33. Define Operations Research, give features of OR. Briefly discuss techniques or tools of OR. [G.N.D.U. B.Com. III 2001]

34. Give the essential characteristics of the following types of process :
(a) Allocation (b) Competitive Games
(c) Inventory (d) Waiting line. [G.N.D.U. B.Com. III 1995, 99]

35. Discuss briefly the limitations of Operation Research.

36. Define OR. Give the main characteristics of Operations Research.

37. Discuss the importance of Operations Research in decision-making process. [Calicut B.Tech. 86, G.N.D.U. B.Com.III 1995, 2006]

38. What is the role of Operations Research in decision making ?

39. Operations Research increases creative and judicious capabilities of a decision maker. Comment.

40. Discuss the meaning, significance and scope of OR. Describe some method of OR. [Panjab Univ. 2002]

41. Discuss the role and scope of OR in scientific decisions making in a business environment.

42. Briefly describe the importance of OR in the following functional areas of management
(a) Finance (b) Marketing (c) Personnel (d) Production (e) R & D.

43. "Operations Research is the applications of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solution to the problems". Discuss. [G. N.D.U. B.Com III 2005(P)]

44. "O.R. advocates a system approach and is concerned with optimisation. It provides quantitative analysis for decision making. It provides quantitative analysis for decision making". Comment on the above statement. [G. N.D.U. B.Com.III 2005]

2

LINEAR PROGRAMMING–I

(FORMULATION)

- Introduction
- History of Linear Programming
- Meaning and Definition of Linear Programming
- General Statement of LPP
- Assumptions of Linear Programming
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  - Maximization Case With All ≤ Constraints
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INTRODUCTION

Programming problems deal with the determination of optimal use of scarce resources or capacities to meet the given objectives. These resources may be in the form of materials, machines, labour, storage space and investment etc. Therefore, the human elements or resources are always acted towards the achievement of those excess or surplus ends with the available scared resources. It means that human resources, generally, aim at the achievement of maximisation of gains (or minimisation of losses). But, both these objectives are subject to certain limitations and these objectives are to be achieved within the limitations of the constraints. This requires optimum utilisation and allocation of limited resources. Linear Programming is one of the commonly used techniques for this purpose. Linear Programming has been applied to a number of industrial, economic, and social problems.

HISTORY OF LINEAR PROGRAMMING

Linear Programming was first formulated by two experts, L. Kantorovich of Russia and F.L. Hitchcock separately and independently without knowing the work done by each other. In 1939 Kantorovich presented a paper on management and planning technique at a Leningrad seminar where he pointed out that there are many classes of production problems which can be defined mathematically and solved numerically. This paper provided formulation of an activity analysis model and outlined a computational technique for solving linear programming problems. But, American expert G.B. Dantzig was the person to whom the credit for using the term Linear Programming goes. He not only gave the general model of Linear Programming but also gave the Simplex Method of solving the LPP. In 1947, Dantzig and his associates while working in U. S. department of Air force, observed that many military programming problems could be formulated and solved as optimising a linear form of profit/cost function.

MEANING AND DEFINITION OF LINEAR PROGRAMMING

Linear programming is a quantitative or mathematical technique for determining the optimal allocation of resources and obtaining a particular objective when there are alternative uses of resources. The objective in resource allocation may be **minimization** (cost) or **maximation** (profit, production). The technique of linear programming is applicable to problem in which the total effectiveness can be shown as a linear function of individual allocations, and the limitations on resources give rise to **Linear** equalities or inequalities of single allocations. The adjective linear, is to be particularly noted. It states that all the constraints and the objective must afford expression as linear function.

The word Linear programming consist of two words :

1. **Linear :** This word is used to show the relationship between decision variables which are directly proportional. For example if we increase the production of product, it will proportionate increase the profit, this is called a linear relationship.

2. **Programming :** It implies the planning of activities in such a way that the activities with available resources yield optimal results *i.e.* If criterion of effectiveness is profit, then it will maximize and if the criterion of effectiveness is cost, then it will minimize.

Thus, 'Linear programming' states the planning of decision variables which are directly proportional to achieve desired optimal results or **Linear Programming involves the planning of activities to obtain an optimum result.** The following definitions will make the meaning of Linear Programming more clear :

"A method of planning and operation involved in the construction of a model of a real situation containing the following elements : (a) variables representing the available choices, and (b) mathematical expressions (i) relating the variables to the controlling conditions, and (ii) reflecting the criteria to be used in measuring the benefits derivable from each of the several possible plans, and (iii) establishing the objective. The method may be so devised as to ensure the selection of the best of a large number of alternatives."

Kohlar

"Linear Programming is the analysis of problems in which a linear function of a number of variables is to be maximised (or minimized) when those variables are subject to a number of restraints in the form of linear inequalities".

Samuelson and Solow

"Linear Programming is only one aspect of what has been called a system approach to management wherein all programmes are designed and evaluated in terms of their ultimate effects in the realisation of business objectives."

Loomba

From the above definitions, it is clear that a Linear Programming Model has the following basic requirements :

1. **Objective Function :** In order to optimise (maximize or minimise) results, an LPP must shows a clearly defined objective or goal.

2. **Alternative Course of Action :** One optimal course of action is to be selected out of available courses of action.

3. **Constraints :** There must be some restrictions or constrains or limitations on the resources for which the various activities compete.

4. **Inter-related Variables :** The decision variable are assumed to be inter-related in terms of ultilisation of resources.

GENERAL STATEMENT OF LINEAR PROGRAMMING PROBLEM

In general terms, *a* L.P.P. can be written as

Maximise or Minimise $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ objective function

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n\ (\le, =, \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n\ (\le, =, \ge)\ b_2 \\ \vdots \\ \vdots \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n\ (\le, =, \ge)\ b_m \end{array}\right\}\ \text{constraint set}$$

$$x_1, x_2 .... x_n \ge 0 \quad \text{Non-negativity restriction}$$

where x_j, s(j = 1, 2, ... n) are decision variables. C_j 's are termed as the profit coefficients. a_{ij} 's (i = 1, 2, ..., m) (j = 1, 2, ..., n) the technological coefficients and b_i 's are the resource values. Expressions (≤, =, ≥) means that each constraint may take only one of the three possible forms.

ASSUMPTIONS OF LINEAR PROGRAMMING

A linear programming model is based on the following assumptions :

1. **Linearity or Proportionality :** A basic assumption of linear programming is that there exis[ts] proportionality in the objective function and the constraints. It implies that if a product gives a profit [of] Rs. 25, the profit earned from the sale of 10 such products will be Rs. 250. Similarly, if one unit take[s] man hours of a certain type, 7 units would require 35 man hours. But, this may not always be true beca[use] of economies of scale.

1 Product 1P = Rs 25
10 25×10 = Rs 250

2. **Continuity :** One more assumption underlying a L.P. model is that the decision variables [are] continuous. It implies, combinations output with fractional values, in the context of production probl[ems] are possible. For instance, the optimum solution to a problem may be to produce 8.75 units of produ[ct] and 9.10 units of product P_2 per day. However, there are problems in which variables are restricted to integral values only. Such problems are frequently solved by L.P. techniques and then the value[s] rounded off to the nearest integers to satisfy the constraints. But such problems may be restated as in[teger] programming problems, forcing the solutions to be in integers only.

