-$  are exponential random variables with mean  $\frac{1}{\eta_1}$  and  $\frac{1}{\eta_2}$ , respectively.

### 4.3 The equilibrium for the jump-diffusion model

**Definition 4.3.1** (Karatzas & Shreve, 1998)

**A utility function** is a concave, non-decreasing, upper semicontinuous function  $U : \mathbb{R} \rightarrow [-\infty, \infty)$  satisfying

- 1) the half-line  $\text{dom}(U) = \{x \in \mathbb{R}; U(x) > -\infty\}$  is a nonempty subset of  $[0, \infty)$ ,
- 2)  $U'$  is continuous, positive and strictly decreasing on the interior of  $\text{dom}(U)$

$$U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0. \quad (4.5)$$

In a rational expectations economy the investors' objective is to solve a utility maximization problem

$$\max_c E \left[ \int_0^\infty U(c(t), t) dt \right], \quad (4.6)$$

where  $U(c(t), t)$  is the utility function of the consumption process  $c(t)$ .

Investors have at their disposal the extern endowment process  $\delta(t)$  and the opportunity to invest in securities without dividends.

Now we are looking for the rational expectations equilibrium price. In the case when  $\delta(t)$  is Markovian, the equilibrium price of the security is given by the Euler equation (Kou, 2008)

$$p(t) = \frac{E(U_c(\delta(T), T)p(T)|F_t)}{U_c(\delta(t), t)}, \quad \forall T \in [t, T_0] \quad (4.7)$$

where  $p(t)$  denotes the equilibrium price,  $U_c$  denotes the partial derivative of  $U$  with respect to  $c$  and  $\delta(t)$  is the exogenous endowment process.

At this price  $p(t)$  investors will never decide to invest in the security. They will just consume the extern endowment so  $c(t) = \delta(t)$ .

In the case when  $\delta(t)$  follows a general jump-diffusion process

$$\frac{d\delta(t)}{\delta(t-)} = \mu_1 dt + \sigma_1 dW_1(t) + d \left( \sum_{i=1}^{N(t)} (\tilde{V}_i - 1) \right), \quad (4.8)$$

$p(t)$  and  $\delta(t)$  do not have to have similar jump dynamics. The analysis of the change of the parameters from  $p(t)$  to  $\delta(t)$  provides information about the risk premiums embedded in jump-diffusion models. The rational expectations equilibrium price is given by

$$p(t) = \frac{E(e^{-\theta T} (\delta(T))^{\alpha-1} p(T) | F_t)}{e^{-\theta t} (\delta(t))^{\alpha-1}}. \quad (4.9)$$

This price is calculated under the special type of the utility function as formulated in (4.10)

$$U(c, t) = \begin{cases} e^{-\theta t} \frac{c^\alpha}{\alpha} & \text{if } \alpha < 1, \\ e^{-\theta t} \log(c) & \text{if } \alpha = 1, \end{cases} \quad (4.10)$$

where  $\theta > 0$  is the discount rate in the utility function.

## 4.4 The Laplace transform for the plain vanilla option pricing

**Definition 4.4.1** (*Widder, 1945*)

**The Laplace transform** of a function  $f(t)$  defined on  $[0, \infty)$ , is the function  $F(s)$ , defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt. \quad (4.11)$$

To value the option under the double exponential jump-diffusion model, we use Laplace transform method, which is widely used in pricing of derivatives.

By using the notation as in Kou (2008), the price of a European Call option with strike price  $K$  and maturity  $T$  is given by

$$\begin{aligned} C_T(k) &= e^{-rT} E^*[(S(T) - K)^+] \\ &= e^{-rT} E^*[(S(0)e^{X(T)} - e^{-k})^+], \end{aligned} \quad (4.12)$$



where  $k = -\log(K)$ .

Analogously, the price of a European put option is given by

$$\begin{aligned} P_T(k') &= e^{-rT} E^*[(K - S(T))^+] \\ &= e^{-rT} E^*[(e^k - S(0)e^{X(T)})^+], \end{aligned} \quad (4.13)$$

where  $k' = \log(K)$  and  $E^*$  is the expectation under the risk-neutral measure. Let us define the moment generating function of  $X(t)$  as

$$E[e^{\theta X(t)}] = e^{G(\theta)t}, \quad (4.14)$$

where

$$\begin{aligned} G(x) &= x\tilde{\mu} + \frac{1}{2}x^2\sigma^2 + \lambda(E[e^{xY}] - 1) \\ &= x\tilde{\mu} + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right), \end{aligned} \quad (4.15)$$

in the case of the double exponential jump-diffusion model. Under the risk-neutral measure we have  $\tilde{\mu} = r - \delta - \frac{1}{2}\sigma^2 - \lambda\zeta$ , with  $\zeta := E^*[e^Y] - 1$ , for the asset that pays continuous dividends with rate  $\delta$ .

For the given prices (4.12) and (4.13) we derive the Laplace transform (Kou, et al., 2005) as follows

$$\begin{aligned} \hat{f}_C(\xi) &:= \int_{-\infty}^{+\infty} e^{-\xi k} C_T(k) dk \\ &= e^{-rT} \frac{S(0)^{\xi+1}}{\xi(\xi+1)} e^{G(\xi+1)T}, \quad \xi > 0, \end{aligned} \quad (4.16)$$

$$\begin{aligned} \hat{f}_P(\xi) &:= \int_{-\infty}^{+\infty} e^{-\xi k'} P_T(k') dk' \\ &= e^{-rT} \frac{S(0)^{-(\xi-1)}}{\xi(\xi-1)} e^{G(-(\xi-1))T}, \quad \xi > 1, \end{aligned} \quad (4.17)$$

with respect to  $k$  and  $k'$ , respectively.

## 4.5 The inversion of the one-dimensional Laplace transform

Because in some probabilistic models any explicit solutions are not available, it is necessary to use numerical algorithms. There are many of algorithms that can be used in the Laplace transform inversion problem, but we prefer the Euler inversion algorithm. Among the most important advantages of this method are

- the very fast convergence,
- the smaller numerical accuracy is required than in other methods,
- it provides error bounds.

According to Abate and Whitt (1991) a real function  $f(\cdot)$ , for any  $t > 0$  can be presented in the form

$$f(t) = \frac{e^{\frac{A}{2}}}{2t} \operatorname{Re} \left( \hat{f} \left( \frac{A}{2t} \right) \right) + \frac{e^{\frac{A}{2}}}{t} \sum_{j=1}^{\infty} (-1)^j \operatorname{Re} \left( \hat{f} \left( \frac{A + 2j\pi i}{2t} \right) \right), \quad (4.18)$$

where  $\hat{f}(\cdot)$  is the Laplace transform of  $f(\cdot)$  with respect to a logarithm of the strike and  $A$  is an arbitrary positive constant.

As an alternative to the Euler algorithm we can also use the Zakian method as

$$f(t) = \frac{2}{t} \sum_{i=1}^5 \operatorname{Re} \left[ k_i \hat{f} \left( \frac{\alpha_i}{t} \right) \right], \quad (4.19)$$

where the parameters are

$$\begin{aligned} \alpha_1 &= 1.283767675 \cdot 10^1 + i1.666063445, \\ \alpha_2 &= 1.222613209 \cdot 10^1 + i5.012718792, \\ \alpha_3 &= 1.09343031 + i8.40967312, \\ \alpha_4 &= 8.77643472 + i1.19218539 \cdot 10^1, \\ \alpha_5 &= 5.22545336 + i1.57295290 \cdot 10^1, \\ k_1 &= -3.69020821 \cdot 10^4 + i1.96990426 \cdot 10^5, \\ k_2 &= 6.12770252 \cdot 10^4 - i9.54086255 \cdot 10^4, \\ k_3 &= -2.89165629 \cdot 10^4 + i1.81691853 \cdot 10^4, \\ k_4 &= 4.65536114 \cdot 10^3 - i1.90152864, \\ k_5 &= -1.18741401 \cdot 10^2 - i1.41303691 \cdot 10^2. \end{aligned}$$

## 4.6 The Laplace transform for the path-dependent option pricing

In this section we focus on pricing of path-dependent options via Laplace transform. As an example of the path-dependent options we take barrier up-and-in call option (UIC). UIC is a call option, which is activated if the price of the underlying asset rises above a certain price level, i.e. the barrier. The price of UIC option is given by (Kou, et al., 2005)

$$UIC(k, T) = E^* [e^{-rT} (S(T) - e^{-k})^+ \mathbb{1}_{\{\tau_b < T\}}], \quad (4.20)$$

where  $k = -\log(K)$ ,  $K$  denotes the strike price,  $b = \log(H/S(0))$ ,  $H > S(0)$  is the barrier level and  $\tau_b$  is the first passage time.