3. **Finite Choices :** Another assumption of a L.P. model is that a limited number of choic[es] available to a decision maker and the decision variables do not have negative values. This assumpti[on is a] realistic one, as it is not possible to produce or use negative quantities.

4. **Additivity :** It means that if m_1 minutes are required to produce a product P_1 on machine A [and m_2] minutes to make product P_2, the total time required to make product P_1 and P_2 on machine A is m[$_1 + m_2$] minutes. This however, may not happen because of the change over time from product P_1 to pro[duct P_2.] Similarly the total profit is determined by the sum of profit is calculated by each of the products sep[arately.]

co efficients of obj function
resource values
and the constraints

5. Certainty : A L.P. model also assumes that the various parameters, such as coefficients of the constraints, objective function coefficients and resource values are known certainly and they do not change with the passage of time. So, availability of materials, labour etc. the cost or profit per unit of the product, market demand of the product are assumed to be known certainly.

APPLICATIONS OF LINEAR PROGRAMMING

Linear Programming has been applied with considerable success to the following important situations.

1. Product Mix : Linear programming helps in determining the quantity of different products to be manufactured knowing the marginal contribution of each product and amount of available resources used by each product. The objective is to maximise the total profit subject to constraints formed by available resources.

2. Transportation Problems : Transportation problems are faced by business organisations which have number of availability centres (Plants, Warehouses etc.) with given capacities which feed number of requirement centres (warehouses, markets etc.) with given requirements. Given the cost of transportation per unit from an origin to a destination the problem then becomes to determine the distribution system that will minimise total transportation cost.

3. Blending Problems : This type of problem is faced by the chemical, food and petroleum industry etc. while deciding production of a product which can be made from a variety of available raw materials of different composition and prices. The solution provides the number of units or quantity of each raw material which are to be blended for making one unit of product.

4. Media Selection : Linear programming techniques are used in selecting an effective advertising mix that will maximise the number of effective exposures subject to the following constraints : the total advertising budget, specified minimum and maximum usage rates of various media (television, radio, magazines, newspaper etc.) and specified exposure rates to different market segments.

5. Diet Problems : It involves determination of combination of different nutrients such as proteins, vitamins, starch, carbohydrates etc. for different foods to satisfy the minimum daily nutritional requirements at the minimum cost and minimises the cost of raising live-stock.

6. Personnel Assignment : The problem of assigning the given number of personnel to different tasks can be tackled with the help of assignment technique. The objective may be to minimise the total time taken to complete all the tasks.

7. Travelling Salesman Problems : This type of problems involve to find the shortest route for a salesman starting from a given city, visiting each of the specified cities and then returning to the origin. Such problems can be easily handled by the assignment technique (with few modifications) of linear programming.

8. Capital Investment : This type of problem arises because of the different ways in which a fixed amount of capital can be allocated to a number of activities. The total return depends upon the manner in which the allocation is made. Thus, the objective may be to find that allocation which maximises the total return.

9. Sub-Contracting : Linear programming has also been applied when problem is faced by a producer in face of capacity limitations and sudden spurt in demand for his products. The producer not being sure of the demand pattern is usually reluctant to add additional capacity and has to take a decision

regarding the product to be produced with his own resources and products to be sub-contracted such that the total cost is minimized. i.e. reluctant to expand production through sub contracting

The above list of applications of linear programming is not complete as it is being used in many other areas such as agriculture, personnel, banking, hospitals and defence etc.

FORMULATION OF LP PROBLEMS

The procedure for mathematical formulation of a LPP consist of the following steps :-

Step 1 : Write down the **decision variables** of the problem. The decision variables refers to the economic (or physical) quantities which shows linear relationship and are competing with one another for sharing the given limited resources. The numerical values of decision variables shows the solution of LPP.

Step 2 : Formulate the **objective function** or objective of the decision maker *i.e.* whether he wants to maximize profits or minimize cost. The objective function of a LPP is a linear function decision. Variable indicates the objective of decision maker.

Step 3 : Formulate other conditions of the problem such as :

(1) Resource Limitation.

(2) Express constraints, and ascertain the **constraints** showing the maximum availability or minimum commitment or equality and represent them as

(1) Maximum availability as '≤' type inequality

(2) Minimum commitment as '≥' type inequality

(3) Equality (equal to) as '=' equality form.

Step 4 : Put the **non-negative constraints** from the consideration so that the negative values of the decision variable do not have any valid physical interpretation. In other words, Non-negativity restriction shows that all decision variables must take on value equal to or greater than zero.

Step 5 : Now **formulate** the LP Problem as under :

Maximize (or Minimize) $Z = x_1 + x_2 + - - - - x_n$

$x_1 + x_2 - - x_n \leq$ – (Maximum availability)

$x_1 + x_2 - - x_n \geq$ – (Minimum commitment)

$x_1 + x_2 - - x_n =$ – (equality)

$x_1, x_2, - - x_n \geq 0$ – (Non-negativity constraints)

Blending
Media Selec
Cap Invest
Sub-Contra

We divide the formulation problems in three categories *i.e.*

1. Case with " ≤ " type of in-equality
2. Case with " ≥ " type of in-equality
3. Case with " ≥ ", " ≤ " type of in-equality and "=" type of equality.

Now, we will discuss all three cases step by step.

Case I.

CASE WITH ≤ TYPE OF IN-EQUALITY

Example 1. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space at the most for 20 items. A fan cost him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 25 and a sewing machine at a profit of Rs. 16. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profits ? Formulate the problem.

Formulation :

Decision Variables : Let x_1 be the number of fans and x_2 be the number of sewing machines bought.

Objective function : Since the profit on both items are given, we have to maximize the profits viz.

$$\text{Max } Z = 25x_1 + 16x_2$$

Constraints : There are two constraints one for total investment limit and other for space.

$x_1 + x_2 \le 20$ [Since he has space at the most for 20 items]

$360x_1 + 240x_2 \le 5760$ [Since the total investment cannot exceed the amount in hand]

$x_1 \ge 0,\ x_2 \ge 0$ [Since the dealer cannot buy negative quantities]

Finally, we have

$$\text{Max } Z = 25x_1 + 16x_2$$

Subject to constraints

$$x_1 + x_2 \le 20$$

$$360x_1 + 240x_2 \le 5760 \quad \text{or} \quad 3x_1 + 2x_2 \le 48$$

Where $x_1 \ge 0,\ x_2 \ge 0$

Example 2. A firm has 240, 370 and 180 kg. of wood, plastic and steel respectively. The firm produces two products A and B. Each unit of A requires 1, 3 and 2 kg. of wood, plastic and steel respectively. The corresponding requirement for each unit of B is 3, 4 and 1 kg. respectively. If A sells for Rs. 4 and B sells for Rs. 6 per unit, then what product mix should the firm produce in order to have maximum gross income ? Formulate this as a LPP.

(G.N.D.U. B.Com. (P) III, April 2004)

Formulation :

Let X_1 and X_2 be the number of products A and B produced respectively.

Objective function : Since the objective is to maximize the profits, the objective function can be stated as :

Maximize $Z = 4X_1 + 6X_2$

Constraints : Availability of wood, plastic and steel is limited.

So the respective constraints are

$$X_1 + 3X_2 \le 240$$

$$3X_1 + 4X_2 \le 370$$

$$2X_1 + X_2 \le 180$$

$$X_1X_1 \ge 0$$

Objective function always have the co-efficients of Profit only:

Example 3. A company manufactures two products A and B. These products are processed in the same machine. It takes 10 minutes to process one unit of product A and 2 minutes for each unit of product B and the machine operates for a maximum of 35 hours in a week. Product A requires 1 kg. and B 0.5 kg. of raw material per unit, the supply of which is 600 kg. per week. Market constraint on product B is not exceeding 800 unit every week. Product A costs Rs. 5 per unit and sold at Rs. 10. Product B costs Rs. 6 per unit and can be sold in the market at a unit price of Rs. 8. Formulate the LP problem.

(MU, BE, Oct. 1996)

∴ P = SP − CP = 10 − 5
to make obj function to max. Profit

Formulation :

Decision Variables : Let X_1 = number of units of product A

X_2 = number of units of product B

Max Z = 5X₁ + 2X₂

Objective function : Cost of product A per unit = Rs. 5

Selling price of it is Rs. 10 per unit

Profit (SP – CP) on product A per unit is Rs. 5 (10 – 5).

Cost of product B per unit = Rs. 6

Selling price of it is Rs. 8 per unit

Profit (SP – CP) on product B per unit is Rs. 2.

10x + 2y ≤ 35×60
x + 0.5y ≤ 600
y ≤ 800

Therefore objective function is

Maximise $\quad Z\text{ (Profit)} = 5X_1 + 2X_2$

Constraints : Time requirement constraint is given by Max Z = 5x₁ + 2x₂

$$10X_1 + 2X_2 \le 2100\ (35 \times 60) \text{ min}$$

Hrs

Raw material constraint is

$$X_1 + 0.5X_2 \le 600$$

Market demand constraint is

$$X_2 \le 800$$

Finally, we have

$$\text{Max } Z = 5X_1 + 2X_2$$

Subject to restriction

$$10X_1 + 2X_2 \le 2100$$

$$X_1 + 0.5X_2 \le 600$$

$$X_2 \le 800$$

$$X_1,\ X_2 \ge 0$$

1 Kg = 1000 g
2.6×100 = 2600 g
1 Kg = 1000 g

Example 4. A garment manufacturer has a production line making two styles of shirts. Style I requires 20 grams of cotton thread, 30 grams of dacron thread and 30 grams of linen thread. Style II requires 20 grams of cotton-thread, 20 grams of dacron thread and 10 grams of linen thread. The manufacturer makes a net profit of Rs. 200 on Style I and Rs. 150 on Style II. He has in hand an inventory of 2.4 kg of cotton thread, 2.6kg of dacron thread and 2.2 kg of linen thread. His immediate problem is to determine a production schedule, given the current inventory to make maximum profits. Formulate LPP.

Max Z = 200x₁ + 150x₂
20x₁ + 20x₂ ≤ 2.4 × 1000
30x₁ + 20x₂ ≤ 2.6 × 1000
30x₁ + 10x₂ ≤ 2.2 × 1000

Formulation :

Decision Variable : Let x_1 = Number of Style I Shirts and x_2 = Number of Style II Shirts.

Objective Function : Since the objective is to maximize the profits, the objective function is given as :

Maximize $Z = 200x_1 + 150x_2$

Constraints :

$20x_1 + 20x_2 \le 2400$ (Maximum quantity of Cotton Thread)

$30x_1 + 20x_2 \le 2600$ (Maximum quantity of Dacron Thread)

$30x_1 + 10x_2 \le 2200$ (Maximum quantity of Linen Thread)

$x_1 \ge 0,\ x_2 \ge 0$ (Non–Negativity Constraint)

Example 5. A manufacturer produces two types of models M_1 and M_2. Each model of the type M_1 requires 24 hours of grinding and 12 hours of polishing ; where as each model of the type M_2 requires 12 hours of grinding and 15 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on M_1 model is Rs. 300 and on model M_2 is Rs. 400. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week. Formulate the problem.

Formulation :

Decision Variables : Let us X_1 = number of units of M_1

X_2 = number of unit of M_2

Objective function : The profit on both the models are given

Max $Z = 300\,X_1 + 400\,X_2$

Constraints : There are two constraints one for grinding and the other for polishing. Number of hours available on each grinder for one week is 40 hours. There are 2 grinders. Hence the manufacturer does not have more than $2 \times 40 = 80$ hours of grinding. M_1 requires 24 hours of grinding and M_2 requires 12 hours of grinding. Hence the **grinding constraints** is as $24\,X_1 + 12\,X_2 \le 80$

Since there are 3 polishers, the available time for polishing in a week is given by $3 \times 60 = 180$ hours. M_1 requires 12 hours of polishing and M_2 requires 15 hours of polishing. Hence we have $12\,X_1 + 15\,X_2 \le 180$, type of **polishing constraint.**

Finally, we have

Max $Z = 300\,X_1 + 400\,X_2$

Subject to constraints

$24\,X_1 + 12\,X_2 \le 80$

$12\,X_1 + 15\,X_2 \le 180$

$X_1, X_2 \ge 0$

Net SP = Sales Proceeds – cost of maint

Example 6. Old hens can be bought for Rs. 2 each but young ones cost Rs. 5 each. The old hens lay 3 eggs per week and young ones 5 eggs per week, each egg being worth 30 paise. A hen costs Re. 1 per week to feed. If a person has only Rs. 80 to spend on the hens, how many of each kind should he buy to get a maximum profit per week, assuming that it is not possible to house more than 20 hens at a time.

$2x_1 + 5x_2 \le 80$

Max $Z = 3 \times .30x_1 + 5 \times .30x_2 - x_1 x_2$

$x_1 + x_2 \le 20$

Formulation :

Let X_1 and X_2 be the number of old and young hens respectively.

Maximize $Z = 0{\cdot}3\,(3X_1 + 5X_2) - (X_1 + X_2) = 0{\cdot}5X_2 - 0{\cdot}1X_1$

Subject to constraints

$2X_1 + 5X_2 \le 80$

$X_1 + X_2 \le 20$

where $X_1, X_2 \ge 0$

Example 7. A company produces two types of hats. Each hat of the first type requires twice as much labour time as the second type, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second types to 150 and 250 hats. Assuming that the profits per hat are Rs. 8 for type I and Rs. 5 for type II. Formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

(BBA III GNDU April 2002)

$x_1 = 2x_2$

$x_1 + x_2 \le 500$

Max $Z = 8x_1 + 5x_2$

$x_1 \le 150$

$x_2 \le 250$

Formulation :

Let P is number of hats type I to be produced and Q is number of hats of type II to be produced.