**Definition 4.6.1** (*Kou, 2008*)

The first passage time of a jump-diffusion process  $X(t)$  to a flat boundary  $b$  is

$$\tau_b := \inf \{t \geq 0; X(t) \geq b\}, \quad b > 0, \quad (4.21)$$

where  $X(\tau_b) := \limsup_{t \rightarrow \infty} X(t)$ , on the set  $\{\tau_b = \infty\}$ .

**Theorem 4.6.1** (*Kou, et al., 2005*)

For  $\xi$  and  $\alpha$  such that  $0 < \xi < \eta_1 - 1$  and  $\alpha > \max(G(\xi + 1) - r, 0)$ , the Laplace transform with respect to  $k$  and  $T$  of  $UIC(k, T)$  is given by

$$\begin{aligned} \hat{f}_{UIC}(\xi, \alpha) &= \int_0^\infty \int_{-\infty}^\infty e^{-\xi k - \alpha T} UIC(k, T) dk dT \\ &= \frac{H^{\xi+1}}{\xi(\xi+1)} \frac{1}{r + \alpha - G(\xi+1)} \left( A(r + \alpha) \frac{\eta_1}{\eta_1 - (\xi+1)} + B(r + \alpha) \right), \end{aligned} \quad (4.22)$$

where

$$A(h) : = E^*[e^{-h\tau_b} \mathbb{1}_{\{X(\tau_b) > b\}}] = \frac{(\eta_1 - \beta_{1,h})(\beta_{2,h} - \eta_1)}{\eta_1(\beta_{2,h} - \beta_{1,h})} [e^{-b\beta_{1,h}} - e^{-b\beta_{2,h}}], \quad (4.23)$$

$$B(h) : = E^*[e^{-h\tau_b} \mathbb{1}_{\{X(\tau_b) = b\}}] = \frac{\eta_1 - \beta_{1,h}}{\beta_{2,h} - \beta_{1,h}} e^{-\beta_{1,h}} + \frac{\beta_{2,h} - \eta_1}{\beta_{2,h} - \beta_{1,h}} e^{-\beta_{2,h}}, \quad (4.24)$$

with  $b = \log(H/S(0))$ ,  $G(x) = \tilde{\mu} + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{px_1}{\eta_1 - x} + \frac{qx_2}{\eta_2 + x} - 1\right)$  and  $\beta_{1,h}, \beta_{2,h}$  denote two positive roots of the equation  $G(x) = h$ .

## 4.7 The inversion of the two-dimensional Laplace transform

The price of the UIC option can be obtained by the inverting the two-dimensional Laplace transform (4.22) with respect to the logarithm of the

strike price and maturity ( $k$  and  $T$ , respectively). Because of the quite complicated expressions it is not easy to find the inverse by using standard functions, so we have to use a numerical inversion (Abate, et al., 1998). Again we investigate the Euler algorithm, more precisely in this case - the two-sided Euler algorithm (Choudhury, 1993).

Let us consider a real function  $f(t_1, t_2)$  defined in  $\mathbb{R}^2$ . The following formula can be obtained (Petrella, 2003)

$$\begin{aligned}
f(t_1, t_2) = & \frac{\exp(A_1/(2l_1) + A_2/(2l_2))}{4t_1l_1t_2l_2} \times \left( \hat{f}\left(\frac{A_1}{2l_1t_1}, \frac{A_2}{2l_2t_2}\right) \right. \\
& + 2 \sum_{k_1=1}^{l_2} \sum_{k=0}^{\infty} (-1)^k \operatorname{Re} \left[ \exp\left(-\frac{ik_1\pi}{l_2}\right) \hat{f}\left(\frac{A_1}{2l_1t_1}, \frac{A_2}{2l_2t_2} - \frac{ik_1\pi}{l_2t_2} - \frac{ik\pi}{t_2}\right) \right] \\
& + 2 \sum_{j_1=1}^{l_1} \sum_{j=0}^{\infty} (-1)^j \operatorname{Re} \left[ \sum_{k_1=1}^{l_2} \sum_{k=0}^{\infty} (-1)^k \exp\left(-\left(\frac{ij_1\pi}{l_1} + \frac{ik_1\pi}{l_2}\right)\right) \right. \\
& \times \left. \hat{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij\pi}{t_1}, \frac{A_2}{2l_2t_2} - \frac{ik_1\pi}{l_2t_2} - \frac{ik\pi}{t_2}\right) \right] \\
& + 2 \sum_{j_1=1}^{l_1} \sum_{j=0}^{\infty} (-1)^j \operatorname{Re} \left[ \exp\left(-\frac{ij_1\pi}{l_1}\right) \times \hat{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij\pi}{t_1}, \frac{A_2}{2l_2t_2}\right) \right. \\
& + \sum_{k_1=1}^{l_2} \sum_{k=0}^{\infty} (-1)^k \exp\left(-\left(\frac{ij_1\pi}{l_1} - \frac{ik_1\pi}{l_2}\right)\right) \\
& \left. \times \hat{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij\pi}{t_1}, \frac{A_2}{2l_2t_2} + \frac{ik_1\pi}{l_2t_2} + \frac{ik\pi}{t_2}\right) \right], \tag{4.25}
\end{aligned}$$

where  $\hat{f}(\cdot, \cdot)$  denotes the Laplace transform of the function  $f(\cdot, \cdot)$ ,  $l_1, l_2$  are integers,  $A_1$  and  $A_2$  are arbitrary constants. For example, the UIC, the Laplace transform in Equation (4.22) can be inverted by using  $l_1 = l_2 = 1$ ,  $A_1 = 40$  and  $A_2 = 18.4$ , which provide the best results in practice (Petrella, 2003).

# Chapter 5

## The application of the Kou model

### 5.1 The market and a data specification

For the application of the double exponential jump-diffusion model we consider the NASDAQ OMX Stockholm Market whose main index is OMX Stockholm 30 (OMXS30). This index consists of the 30 most traded stocks on the Stockholm Stock Exchange e.g. Scania, Ericson, Electrolux, Tele2, Nordea, Swedbank and SEB.

We start with the historical background of the creating of the NASDAQ OMX Group, whose one of the part is NASDAQ OMX Stockholm Market. In 1998 the futures exchange Optionsmäklarna (OM AB), founded in 1980s, acquired the Stockholm Stock Exchange. In September 2003 the Helsinki Stock Exchange (HEX) merged with OM AB and as the result of this the company OM HEX was raised, which later was renamed to OMX. In January 2005 OMX aquired the Copenhagen Stock Exchange, in September 2006 Iceland Stock Exchange and also Armenian Stock Exchange in October 2006. Finally on May 2007, NASDAQ bought OMX to form NASDAQ OMX Group.

In our practical part we decided to investigate the stock of the Skandinaviska Enskilda Banken (SEB). The SEB is one of the biggest Nordic financial groups, formed in 1972 as the result of the merger of the Stockholm Enskilda Bank and the Skandinaviska Banken. The main reason for the merger was that the Swedish government set the certain limit of a profit which was in contradiction with the expansion of the Stockholm Enskilda Bank. The fusion with the Skandinaviska Banken was the only way to create better positioned bank which could be able to expand. Currently the SEB is one of the Northern Europes leaders in the branch of financial companies.

The company serves 2,500 large customers and institutions, 400,000 small and medium sized companies and also five million private individuals. The SEB is represented in 21 countries including the Baltic area but also in USA, China, Great Britain, Russia, Singapore and several other countries. The number of employees exceeds 21,000 worldwide.

The data were collected by using a professional tool - the *SIXEdge<sup>TM</sup>*, provided by the Halmstad University. We have to our disposal data from period 6.03.2009 - 4.06.2009 for time intervals 1 minute, 5 minutes, 10 minutes, 20 minutes, 30 minutes, 1 hour and daily. The data contain the asset prices and volatilities for the period specified above. For our computations we use also the fact that in 2009 there were 252 trading days.