Profit per hat is Rs. 8 in hat type P and Rs. 5 in hat type Q. So we get the following

Maximize $Z = 8P + 5Q$

Subject to constraints :

$2P + Q \le 500$

$P \le 150$

$Q \le 250$

Whereas $P \ge 0,\ Q \ge 0$

Example 8. A company produces two products X and Y, each of which requires three types of processing. The length of time in hours for Processing each unit and the profit per unit are given in the following table.

	Product X	Product Y	Availability of time (Hours)
Process I	12	12	840
Process II	3	6	300
Process III	8	4	480
Profit/Unit	5	7	

Formulate the LPP.

Formulation :

Let X = Number of units of type x

Y = Number of units of type y.

Since the objective is to maximize profit. Hence objective function is :

Maximize $Z = 5x + 7y$

$3 - (3 \times 2 + 3 \times 1)$

Subject to constraints

$$12x + 12y \le 840$$
$$3x + 6y \le 300$$
$$8x + 4y \le 480$$
$$x \ge 0,\ y \ge 0.$$

Example 9. Wordworth Ltd. has 3 departments (assembly, finishing and packing) with capability to make three products Table T at Rs. 2/- per unit profit, chair C at Rs. 4/unit profit and book case, B at Rs. 3/unit profit. One table requires 3 hours of assembly, 2 hours of finishing and 1 hour of packing time – one chair requires 4 hours, 1 hour and 3 hours of assembly, finishing and packing time respectively. One book case requires 2 hours each of assembly, finishing and packing time. Total time available for assembly, finishing and packing are 60 hours, 40 hours and 80 hours respectively. Formulate the LPP.

Formulation :

Maximize $Z = 2X_1 + 4X_2 + 3X_3$

Subject to constraints

$$3X_1 + 4X_2 + 2X_3 \le 60$$
$$2X_1 + 1X_2 + 2X_3 \le 40$$
$$1X_1 + 3X_2 + 2X_3 \le 80$$
$$X_1, X_2, X_3 \ge 0.$$

Example 10. A firm can produce three types of cloth, say : A, B, and C. Three kinds of wool are required for it, say : red, green and blue wool. One unit length of type A cloth needs 2 meters of red wool and 3 meters of blue wool; one unit length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool; and one unit of type C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 8 meters of red wool, 10 meters of green wool and 15 meters of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3.00, of type B cloth is Rs. 5.00, and of type C cloth is Rs. 4.00.

Formulate the LP problem.

$2x_1 + 3x_2 + 0x_3 \le 8$

$3x_1 + 2x_2 + 3x_3 \le 10$

$0x_1 + 4x_1 + 5x_3 \le 15$

Formation :

Let x_1 = quantity (in meters) produced of cloth type A

x_2 = quantity (in meters) produced of cloth type B

x_3 = quantity (in meters) produced of cloth type C

The total income from the finished cloth is given by Maximize $Z = 3x_1 + 5x_2 + 4x_3$. Since 2 meters of red wool are required for each meter of cloth A and x_1 meters of this type of cloth are produced, so $2x_1$ meters of red wool will be required for cloth A. Similarly, cloth B requires $3x_2$ meters of red wool and cloth C does not require red wool. Thus, total quantity of red wool becomes :

$$2x_1 + 3x_2 + 0x_3 \ (red\ wool)$$

Following similar arguments for green and blue wool.

$$0x_1 + 2x_2 + 5x_3 \ (green\ wool)$$
$$3x_1 + 2x_2 + 4x_3 \ (blue\ wool)$$

Since not more than 8 meters of red, 10 meters of green and 15 meters of blue wool are available, the variable x_1, x_2, x_3 must satisfy the following restrictions :

$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$

Finally we have

Max. $Z = 3x_1 + 5x_2 + 4x_3$

Subject to constraints

$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0 \text{ (Non-negativity constraints)}$$

$2x_1 + 3x_2 + 0 \le 8$

$3x_1 + 2x_2 + 4x_3 \le 4$

$0 + 2x_2 + 5x_3 \le 10$

Example 11. Upon completing the construction of his house, Mr. Malhotra discovers that 1000 square feet of plywood scrap and 800 square feet of white pine scrap are in usable form for the construction of chairs and cupboards. It takes 32 square feet of plywood and 32 square feet of white pine to construct a chair. It takes 40 square feet of plywood and 40 square feet of white pine to construct a cupboard. By selling the finished products to a local furniture store, Mr. Malhotra can realise a profit of Rs. 125 on each chair and Rs. 120 on each cupboard. Formulate the LPP.

Formulation :

Let x_1 = Number of chairs to be produced

x_2 = Number of cupboard to be produced

Since the objective is to maximize the profit, the objective function is given by –

Maximize $Z = 125x_1 + 120x_2$

Subject to constraints :

$32x_1 + 40x_2 \le 1000$ (Maximum plywood scrap available)

$32x_1 + 40x_2 \le 800$ (Maximum white pine scrap available)

$x_1, x_2 \ge 0$ (Non-negativity constraint)

Example 12. Two products A and B are to be manufactured. A single unit of product A requires 2.5 minutes of punch press time and 5 minutes of assembly time. The profit for product A is Rs 0.60 per unit. A single unit of product B requires 3 minutes of punch press time and 2.5 minutes of welding time. The profit for product B is Rs. 0.70 per unit. The capacity of the punch press department available for these products is 1,200 minutes/week. The welding department has an idle capacity of 600 minutes/week and assembly department has 1,500 minutes/week. Formulate the problem as linear programming problem.

Max Z = $0.60x_1 + 0.70x_2$

Formulation :

Let x_1 = Number of product A to be manufactured

x_2 = Number of product B to be manufactured

$2.5x_1 + 3x_2 \le 1200$

$5x_1 + 0x_2 \le 1500$

$0x_1 + 2.5x_2 \le 600$

Since the objective is to maximize the profit, the objective function is given by –

Maximize $Z = .60x_1 + .70x_2$

Subject to constraints :

$2.5x_1 + 3x_2 \le 1200$ (Availability of punch press time)

$5x_1 \le 1500$ (Availability of assembly time)

$2.5x_2 \le 600$ (Availability of welding time)

$x_1, x_2 \ge 0$ (Non-negativity constraint)

Example 13. A paper mill produces two grades of paper namely X and Y. Because of raw material restrictions, it cannot produce more than 400 tons of grade X and 300 tons of grade Y in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce a ton of products X and Y respectively with corresponding profits of Rs. 200 and Rs. 500 per ton. Formulate the above as a LPP to maximize profit.