In Figure 5.1 one can see the changes in the prices of the SEB A stock for 5 minutes data. In this figure we can notice places of a potential existence of jumps, i.e. places where the price rapidly goes up or down.



Figure 5.1: The changes in the price of the SEB A stock (6.03.2009 - 4.06.2009, interval 5 minutes).

From the analytical point of view, more interesting it is the comparison of changes in the volatility with the changes in the asset prices, which one can see in Figure 5.2.



Figure 5.2: The changes in the price compared with the changes in the volatility of the SEB A stock (6.03.2009-4.06.2009, interval one day).

For the jump-detection and the estimation of the parameters for the Kou model we analyse the 5-minute data. For more detailed information about the available data, the descriptive statistics are presented in Table 5.1.

No. of obs.	6342
Mean	0,000083
Minimum	-0,091231
Maximum	0,053843
Variance	0,000041
Std.dev.	0,006421
Skewness	0,099173
Kurtosis	15,818907

Table 5.1: The descriptive statistics for the log returns for the 5-minute data.

## 5.2 The programme description

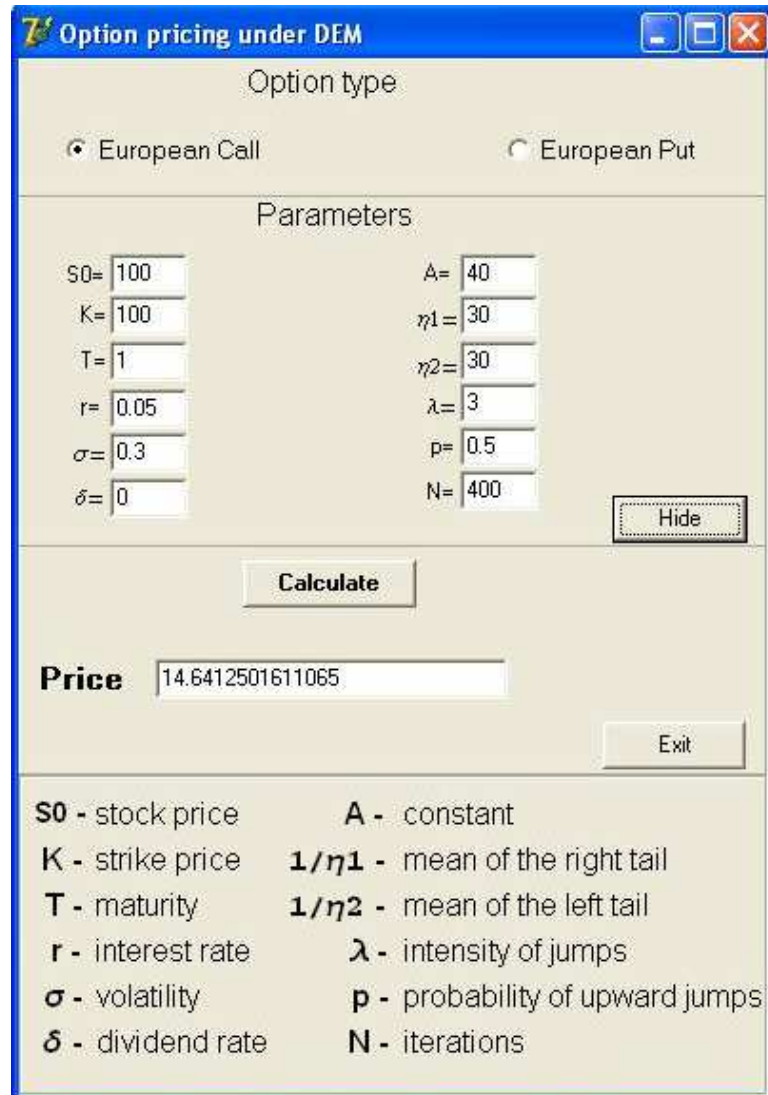


Figure 5.3: An example of the option pricing by attached programme.

In Figure 5.3 we present the layout window of our programme. By using this programme we can simply calculate prices of European call and put options under the double exponential jump-diffusion model via the Laplace transform. The input data are the jump parameters  $\eta_1, \eta_2, \lambda, p$  and the option parameters  $S(0), K, T, r, \sigma, \delta$ . The meaning of this parameters is explained in Figure 5.3.



To write this programme we used the theory concerning the double exponential jump-diffusion model presented in Chapter 4. We also exploited the Laplace transform as a method for the option pricing and the one-dimensional Euler algorithm as a method of inverting the Laplace transform. The basic idea was to use in practice the formulas (4.16) (4.17) and (4.18).

The programme is written in the Pascal language in programming environment Delphi 7. We use package the CMATH which is available on the website <http://www.optivec.com/>. CMATH is a comprehensive library for complex-number arithmetics and mathematics, both in cartesian and in polar coordinates. The complete programme code is presented in the Appendix A.

### 5.3 The estimation of the parameters for the Kou model

To use the double exponential jump-diffusion model for the options pricing on the SEB A stock we have to estimate the parameters of the model  $\eta_1, \eta_2, \lambda$  and  $p$ . To this purpose we use the 5-minute data. We dispose only the asset prices  $S(t)$  and the volatility. Firstly we calculate the log returns  $r(t)$  as follows

$$r(t) = \ln(S(t)/S(t-1)). \quad (5.1)$$

The values of the log returns  $r(t)$  are crucial for the jump-detection. We assume that the positive jump appears, when the return  $r(t)$  is larger than four times standard deviation of all returns while the negative jump appears, when the return  $r(t)$  is smaller than minus four times standard deviation. The substantiation of this choice we present in Section 5.4. In the case of SEB A stock we find 56 jumps during examined period, among which we have 32 positive jumps and 24 negative jumps. The plot of log returns (5.1) with the highlighted level of the standard deviation multiplied by the constant, one can see in Figure 5.4.

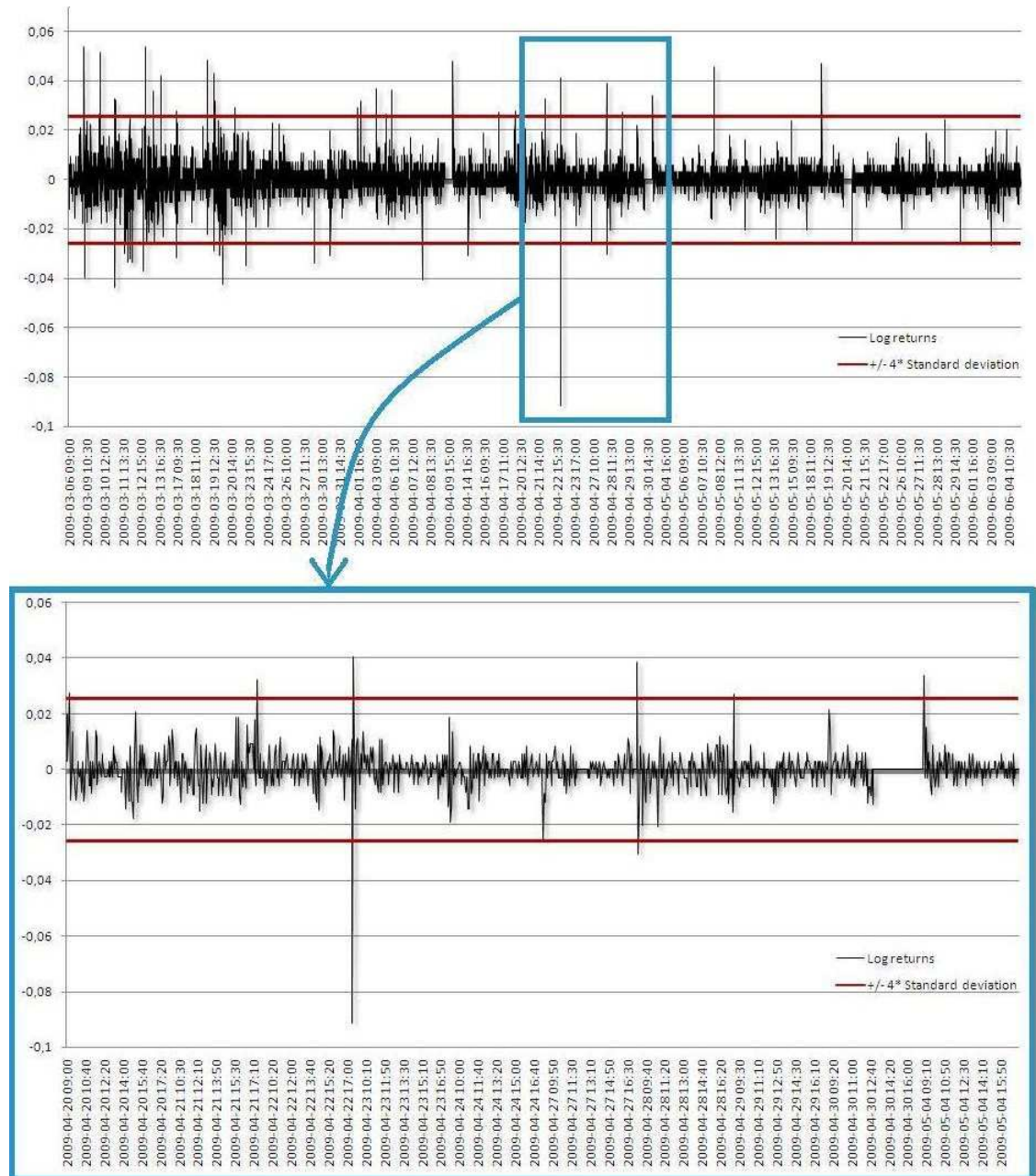


Figure 5.4: The log returns of the SEB A stock (6.03.2009-4.06.2009, interval 5 minutes) are presented in the upper part. The lower part is the zoom of the framed area.

In the next step we calculate the daily intensity of the jumps  $\lambda$  by using the following relation

$$\lambda = \frac{\text{number of jumps}}{\text{number of observations}} * \text{number of observations during a day.} \quad (5.2)$$

In the case of the SEB A stock,  $\lambda$  is estimated as  $\frac{56}{6324} * 102 \approx 0,903229$ .

Later we calculate the probability  $p$  of the upward jumps

$$p = \frac{\text{number of positive jumps}}{\text{number of all jumps detected}}. \quad (5.3)$$

According to the data used in this thesis  $p = \frac{32}{56} \approx 0,571429$ .

The parameters still to be estimated are  $\eta_1$  and  $\eta_2$ , whose reverses are the means of the right tail and left tail of distribution of  $r(t)$ , respectively. To find  $\eta_1$  we calculate the arithmetic average of the positive jumps and exploit the fact that  $\eta_1 \sim \frac{1}{\text{average}}$ . For the SEB A stock we obtain  $\eta_1 = 99,39$ . By analogy we derive  $\eta_2 = 108$ .

## 5.4 The real prices versus the Kou model

In this section we present the comparison of the prices derived by the double exponential jump-diffusion model with the market prices. For this purpose we choose options starting on 15.05.2009 with maturity dates 18.06.2009, 21.08.2009 and 16.10.2009. In other words we consider options with 23, 68 and 107 days to expiration.

The parameters to the Kou model were chosen in the way we developed in Chapter 5.3. As  $\sigma$  we take the standard deviation of the log returns  $r(t)$  which in this case is equal to 0,7324. The price  $S(0)$  is the closing price from the 14.05.2009 which is equal to 33,6. Due to the length of the period to maturity we have different sets of parameters, as follows

$$\begin{cases} r = 0,005 \\ T = 0,0912698 \end{cases}, \text{ for 18.06.2009,}$$

$$\begin{cases} r = 0,00502 \\ T = 0,2698413 \end{cases}, \text{ for 21.08.2009,}$$

$$\begin{cases} r = 0,005 \\ T = 0,4246032 \end{cases}, \text{ for 16.10.2009.}$$

Complete results of our calculations are presented in the Appendix B, while here we display the graph of the prices.

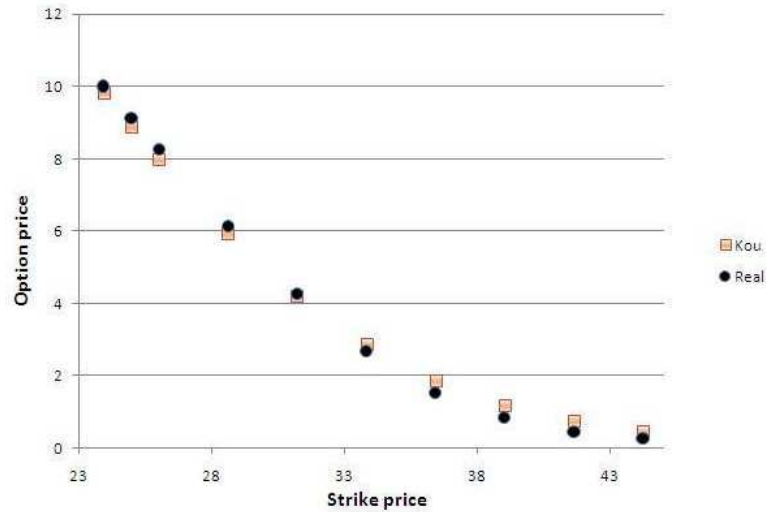


Figure 5.5: A relationship between the real prices and results for the Kou model for call option with 23 days to expiry.

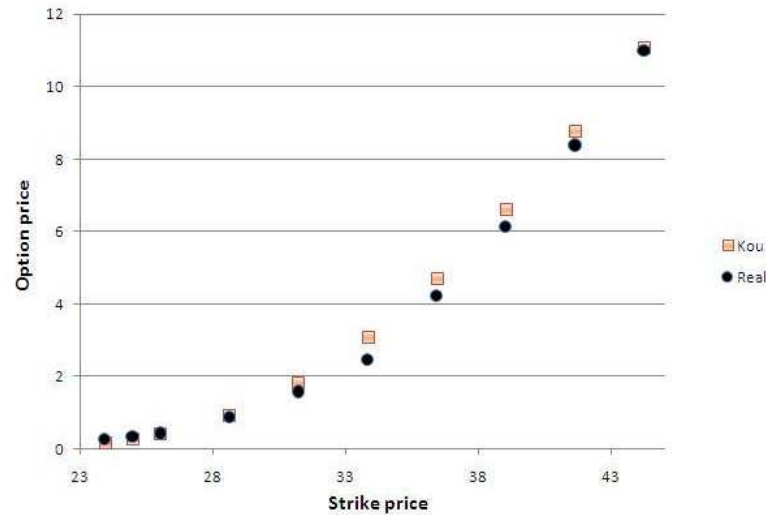


Figure 5.6: A relationship between the real prices and results for the Kou model for put option with 23 days to expiry.

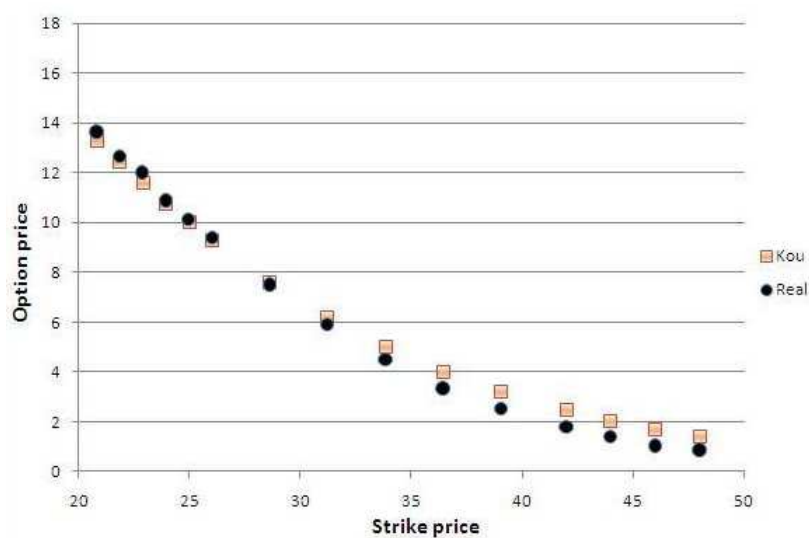


Figure 5.7: A relationship between the real prices and results for the Kou model for call option with 68 days to expiry.