$x_1 + x_2 \le 400$ 600

Formulation :

$0.2x_1 + 0.4x_2 \le 160$

Let x_1 = the number of units of two grades of paper of X

x_2 = the number of units of two grades of paper of Y

$\text{Max } Z = 200x_1 + 500x_2$

The objective function is to maximize the profit.

$$\text{Max } Z = 200\, x_1 + 500\, x_2$$

There are 2 constraints one with respect to raw material and the other with respect to production hours.

$$\text{Max } Z = 200\, x_1 + 500\, x_2$$

Subject to constraints

$x_1 \le 400$

$x_2 \le 300$

$0.2\, x_1 + 0.4\, x_2 \le 160$

$x_1, x_2 \ge 0$ (Non-negative restriction)

Example 14. A Gear Manufacturing Company, received an order for three Special Type of gears for regular supply. The management is considering to devote the available excess capacity to one or more of the three gears, say A, B and C. The available capacity on the machines which might limit output and the number of hours required for each unit of the respective gear is also given below :

Machine Type	Available Machine Hrs/Weak	Productivity is M/C Hrs/unit		
		Gear A	Gear B	Gear C
Gear Hobbing machine	250	8	2	3
Gear Shaping machine	150	4	3	0
Gear Grinding machine	50	2	–	1

The unit profit would be Rs. 20, Rs. 6 and Rs. 8 respectively for the Gears A, B and C respectively. Formulate the LPP.

(I.C.W.A. (Final) December 1983)

Formulation :

The mathematical formulation of given LPP is :

Max. $Z = 20x_1 + 6x_2 + 8x_3$

Subject to constraints

$8x_1 + 2x_2 + 3x_3 \le 250$

$4x_1 + 3x_2 + 0x_3 \le 150$

$2x_1 + 0x_2 + 1x_3 \le 50$

and $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

EXERCISE 2.1

1. A manufacturer makes 2 products P_1 and P_2 using 2 machines M_1 and M_2. Product P_1 requires 2 hrs. on machine M_1 and 6 hours on machine M_2. Product P_2 requires 5 hrs. on machine M_1 and no time on machine M_2. There are 16 hours of time per day available on machine M_1 and 30 hours on M_2. Profit margin from P_1 and P_2 is Rs. 2 and Rs. 10 per unit respectively. What should be the daily production run to optimise profit ? Also determine the maximum daily profit.

2. The ABC manufacturing company can make 2 products P_1 and P_2. Each of the products requires time on a cutting machine and a finishing machine. Relevant data are :

$x_1, x_2 \ge 0$ (non negativity constraint)

	Product	
	P_1	P_2
Cutting hours (per unit)	20	10
Finishing hours (per unit)	30	30
Profit (Rs. per unit)	60	40
Maximum Sales (unit per week)		200

The number of cutting hours available per week is 390 and the number of finishing hours available per week is 810. How much should be produced of each product in order to achieve maximum profit for the company.

3. A company produces two types of presentation goods A and B that require gold and silver. Each unit of type A requires 3 gm of silver 1 gm of gold while that of B requires 1 gm of silver and gm of gold. The company can produce 9 gm of silver and 8 gm of gold. If each unit of type brings a profit of Rs. 40 and that of type B Rs. 50 determine the number of units of each type the company should produce to maximize the profit. What is the maximum profit ?

4. A company produces two products X and Y each of which requires three types of processing. The length of time for processing each unit and the profit per unit are given in the following table.

	Product X (hr/unit)	Product Y (hr/unit)	Available Capacity per day (hr)
Process I	12	12	840
Process II	3	6	300
Process III	8	4	480
Profit per unit (Rs.)	5	7	

How many units of each product should the company produce per day in order to maximize profit ?

5. A manufacturer produces two product A and B and has his machines in operation 24 hours a day. Production A requires 2 hours of processing in machine M_1 and 6 hours in machine M_2. Production of B requires 6 hours of processing in machine M_1 and 2 hours in machine M_2. The manufacturer earns a profit of Rs. 5 on each unit of A and Rs. 2 on each unit of B. How many units of each product should be produced in order to achieve maximum profit ?

6. A firm manufactures 2 products A and B on which profits earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on 2 machines M_1 and M_2. The product A requires one minute of processing time on M_1 and 2 minutes on M_2 while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for use for not more than 7 hours 30 minutes, while M_2 is available for 10 hours during any working day. Find the number of units of products A and B to be manufactured to get the maximum profit.

7. A company produces 2 products X and Y each of which requires processing in 3 machines. The first machine can be used at most 70 hours, the second machine at most 40 hours and the third machine at most 90 hours. The products X requires 2 hours on machine 1, 1 hour on machine 2 and 1 hour on machine 3, the product Y requires 1 hour each on machines 1, 2 and 3 hours on the third machine. The profit is Rs. 50 per unit of X and Rs. 30 per unit of Y. How many units of each product should the company produce to maximize profit ?

8. A farmer has 2000 acres of land on which he can grow corn, wheat and soyabeans. Each acre of corn costs Rs. 2000 for preparation, requires T man-days of work and yields a profit of Rs. 600. An acre of wheat costs Rs. 2400 for per partion, requires 10 man-days of work and yields a profit of Rs. 800. An acre of soyabeans costs Rs. 1400 to prepare requires 8 man days of work and yields a profit of Rs. 400. If the farmer has Rs. 2,00,000 for preparation and can count on 16000 man-days of work how many acres should be the allocated to each crop to maximise profits. Formulate an LP model.

9. A firm makes 2 types of furniture : Chairs and table. The contribution for each product as calculated by the accounting department is Rs. 20 per chair and Rs. 30 per table. Both products are processed on 3 machines M_1, M_2, M_3. The time required by each product and total time available per week on each machine are as follows :

Machine	Chair	Table	Available hours
M_1	3	3	35
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximum contribution ?

Case II.

CASE WITH ≥ TYPE OF IN-EQUALITY

Example 15. A product is manufactured by blending three different raw materials. The finished product should meet certain requirements. Given the following data what is your recommendation with regard to quantity for raw materials to be blended, which will meet the quality requirements with minimum cost.

Quality	Contribution of quality			Minimum quality Requirement
	A	B	C	
1.	3	0	1	10
2.	5	1	2	15
3.	1	2	0	8
Cost of raw material per unit in Rs.	2	5	3	

Formulate the LPP.

Formulation :

Let x_1, x_2 and x_3 be the quantity of raw material A, B and C, respectively.

The above problem can be presented as a LPP as below :

Minimize $\quad Z = 2x_1 + 5x_2 + 3x_3$

Subject to constraints ;

$$3x_1 + 0x_2 + x_3 \geq 10$$
$$5x_1 + x_2 + 2x_2 \geq 15$$
$$1x_1 + 2x_2 + 0x_3 \geq 8$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Example 16. A person requires 10, 12 and 12 units chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per jar. A dry product contains 1, 2 and 4 units of A, B, C per carton. If the liquid product sells whose cost price is Rs. 3 per jar and the dry product cost for Rs. 2 per carton, how many of each should be purchased, in order to minimize the cost and meet the requirements ? Formulate the LPP.

Formulation :

Decision Variables : Let X_1 = number of units of liquid products

X_2 = number of unit of dry products

Objective function : As the cost for the product are given we have to minimize the cost

$$\text{Min } Z = 3X_1 + 2X_2$$

Constraints : As there are 3 chemicals and its requirements are given. We have three constraints for these three chemicals.

$$5X_1 + X_2 \geq 10$$
$$2X_1 + 2X_2 \geq 12$$
$$X_1 + 4X_2 \geq 12$$

$5x_1 + 1x_2 \geq 10$
$2x_1 + 2x_2 \geq 12$
$1x_1 + 4x_2 \geq 12$

Finally, we have

$$\text{Minimise } Z = 3X_1 + 2X_2$$

Subject to constraints

$$5X_1 + X_2 \geq 10$$
$$2X_1 + 2X_2 \geq 12$$
$$X_1 + 4X_2 \geq 12$$
$$X_1, X_2 \geq 0$$

Example 17. A Diet for a sick person must contain at least 4000 units of vitamin, 50 units of minerals and 1400 calories. Two foods A and B are available at a cost of Rs. 4 and Rs. 3 per unit respectively. If one unit of Food A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamin, 2 units of mineral and 40 calories. Formulate the LPP.

Formulation :

Let x_1 = Food A

x_2 = Food B

Since objective function is cost (minimize), It is given as :

Minimize $Z = 4x_1 + 3x_2$

Subject to constraints

least requires

$$200x_1 + 100x_2 \geq 4000$$
$$x_1 + 2x_2 \geq 50$$
$$40x_1 + 40x_2 \geq 1400$$
$$x_1 \geq 0, x_2 \geq 0$$

EXERCISE 2.2

1. The vitamins A and B are found in 2 different foods, F_1 and F_2. The respective prices per unit of each food are Rs. 3 and Rs. 2.5. One unit of F_1 contains 2 units of vitamin A and 3 units of vitamin B. Similarly one unit of F_2 contains 4 units of vitamin A and 2 units of vitamin B. Daily requirements of vitamin A is at least 60 units and vitamin B at least 75 units.

   The problem is to determine optimal quantities of foods F_1 and F_2 to be bought so that the daily vitamins requirements are met and simultaneously the cost of buying the food is minimized.

2. A company that produces soft drinks has a contact that requires that a minimum of 80 units of the chemical A and 60 units of the chemical B go into each bottle of the drink. The chemicals are available in a prepared mix from 2 different suppliers. Supplier S_1 has a mix of 4 units of A and 2 units of B that costs Rs. 10 and supplier S_2 has a mix of 1 unit of A and 1 unit of B that costs Rs. 4. How many mixes from supplier S_1 and supplier S_2 should the company purchase to honour contract requirement and yet minimize cost ?

3. A tonic feed for cattle is to be prepared using 2 types of grain X_1 and X_2. Each unit of grain X_1 weights 4 gm and contain 2 gm of protein. Each unit of grain X_2 weights 5 gm and contain 1 gm of protein. The feed is to weight at least 40 gm and must contain at least 10 gm of protein. If the grain X_1 costs 30 paise a unit and that X_2 costs 10 paise a unit, determine by graphical method the amount of 2 types of grain to be used in the preparation of the feed to minimize the cost yet meeting the requirements. Also determine the minimum cost.

4. A small township of 15000 people requires, on an average 300,000 gallons of water daily. The city is supplied from a central water works where the water is purified by such conventional methods as filtration and chlorination. In addition 2 different chemical compounds (*i*) softening chemical (*ii*) health chemical are needed for softening the water and for health purposes. The water works plan to purchase 2 popular brands that contain these chemicals. One unit of chemical corporations product gives 8 pounds of softening chemical and 3 pounds of health chemical. One unit of Indian Chemical's product contains 4 pounds and 9 pounds per unit, respectively for the same purposes.

   To maintain the water at a minimum level of softness and to meet a minimum in health protections experts have decided that 150 and 100 pounds of the 2 chemicals that make up each product must be added to water daily. At a cost of Rs. 8 and Rs. 10 per unit respectively for Chemico's and Indian Chemical products, what is the optimal quantity of each product that should be used to meet that minimum level of softness and minimum health standard ?

5. Shivani Paper Company produces rolls of paper used in accounting registers. Each role of paper is 500 ft. in length and can be produced in widths of 1″, 2″, 3″ and 5″. The company's production process results in 500 rolls that are 12″ in width. Hence company must cut its 12″ roll to the desired widths. It has 6 cutting alternatives :

Alternatives	Number of Rolls				Waste
	1″	2″	3″	4″	
1	4	2	0	0	0
2	0	2	4	0	0
3	2	1	1	2	2
4	0	0	4	2	2
5	4	2	2	0	2
6	3	1	1	0	2

Max Z = $40x_1 + 30x_2$

$2x_1 + x_2 \leq 10000$

$x_1 + x_2 \leq 8000$

$x_2 \leq 800$

The minimum demand requirement for the 4 rolls are 600, 200, 400 and 300 respectively. The company wishes to minimise the waste in the production process. Formulate the problem.

Max Z = $40x_1 + 30x_2$

$x_1 \leq 400$

$x_2 \leq 700$

$x_1 + x_2 \leq 800$

Case III.

CASE WITH '≥', '≤' AND '=' TYPE CONSTRAINTS

Example 18. A company makes two kinds of leather belts. Belt A is high quality belt, and belt B is of lower quality. The respective profits are Rs. 40 and Rs. 30 per belt. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 10,000 per day. The supply of leather is sufficient for only 8000 belts per day (both A and B combined). Belt A requires a fancy buckle and only 4000 per day are available. There are only 7000 buckles a day available for belt B. What should be the daily production of each type of belt ? Formulate :

$2x_1 + x_2 \leq 10000$

It is a single constraint as line is in continuation

Formulation :

Let x_1 = Number of Belts A to be produced

x_2 = Number of Belts B to be produced

Since the objective is to maximize the profit, the objective function is given by–

Maximize $Z = 40x_1 + 30x_2$

Subject to Constraints :

$2x_1 + x_2 \le 10{,}000$ (Total availability of time)

$x_1 + x_2 \le 8000$ (Total availability of Leather)

$x_1 \le 4000$ (Availability of buckles for belt A)

$x_2 \le 7000$ (Availability of buckle for belt B)

$x_1 \ge 0,\ x_2 \ge 0$ (Non-negativity constraints)

Example 19. An animal feed company must produce 2000kg of mixture consisting of ingredients x_1 and x_2 daily. x_1 cost Rs. 30 per kg and x_2 cost Rs. 80 per kg. Not more than 800 kg of x_1 can be used and at least 600 kg of x_2 must be used. Find how much of each ingredient should be used if the company wants to minimize cost. Formulate the LPP.

equal to

$x_1 + x_2 \ge 2000$

Min $Z = 30x_1 + 80x_2$

$x_1 \le 800$

$x_2 \ge 600$

Formulation :

Let x_1 = kg of ingredient x_1, to be used

x_2 = kg of ingredient x_2, to be used

Since the objective is to minimise the cost, the objective function is given as

Minimise $Z = 30x_1 + 80x_2$

Subject to restrictions

$x_1 + x_2 = 2000$ (Total mixture to be produced)

$x_2 \ge 600$ (Minimum use of x_2)

$x_1 \le 800$ (Maximum use of x_1)

$x_1 \ge 0, x_2 \ge 0$ (Non-negativity constraints)

Example 20. A firm produces an alloy having the following specifications :.