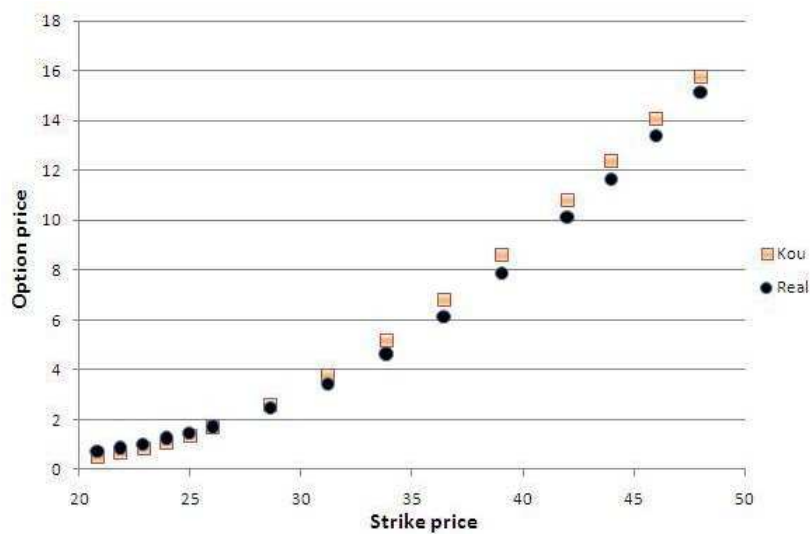


Figure 5.8: A relationship between the real prices and results for the Kou model for put option with 68 days to expiry.

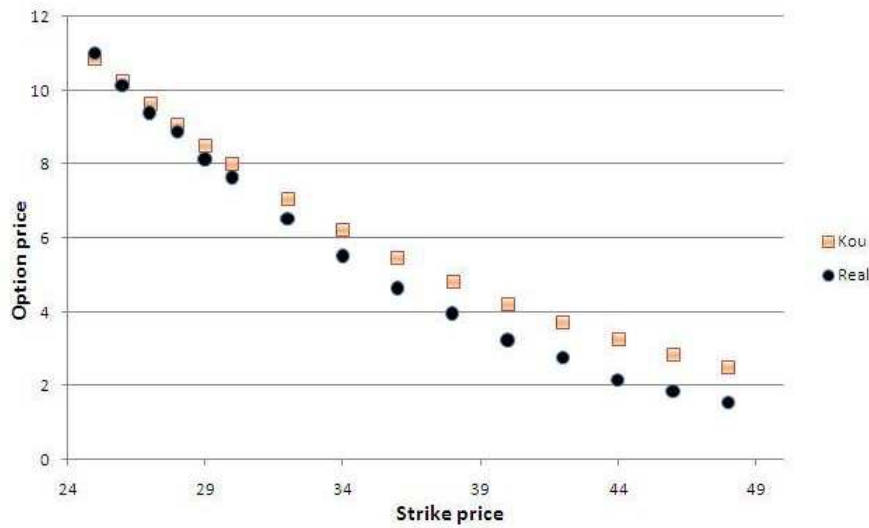


Figure 5.9: A relationship between the real prices and results for the Kou model for call option with 107 days to expiry.

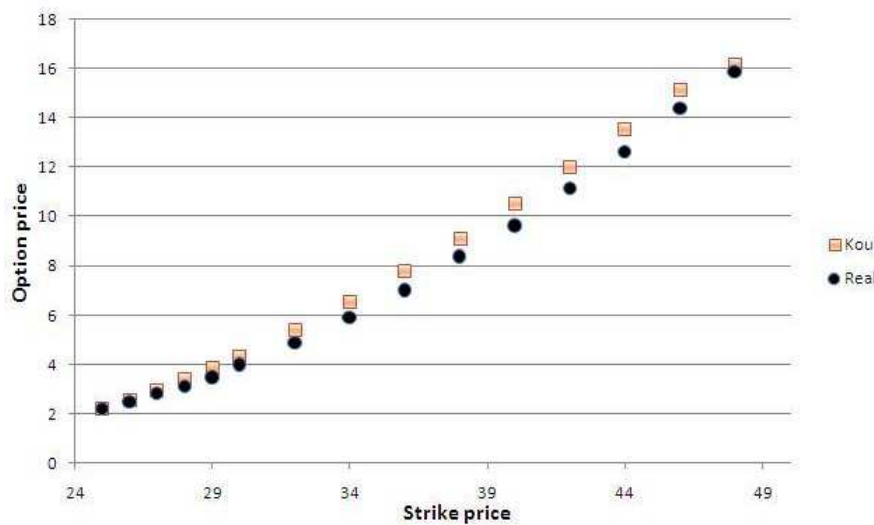


Figure 5.10: A relationship between the real prices and results for the Kou model for put option with 107 days to expiry.

On the basis of Figures 5.5-5.10 we conclude that the prices obtained by the double exponential jump-diffusion model are rather close to the real prices. To show the differences between prices we calculate the average relative error by using the formula

$$ERROR = \frac{1}{N} \sum_{i=1}^N \left| \frac{Kou\ price - Real\ price}{Real\ price} \right|, \quad (5.4)$$

where  $N$  is the number of options. Result of this calculation is presented in Table 5.2.

<b>T</b>	<b>Call</b>	<b>Put</b>
23 days	22,17196 %	12,92048 %
68 days	17,58518 %	14,65834 %
107 days	20,90738 %	7,87304 %

Table 5.2: The average relative errors for the Kou model.

As one can see in Table 5.2 errors do not exceed 25% which is a satisfying result.

In Figure 5.5 - Figure 5.10 we notice that in general the prices given by the Kou model are higher than the real prices. The price of a stock that is interfered with the jumps should be higher than a stock without jumps, as it has a higher volatility which reflects a bigger risk.

For more detailed analysis we compared results of the Kou model with results obtained with the classical Black-Scholes model. The complete comparison is presented in the Appendix B. One example of results of the Kou model, Black-Scholes model and real prices is presented in Figure 5.11.

As we can notice in Figure 5.11, in the case of the SEB A stock there is no significant difference between prices obtained using both models. The factor which can have an influence here is probably the high volatility of the log returns. In the case of the high level of the volatility, the detection of jumps can be complicated and disturbed. As the result the advantages of using the Kou model are decreased.

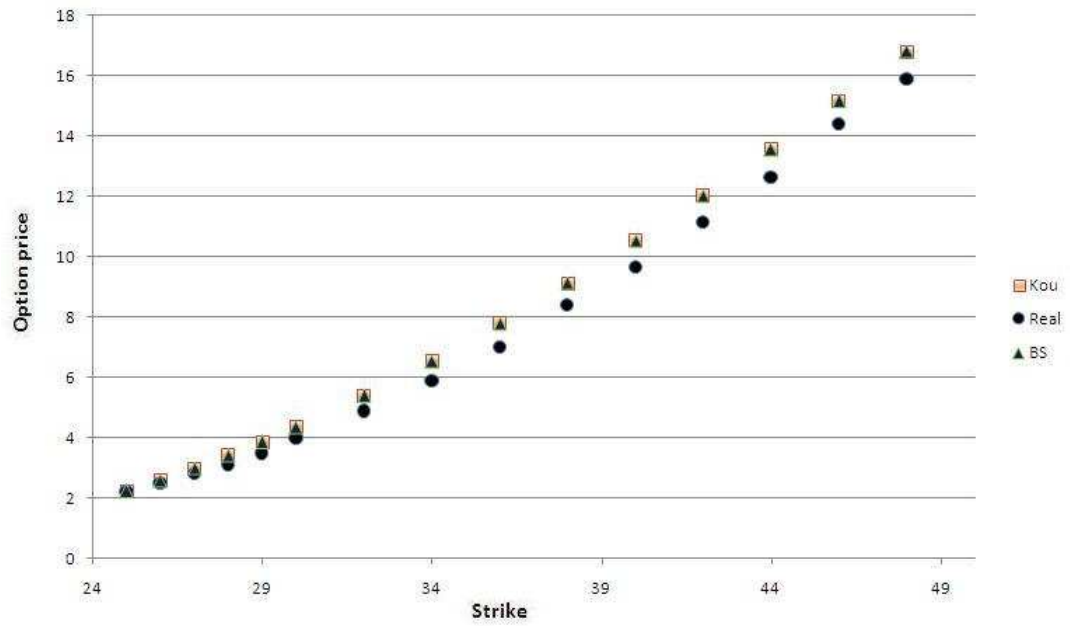


Figure 5.11: A relationship between the real prices, results of the Kou model and results of the Black-Scholes model for put option with 68 days to expiry.