(*i*) Specific gravity $\le 0{\cdot}98$.

(*ii*) Chromium $\ge 8\%$

(*iii*) Melting point $\ge 450°C$

Raw materials A, B and C having the properties shown below can be used to make the alloy.

Property	Properties of Raw Material		
	A	B	C
Specific gravity	0·92	0·97	1·04
Chromium	7%	13%	16%
Melting Point	440°C	490°C	480°C

Cost of various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C. Find the proportion in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum. Formulate this problem as an LPP. (*G.N.D.U., BBA-III April 2006*)

Formulation :

Objective function : Objective is to minimize the cost of raw materials.

Thus,

Minimize $Z = 90X_1 + 280X_2 + 40X_3$

The constraints relevant to this problem can be stated as

$0{\cdot}92X_1 + 0{\cdot}97X_2 + 1{\cdot}04X_3 \le 0{\cdot}98$

$7X_1 + 13X_2 + 16X_3 \ge 8$

$440X_1 + 490X_2 + 480X_3 \ge 450$

$X_1, X_2, X_3 \ge 0$

where

max. land available

$P = SP - CP$

Example 21. A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is Re. 1.00 per kg. for tomatoes, Rs. 0.75 a head for lettuce and Rs. 2.00 per kg for radishes. The average yield per acre is 2,000 kg of tomatoes, 3000 heads of lettuce, and 1000 kgs of radishes. Fertilizer is available at Rs. 0.50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20.00 per man-day.

SP price is obtained by selling the output

Formulate this problem as a linear programming model to maximize the farmer's total profit.

of fertilizer

for SP

$x_1 + x_2 + x_3 \le 100$

Formation :

Let x_1 = acre of land to grow tomatoes

x_2 = acre of land to grow lettuce

x_3 = acre of land to grow radishes

Since the farmer has only 100 acres of land available, the profit on each of tomatoes, lettuce and radishes is given below :

	Tomatoes	Lettuce	Radishes
Selling Price	1 × 2000 = 2000	0·75 × 3000 = 2250	2 × 1000 = 2000
Cost Price – Fertilizer	0·50 × 100 = 50	0·50 × 100 = 50	0·50 × 50 = 25
Labour Cost	20 × 5 = 100	20 × 6 = 120	20 × 5 = 100
Profit	2000 – 150 = 1850	2250 – 170 = 2080	2000 – 125 = 1875

Finally, we have

Maximise $Z = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$

Subject to constraints

$x_1 + x_2 + x_3 \le 100$ ⇒ farm available in all

$5x_1 + 6x_2 + 5x_3 \le 400$

and $x_1,\ x_2,\ x_3 \ge 0$

$x_1 + x_2 + x_3 \le 100$

A is means =

Example 22. A rubber company is engaged in producing three different kinds of tyres X, Y and Z. These three different tyres are produced at the company's two different plants with different production capacities. In a normal 18 hours working day, Plant 1 produces 500, 1000 and 1000 tyres of type X, Y and Z respectively. Plant 2, produces X, 1000 and 1500 and 2000 tyres of type X, Y and Z respectively. The monthly demand for type X, Y and Z is 25,000, 3,0000 and 70,000 units, respectively. The daily cost of operation of Plant 1 and Plant 2 is Rs. 12,500 and Rs. 13,500, respectively. Form LP Model to determine the minimum number of days of operation per month at two different plants to minimise the total cost while meeting the demand. Formulate the LPP.

Formation :

Let x_1 = Number of days of operation in Plant 1

x_2 = Number of days of operation in Plant 2.

Since, the objective is to minimise the cost, the objective function is given as :

Minimise $Z = 12{,}500\,x_1 + 13{,}500\,x_2$

Subject to constraints :

$$500\,x_1 + 1000\,x_2 = 25000$$
$$1000\,x_1 + 1500\,x_2 = 30000$$
$$1000\,x_1 + 2000\,x_2 = 70000$$

where $x_1, x_2 \geq 0$

$\text{Min } Z = 12500x_1 + 13500x_2$

$\cancel{x_1 + x_2 \leq 18}$

$500x_1 + 1000x_2 \geq 25000$

$1000x_1 + 1500x_2 \geq 30000$

$1000x_1 + 2000x_2 \geq 70000$

Example 23. A publisher of textbooks is in the process of presenting a new book to the market. The book may be bound by either cloth or hard paper. Each cloth bound book sold contributes Rs. 124, and each paper-bound book contributes Rs. 123. It takes 100 minutes to bind a cloth cover, and 90 minutes to bind a paperback. The total available time for binding is 1000 hours. After considerable market survey, it is predicted that the cloth-cover sales will exceed at least 1000 copies, but the paperback sales will be not more than 6,000 copies. Formulate the problem as a LP problem.

1hr = 60 min
1000 hrs = 60,000 min.

Formulation :

Let x_1 = Number of books bound by cloth

x_2 = Number of books bound by hard paper

Since the objective is to maximize the profit, the objective function is given by –

Maximize $Z = 124x_1 + 123x_2$

Subject to Constraints :

$100x_1 + 90x_2 \leq 60000$ (1000×60) min (Total time available in minutes)

$x_1 \geq 1000$ (Minimum sales of cloth-cover books)

$x_2 \leq 6{,}000$ (Maximum sales of hard paper books)

$x_1 \geq x_2 \geq 0$ (Non-negativity constraint)

Example 24. A medical scientist claims to have found a cure for the common cold that consists of three drugs called A, B and C. His results indicate that the minimum daily adult dosage for effective treatment must be 100mg. of drug A, 60mg. of drug B and 80mg. of drug C. Two substances are readily available for

A must, is means =

preparing pills or drugs. Each unit of substance X contains 6 mg. 1mg and 2 mg. of drugs A, B and C respectively, and each unit of substance Y contains 20mg. 30 mg and 20 mg of the same drugs. Substance X cost Rs. 30 per unit and substance Y costs Rs. 50 per unit. Formulate the LPP.