# Chapter 6

## Conclusions

The main goal of our work was to investigate the double exponential jump-diffusion model with its assumptions and to use the Laplace transform as the method of the pricing of options under this model. We chose the Kou model as it incorporates the leptokurtic feature and the volatility smile phenomenon, what is not provided in many other models. The gained knowledge we used to create the programme for the pricing of european-type options on the example of the SEB A stock. For analysis we used the 5-minute data, as it ensures a sufficient frequency of the prices. We compared the obtained results with the real prices. It turned out that the estimation of the prices under the Kou model is rather satisfying. We next compared the results with the prices obtained by using the Black-Scholes model. The outcome was surprising, because in the case of the SEB stock with a high level of the volatility, the Kou model does not perform better than the classical Black-Scholes model.

As the implication, the double exponential jump-diffusion model is not suitable for the analyzed data. Considering only the SEB A stock we cannot surely state that the double exponential jump-diffusion model is not appropriate for the Nordic market. For further investigation of the problem we advice to examine other stocks, with smaller average volatility of their log returns. As the idea for the next research we also propose to investigate how the Kou model performs in the case of other types of options, like path-dependent options, e.g. barrier or lookback options.



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# Appendix

## APPENDIX A:

**Program code for pricing European options under DEM model via the Laplace transform.**

*authors : Natalia Nadratowska and Damian Prochna  
unitMain;*

*interface*

*uses*

*Windows, Messages, SysUtils, Variants, Classes, Graphics, Controls, Forms,  
Dialogs, StdCtrls, VarCmplx, Math, CMATH, ExtCtrls;*

*functionFunG(alfa : fComplex) : fComplex;*

*functionFunFC(xi : fComplex) : fComplex;*

*functionFunSklad1 : double;*

*functionFunSklad2 : double;*

*procedureInput;*

*procedurenormal\_color;*

*type*

*TForm1 = class(TForm)*

*GroupBox3 : TGroupBox;*

*Label1 : TLabel;*

*Label2 : TLabel;*

*Panel1 : TPanel;*

*Edit5 : TEdit;*

*Button1 : TButton;*

*Label5 : TLabel;*

*Button2 : TButton;*

*Panel2 : TPanel;*

*Label3 : TLabel;*

*Edit8 : TEdit;*  
*Label9 : TLabel;*  
*Edit17 : TEdit;*  
*Label17 : TLabel;*  
*Edit18 : TEdit;*  
*Label18 : TLabel;*  
*Edit9 : TEdit;*  
*Label8 : TLabel;*  
*Edit10 : TEdit;*  
*Label10 : TLabel;*  
*Edit14 : TEdit;*  
*Label14 : TLabel;*  
*Edit15 : TEdit;*  
*Label15 : TLabel;*  
*Edit12 : TEdit;*  
*Label12 : TLabel;*  
*Edit13 : TEdit;*  
*Label13 : TLabel;*  
*Edit7 : TEdit;*  
*Label7 : TLabel;*  
*Edit11 : TEdit;*  
*Label11 : TLabel;*  
*Edit21 : TEdit;*  
*Label21 : TLabel;*  
*Panel3 : TPanel;*  
*Label4 : TLabel;*  
*RadioButton1 : TRadioButton;*  
*RadioButton2 : TRadioButton;*  
*Label6 : TLabel;*  
*Label16 : TLabel;*  
*Label19 : TLabel;*  
*Label20 : TLabel;*  
*Label22 : TLabel;*  
*Label23 : TLabel;*  
*Label24 : TLabel;*  
*Label25 : TLabel;*  
*Label26 : TLabel;*  
*Label27 : TLabel;*  
*Button3 : TButton;*  
*Label28 : TLabel;*  
*Label29 : TLabel;*



```

Label30 : TLabel;
Label31 : TLabel;
Label32 : TLabel;
Label33 : TLabel;
Label34 : TLabel;
Label35 : TLabel;
Label36 : TLabel;
Label37 : TLabel;
Label38 : TLabel;
Label39 : TLabel;
Label40 : TLabel;
Label41 : TLabel;
Label42 : TLabel;
Label43 : TLabel;
Label44 : TLabel;
procedure Button1Click(Sender : TObject);
procedure Button2Click(Sender : TObject);
procedure FormCreate(Sender : TObject);
procedure FormClose(Sender : TObject; var Action : TCloseAction);
procedure Label17Click(Sender : TObject);
procedure Label18Click(Sender : TObject);
procedure Label7Click(Sender : TObject);
procedure Label15Click(Sender : TObject);
procedure Label21Click(Sender : TObject);
procedure Label9Click(Sender : TObject);
procedure Label8Click(Sender : TObject);
procedure Label10Click(Sender : TObject);
procedure Label11Click(Sender : TObject);
procedure Label12Click(Sender : TObject);
procedure Label13Click(Sender : TObject);
procedure Label14Click(Sender : TObject);
procedure Label5Click(Sender : TObject);
procedure Button3Click(Sender : TObject);
private
Privatedeclarations

public
Publicdeclarations
end;

var

```

```

Form1 : TForm1;

implementation
vari : fComplex;
q, zeta, skladnik1, skladnik2, Wynik : double;
lambda, S0, r, sigma, p, n1, n2, delta, A, kp, teta, X, T, K, kk : double;
N, para : integer;
ch : char;

{$R * .dfm}
//Startbutton
procedure TForm1.Button1Click(Sender : TObject);
begin
Input;
i.Re := 0; //definiциони
i.Im := 1;
q := 1 - p;
zeta := (p * n1)\(n1 - 1) + (q * n2)\(n2 + 1) - 1;

skladnik1 := FunSklad1;
skladnik2 := FunSklad2;
Wynik := skladnik1 + skladnik2;

Form1.Edit5.Text := floattostr(Wynik);

end;

//input
procedure Input;
var temp : double;
begin

lambda := strtofloat(Form1.Edit7.Text);
S0 := strtofloat(Form1.Edit8.Text);
r := strtofloat(Form1.Edit9.Text);
sigma := strtofloat(Form1.Edit10.Text);
p := strtofloat(Form1.Edit11.Text);
n1 := strtofloat(Form1.Edit12.Text);
n2 := strtofloat(Form1.Edit13.Text);
delta := strtofloat(Form1.Edit14.Text);
A := strtofloat(Form1.Edit15.Text);

```

```

K := strtfloat(Form1.Edit17.Text);
T := strtfloat(Form1.Edit18.Text);
N := strtoint(Form1.Edit21.Text);

//EuropeanCall
if Form1.RadioButton1.Checked = True then
begin
teta := A/(2/(sigma * sqrt(T)));
temp := A/4;
ifteta > temp then
begin
teta := temp; //min( )
end;

kp := A/teta;
temp := A/n1;
if kp < temp then
begin
kp := temp; //max()
end;
X := S0 * Exp(kp); //rescalingparameter
kk := Ln(X/K);
end;

//EuropeanPut
if Form1.RadioButton2.Checked = True then
begin
teta := 2/(sigma * sqrt(T));
temp := 4;
ifteta < temp then
begin
teta := temp; //max()
end;

kp := A/teta;
temp := A/n2;
if kp < temp then
begin
kp := temp; //max()
end;
X := S0/Exp(kp); //rescalingparameter

```

```

kk := Ln(K/X);
end;
end;

//FunkcjaG(alfa)
functionFunG(alfa : fComplex) : fComplex;
varCT1, CT2, CT3, CT4, Cmian1, Cmian2, minal : fComplex;
temp : double;
begin
temp := r - delta - 0.5 * sigma * sigma - lambda * zeta;
cf_mulRe(CT1, alfa, temp); //firstterm
temp := 0.5 * sigma * sigma;
cf_mul(CT2, alfa, alfa); //alfa^2
cf_mulRe(CT2, CT2, temp); //secondterm
cf_mulRe(minal, alfa, -1); // - alfa
cf_addRe(Cmian1, minal, n1); //n1 - x
cf_addRe(Cmian2, alfa, n2); //n2 + x
cf_divrRe(CT3, Cmian1, p * n1); //firstfraction
cf_divrRe(CT4, Cmian2, q * n2); //secondfraction
cf_add(CT3, CT3, CT4); //sumof fractions
cf_addRe(CT3, CT3, -1); //lastterm
cf_mulRe(CT3, CT3, lambda); //lambdatimeslastterm
cf_add(CT1, CT1, CT2); //sumof firsttwo terms
cf_add(CT1, CT1, CT3); //pluslastterm
FunG := CT1;
end;

//FunkcjaFCall(xi)
functionFunFC(xi : fComplex) : fComplex;
varCG, XXi, CT3, CT2, mian, CT : fComplex;
CT1 : double;
begin
cf_addRe(XXi, xi, 1); //XXi = xi + 1
CG := FunG(XXi); //G[xi + 1]
cf_powReBase(CT2, S0/X, XXi); //(S0/x)^(xi + 1)
cf_mul(mian, xi, XXi); //xi(xi + 1)
cf_div(CT2, CT2, mian); //secondterm
CT1 := Exp(-r * T) * X; //e^(-rT) * X
cf_mulRe(CT3, CG, T); //G[xi + 1] * T
cf_exp(CT3, CT3); //e^G[xi + 1] * T

```

```

cf_mulRe(CT, CT2, CT1);
cf_mul(CT, CT, CT3);
FunFC := CT;
end;

//FunkcjaFPut(xi)
functionFunFP(xi : fComplex) : fComplex;
varCG, XXi, CT3, CT2, mian, CT : fComplex;
CT1 : double;
begin
cf_mulRe(XXi, xi, -1); //XXi = -xi
cf_addRe(XXi, XXi, 1); //XXi = -xi + 1
cf_powReBase(CT2, S0/X, XXi); //(S0/x)(-xi + 1)

CG := FunG(XXi); //G[-xi + 1]
cf_mulRe(CT3, CG, T); //G[-xi + 1] * T
cf_exp(CT3, CT3); //eG[-xi + 1] * T

cf_addRe(XXi, xi, -1); //xi - 1
cf_mul(mian, xi, XXi); //xi(xi - 1)
cf_div(CT2, CT2, mian); //secondterm
CT1 := Exp(-r * T) * X; //e(-rT) * X

cf_mulRe(CT, CT2, CT1);
cf_mul(CT, CT, CT3);

FunFP := CT;
end;

//FunkcjaFsklad1
functionFunSklad1 : double;
varCF, arg : fComplex;
wyn, C1, real : double;
begin
arg.Re := A/(2 * kk);
arg.Im := 0;

//EuropeanCall
ifForm1.RadioButton1.Checked = Truethen
begin

```

```

CF := FunFC(arg);
end;

//EuropeanPut
if Form1.RadioButton2.Checked = True then
begin
CF := FunFP(arg);
end;
C1 := Exp(A/2)/(2 * kk);
real := cf_real(CF);
wyn := C1 * real;
Funsklad1 := wyn;
end;

//FunkcjaFsklad2
function FunSklad2 : double;
var CF, arg : fComplex;
wyn2, real, C1, wyn : double;
j : integer;
begin
C1 := Exp(A/2)/kk;
wyn2 := 0;
for j := 1 to N do
begin
cf_mulRe(arg, i, -2 * j * Pi);
cf_AddRe(arg, arg, A);
cf_divRe(arg, arg, 2 * kk);
end;
end;

//EuropeanCall
if Form1.RadioButton1.Checked = True then
begin
CF := FunFC(arg);
end;

//EuropeanPut
if Form1.RadioButton2.Checked = True then
begin
CF := FunFP(arg);
end;
real := Power(-1, j) * cf_real(CF);
wyn2 := wyn2 + real;

```

```
end;  
wyn := C1 * wyn2;  
FunSklad2 := wyn;  
end;
```

```
procedure TForm1.Button2Click(Sender : TObject);  
begin  
close;  
end;
```

```
procedure TForm1.FormCreate(Sender : TObject);  
begin  
ch := DECIMALSEPARATOR;  
DECIMALSEPARATOR := ' ';  
Form1.GroupBox3.Visible := False;  
para := 0;  
end;
```

```
procedure TForm1.FormClose(Sender : TObject; var Action :  
TCloseAction);  
begin  
DECIMALSEPARATOR := ch;  
end;
```

```
procedure TForm1.Label17Click(Sender : TObject);  
begin  
normal_color;  
Form1.Label2.Font.Color := clRed;  
Form1.Label29.Font.Color := clRed;  
Form1.GroupBox3.Visible := True;  
para := 1;  
Form1.Button3.Caption := ' Hide';  
end;
```

```
procedure TForm1.Label18Click(Sender : TObject);  
begin  
normal_color;  
Form1.Label6.Font.Color := clRed;  
Form1.Label30.Font.Color := clRed;
```

```

Form1.GroupBox3.Visible := True;
para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label7Click(Sender : TObject);
begin
normal_color;
Form1.Label25.Font.Color := clRed;
Form1.Label37.Font.Color := clRed;
Form1.GroupBox3.Visible := True;
para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label15Click(Sender : TObject);
begin
normal_color;
Form1.Label22.Font.Color := clRed;
Form1.Label34.Font.Color := clRed;
Form1.GroupBox3.Visible := True;
para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label21Click(Sender : TObject);
begin
normal_color;
Form1.Label27.Font.Color := clRed;
Form1.Label39.Font.Color := clRed;
Form1.GroupBox3.Visible := True;
para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label9Click(Sender : TObject);
begin
normal_color;
Form1.Label1.Font.Color := clRed;
Form1.Label28.Font.Color := clRed;
Form1.GroupBox3.Visible := True;

```



```

para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label8Click(Sender : TObject);
begin
normal_color;
Form1.Label16.Font.Color := clRed;
Form1.Label31.Font.Color := clRed;
Form1.GroupBox3.Visible := True;
para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label10Click(Sender : TObject);
begin
normal_color;
Form1.Label19.Font.Color := clRed;
Form1.Label32.Font.Color := clRed;
Form1.GroupBox3.Visible := True;
para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label11Click(Sender : TObject);
begin
normal_color;
Form1.Label26.Font.Color := clRed;
Form1.Label38.Font.Color := clRed;
Form1.GroupBox3.Visible := True;
para := 1;
Form1.Button3.Caption := ' Hide';
end;

procedure TForm1.Label12Click(Sender : TObject);
begin
normal_color;
Form1.Label35.Font.Color := clRed;
Form1.Label23.Font.Color := clRed;
Form1.GroupBox3.Visible := True;
para := 1;

```

```
Form1.Button3.Caption := ' Hide';  
end;
```

```
procedure TForm1.Label13Click(Sender : TObject);  
begin  
normal_color;  
Form1.Label24.Font.Color := clRed;  
Form1.Label36.Font.Color := clRed;  
Form1.GroupBox3.Visible := True;  
para := 1;  
Form1.Button3.Caption := ' Hide';  
end;
```

```
procedure TForm1.Label14Click(Sender : TObject);  
begin  
normal_color;  
Form1.Label20.Font.Color := clRed;  
Form1.Label33.Font.Color := clRed;  
Form1.GroupBox3.Visible := True;  
para := 1;  
Form1.Button3.Caption := ' Hide';  
end;
```

```
procedure TForm1.Label5Click(Sender : TObject);  
begin  
Form1.GroupBox3.Visible := True;  
para := 1;  
Form1.Button3.Caption := ' Hide';  
end;
```

```
procedure TForm1.Button3Click(Sender : TObject);  
begin  
normal_color;  
if para = 0 then  
begin  
para := 1;  
Form1.GroupBox3.Visible := True;  
Form1.Button3.Caption := ' Hide';  
end  
elseif para = 1 then  
begin
```

```

para := 0;
Form1.GroupBox3.Visible := False;
Form1.Button3.Caption := ' Show';
end;
end;

procedure normal_color;
begin
Form1.Label28.Font.Color := clWindowText;
Form1.Label29.Font.Color := clWindowText;
Form1.Label30.Font.Color := clWindowText;
Form1.Label31.Font.Color := clWindowText;
Form1.Label32.Font.Color := clWindowText;
Form1.Label33.Font.Color := clWindowText;
Form1.Label34.Font.Color := clWindowText;
Form1.Label35.Font.Color := clWindowText;
Form1.Label36.Font.Color := clWindowText;
Form1.Label37.Font.Color := clWindowText;
Form1.Label38.Font.Color := clWindowText;
Form1.Label39.Font.Color := clWindowText;
Form1.Label1.Font.Color := clWindowText;
Form1.Label2.Font.Color := clWindowText;
Form1.Label6.Font.Color := clWindowText;
Form1.Label16.Font.Color := clWindowText;
Form1.Label19.Font.Color := clWindowText;
Form1.Label20.Font.Color := clWindowText;
Form1.Label22.Font.Color := clWindowText;
Form1.Label23.Font.Color := clWindowText;
Form1.Label24.Font.Color := clWindowText;
Form1.Label25.Font.Color := clWindowText;
Form1.Label26.Font.Color := clWindowText;
Form1.Label27.Font.Color := clWindowText;
end;

```

**APPENDIX B:**  
**Calculations related to Figures 5.5- 5.10**

<b>Strike</b>	<b>Real</b>	<b>Kou</b>	<b>BS</b>	<b>Error</b>
Call option				
23,95	10,00	9,8311303921	9,8310093868	0,0171770286
24,99	9,125	8,8882694813	8,8881063038	0,0266340393
26,04	8,25	7,9735329198	7,9733225959	0,0346730970
28,64	6,125	5,9114868032	5,9111553433	0,0361183580
31,24	4,25	4,1973380688	4,1969104542	0,0125465069
33,85	2,675	2,8548829977	2,8544109928	0,0630088861
36,45	1,525	1,8717143985	1,8712540130	0,1852389439
39,05	0,825	1,1859071579	1,1855003120	0,3043300274
41,66	0,45	0,7278072850	0,7274754906	0,3817044577
44,26	0,25	0,4360479393	0,4357941139	0,4266685438
<b>Strike</b>	<b>Real</b>	<b>Kou</b>	<b>BS</b>	<b>Error</b>
Put option				
23,95	0,25	0,1701997896	0,1700787441	0,4688619805
24,99	0,325	0,2668719872	0,2667088659	0,2178123430
26,04	0,425	0,4016436170	0,4014333268	0,0581520085
28,64	0,875	0,9384170092	0,9380854761	0,0675787082
31,24	1,575	1,8230881354	1,8226604869	0,1360812627
33,85	2,45	3,0894388044	3,0889667442	0,2069757147
36,45	4,225	4,7050726668	4,7046121398	0,1020329973
39,05	6,125	6,6180886535	6,6176817858	0,0745062025
41,66	8,375	8,7688059174	8,7684739753	0,0449098681
44,26	11,00	11,0758448154	11,0755906364	0,0068477680

Table 6.1: Summary of the real prices, prices obtained by the Kou model and the Black-Scholes model with a relative error, for the call and put options with 23 days to expiry.

<b>Strike</b>	<b>Real</b>	<b>Kou</b>	<b>BS</b>	<b>Error</b>
Call option				
18,75	15,375	15,1292116814	15,1290318307	0,0162459435
19,79	14,5	14,1959504317	14,1957198073	0,0214180494
20,83	13,625	13,2903192256	13,2900377012	0,0251822976
21,87	12,625	12,4153273499	12,4149905111	0,0168882096
22,91	12,00	11,5735636513	11,5731705180	0,0368457254
23,95	10,875	10,7670635129	10,7666140845	0,0100246912
24,99	10,125	9,9974748711	9,9969708515	0,0127557339
26,04	9,375	9,2590238873	9,2584651060	0,0125257386
28,64	7,5	7,6004547747	7,5997817074	0,0132169426
31,24	5,875	6,1813387877	6,1805831796	0,0495586471
33,85	4,5	4,9839561943	4,9831534644	0,0971028186
36,45	3,325	3,9959534732	3,9951375982	0,1679082296
39,05	2,5	3,1871236239	3,1863229956	0,2155936528
42,00	1,775	2,4534888276	2,4527311461	0,2765404187
44,00	1,4	2,0498688000	2,0491512117	0,3170294606
46,00	1,025	1,7100828177	1,7094111865	0,4006138245
48,00	0,85	1,4249328502	1,4243102751	0,4034806623
<b>Strike</b>	<b>Real</b>	<b>Kou</b>	<b>BS</b>	<b>Error</b>
Put option				
18,75	0,525	0,2538412461	0,2536595745	1,0682218039
19,79	0,55	0,3591442228	0,3589141926	0,5314182020
20,83	0,725	0,4921248760	0,4918424552	0,4732033175
21,87	0,85	0,6557456667	0,6554082053	0,2962342615
22,91	1,00	0,8525596490	0,8521657042	0,1729384580
23,95	1,25	1,0846605205	1,0842101205	0,1524343114
24,99	1,45	1,3536478518	1,3531422840	0,0711796262
26,04	1,7	1,6637855900	1,6632268877	0,0217662721
28,64	2,45	2,6016877252	2,6010147219	0,0583035865
31,24	3,425	3,7790465702	3,7782907429	0,0936867444
33,85	4,625	5,1881256512	5,1873228767	0,1085412515
36,45	6,125	6,7966116059	6,7957958041	0,0988156518
39,05	7,875	8,5842547268	8,5834538003	0,0826227494
42,00	10,125	10,7966401026	10,7958823670	0,0622082515
44,00	11,625	12,3903064629	12,3895888949	0,0617665483
46,00	13,375	14,0478164676	14,0471450579	0,0478947365
48,00	15,125	15,7599440569	15,7593216834	0,0402884715

Table 6.2: Summary of the real prices, prices obtained by the Kou model and the Black-Scholes model with a relative error, for the call and put options with 68 days to expiry.

<b>Strike</b>	<b>Real</b>	<b>Kou</b>	<b>BS</b>	<b>Error</b>
Call option				
25,00	11,00	10,8604329791	10,8597280658	0,0128509629
26,00	10,125	10,2273028371	10,2265508456	0,0100029153
27,00	9,375	9,6247837107	9,6239885250	0,0259521375
28,00	8,875	9,0523878109	9,0515511393	0,0195956928
29,00	8,125	8,5094505623	8,5085758315	0,0451792462
30,00	7,625	7,9951812054	7,9942673541	0,0463005398
32,00	6,5	7,0490746153	7,0481117830	0,0778931484
34,00	5,5	6,2062653701	6,2052667500	0,1137987708
36,00	4,625	5,4583204466	5,4573025924	0,1526697552
38,00	3,95	4,7966521597	4,7956283447	0,1765089757
40,00	3,225	4,2127977153	4,2117809280	0,2344754679
42,00	2,75	3,6986703176	3,6976722480	0,2564895587
44,00	2,15	3,2466848493	3,2457105637	0,3377860495
46,00	1,825	2,8498453990	3,2457105637	0,3377860495
48,00	1,525	2,5017696583	2,5008653722	0,3904314912
<b>Strike</b>	<b>Real</b>	<b>Kou</b>	<b>BS</b>	<b>Error</b>
Put option				
25,00	2,175	2,2074106470	2,2067080890	0,0146826541
26,00	2,475	2,5721492727	2,5713973955	0,0377696869
27,00	2,8	2,9675251824	2,9667276169	0,0564528259
28,00	3,1	3,3930081695	3,3921689807	0,0863564586
29,00	3,475	3,8479414464	3,8470649519	0,0969197301
30,00	3,975	4,3315597706	4,3306500298	0,0823167149
32,00	4,875	5,3812060132	5,3802434121	0,0940692499
34,00	5,875	6,5341634213	6,5331652427	0,1008795432
36,00	7,00	7,7819820544	7,7809643860	0,1004862320
38,00	8,375	9,1160519292	9,1150281088	0,0857677205
40,00	9,625	10,5279590493	10,5269418285	0,0857677205
42,00	11,125	12,0095972785	12,0085973094	0,0736575306
44,00	12,625	13,5533566054	13,5523823585	0,0684964347
46,00	14,375	15,1522793846	15,1513375911	0,0512978520
48,00	15,875	16,7999844934	16,7990808056	0,0550586516

Table 6.3: Summary of the real prices, prices obtained by the Kou model and the Black-Scholes model with a relative error, for the call and put options with 107 days to expiry.

# Glossary

**Ask price** - the price a seller is willing to accept for a security, also known as the offer price.

**Barrier option** - a type of option whose payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price.

**Bid price** - the price at which a market maker is willing to buy a security.

**Call option** - gives the buyer of the option the right but not the obligation to buy the underlying at the strike price.

**Maturity** - the end of the life of a security.

**Put option** - gives the buyer of the option the right but not the obligation to sell the underlying at the strike price.

**Path-dependent option** - an exotic option that is valued according to pre-determined price requirements for its underlying asset or commodity.

**Real price** - an arithmetic average of bid price and ask price.

**Strike price** - the price at which a specific derivative contract can be exercised.