Formulation :

Let x_1 = Substance X

x_2 = Substance Y

Minimum $Z = 30x_1 + 50x_2$

Subject to constraints :

$6x_1 + 20x_2 \geq 100$ (Requirement of drug A)

$1x_1 + 30x_2 \geq 60$ (Requirement of drug B)

$2x_2 + 20x_2 \geq 80$ (Requirement of drug C)

$x_1, x_2 \geq 0$ (Non-negativity constraints)

Example 25. The XYZ company during the festival season combines two factors A and B to form a gift pack which must weigh 10kg. At least 4kg of A and not more than 8 kg of B. Should be used. The net profit contribution to the company is Rs. 10 per kg for A and Rs. 12 per kg for B. Formulate the LPP.

equal to

means weight = 10kg

Must =

Formulation :

Let : x_1 = kgs of Product A

x_2 = kgs of Product B

Since the objective is to maximize the profits, the objective function is given by

Maximize $Z = 10x_1 + 12x_2$

demand =
must =

Subject to constraints

$x_1 + x_2 = 10$ (Total weight of the pack)

$x_1 \geq 4$ (Minimum requirement)

$x_2 \leq 8$ (Maximum requirement)

$x_1 \geq 0,\ x_2 \geq 0$ (Non negativity constraint)

Example 26. A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw material (A and B) of which 4000 and 6000 units respectively are available. The raw material requirements per unit of the three models are given below :

Raw Material	Requirement per unit of given model		
	I	II	III
A	2	3	5
B	4	2	7

17600
200000

$x_1 = 2x_2$, $x_2 = \frac{1}{2}x_1$

$x_1 = 3x_3$, $x_3 = \frac{1}{3}x_1$

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2500 units of model I. A market survey indicates that the minimum demand of the three models are 500, 500 and 375 units respectively. However, the ratios of the number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II and III are rupees 60, 40 and 100 respectively. Formulate the problem as a linear programming model in order to determine the number of unit of each model which will maximize profit.

$x_1 = 2x_2$, $x_2 = \frac{1}{2}x_1$

$x_1 = 3x_3$, $x_3 = \frac{1}{3}x_1$

Formulation :

Let x_1 = units of model I

x_2 = units of model II

x_3 = units of model III

Then, the raw material constraints will be

$$2x_1 + 3x_2 + 5x_3 \le 4{,}000 \quad \text{(for A)}$$

$$4x_1 + 2x_2 + 7x_3 \le 6{,}000 \quad \text{(for B)}$$

Suppose it takes labour time 1 for producing one unit of model I, so by the given problem it will take 1/2 and 1/3 labour time for producing one unit of model II and III, respectively.

As the factory can produce 2500 units of model I, so the restriction on the production time will be

$$1x_1 + (1/2)x_2 + (1/3)x_3 \le 2500, \text{ i.e.,}$$

$$x_1 + 1/2x_2 + 1/3x_3 \le 2500.$$

Also, since at least 500 units of model type I and II each and 375 units of model III are demanded, the constraints of market demand needs,

$$x_1 \ge 500,\ x_2 \ge 500, \text{ and } x_3 \ge 375,$$

But, the ratio of the number of units of different types of models is 3 : 2 : 5, we have $1/3\, x_1 = 1/2\, x_2$ and $1/2\, x_2 = 1/5\, x_3$.

Since the profit per unit on model I, II and III are Rs. 60, Rs. 40 and Rs. 100 respectively, the objective function is to maximize the profit : $Z = 60x_1 + 40x_2 + 100x_3$.

Finally, we have

Maximize : $Z = 60x_1 + 40x_2 + 100x_3$

Subject to the constraints :

$$2x_1 + 3x_2 + 5x_3 \le 4{,}000$$

$$4x_1 + 2x_2 + 7x_3 \le 6{,}000$$

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \le 2{,}500$$

$$\frac{1}{3}x_1 = \frac{1}{2}x_2,\ \frac{1}{2}x_2 = \frac{1}{5}x_3$$

and $x_1 \ge 500,\ x_2 \ge 500,\ x_3 \ge 375.$

$\frac{1}{x_1} : \frac{1}{x_2} = 3 : 2$

$3x_1 = x_3$, $x_1 = \frac{1}{3}x_3$

$2x_1 = x_2$, $x_1 = \frac{1}{2}x_2$

$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \le 2500$

Example 27. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the inputs and outputs per production run are as follows :

Process	Input (Units)		Output (Units)	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amount available for crudes A and B is 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Formulate the LPP.

Formation :

The mathematical formulation of the given LPP is

Maximize $Z = 300x_1 + 400x_2$

Subject to restrictions

$$5x_1 + 4x_2 \le 200$$

$$3x_1 + 5x_2 \le 150$$

$$5x_1 + 4x_2 \ge 100$$

$$8x_1 + 4x_2 \ge 80$$

Where $x_1,\ x_2 \ge 0$

$B = 2A$

$\frac{1}{2}B = A$

Example 28. A company manufactures two products A and B. Each unit of B takes twice as long to produce as one unit of A and if the company were to produce only A it would have time to produce 2000 units per day. The availability of the raw material is sufficient to produce 1500 units per day of both A and B combined. Product B requiring a special ingredient only 600 units can be made per day. If A fetches a profit of Rs. 2 per unit and B a profit of Rs. 4 per unit. Formulate the LPP. [MU, B.E, April 97]

only sufficient and not adequate

$A + 2B \le 2$

$x_1 + 2x_2 \le 2$

Formulation :

Let x_1 = the number of units of the product A

x_2 = the number of units of the product B

The profit by selling these two products is given by the objective function

$$\text{Max } Z = 2x_1 + 4x_2$$

Since the company can produce maximum of 2000 units of the product A in a day and type B requires twice as much time as that of type A, constraint is given by

$$x_1 + 2x_2 \le 2000$$

means $\le$

Since the raw material are sufficient to produce 1500 units per day both A and B combined

We have $x_1 + x_2 \le 1500$

There are special ingredients for the product B we have $x_2 \le 600$.

Hence the problem can be finally put in the form :

Maximize $Z = 2x_1 + 4x_2$

Subject to constraints

$$x_1 + 2x_2 \le 2000$$
$$x_1 + x_2 \le 1500$$
$$x_2 \le 600$$
$$x_1, x_2 \ge 0$$

Example 29. An advertising agency wishes to reach two types of audiences; customers with annual income of more than Rs. 15,000 (target audience A) and customers with annual income of less than Rs. 15,000 (target audience B). The total advertising budget is Rs. 2,00,000. One programme of TV advertising costs Rs. 50,000 and one programme of radio advertising costs Rs. 20,000. For contract reasons, at least 3 programmes have to be on TV and the number of radio programme must be limited to 5. Surveys indicate that a single TV programme reaches 4,50,000 customers in target audience A and 50,000 in the target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in the target audience B. Formulate the media-mix to maximize the total reach.

objective Function

TV 450000 50000 500000

Radio 20000 80000 100000

Formulation :

Let x_1 = Number of advertisement on Television

x_2 = Number of advertisement on Radio

Since the objective is to maximize the total audience, the objective function is given by –

Maximize $Z = 5{,}00{,}000x_1 + 1{,}00{,}000x_2$

quantity only i.e. no. and not amount

Max Z = no. of ad on TV and radio

Exposure

			Total
TV	Customers with Annual Income of More than 15000	4,50,000	
	Customers with Annual Income of Less than 15000	50,000	5,00,000
Radio	Customers with Annual Income of More than 15000	20,000	
	Customers with Annual Income of Less than 15000	80,000	1,00,000

Subject to constraints :

$50{,}000x_1 + 20{,}000x_2 \le 2{,}00{,}000$ (Total amount available)

$x_1 \ge 3$ (Minimum advertisement on TV)

$x_2 \le 5$ (Maximum advertisement on Radio)

$x_1, x_2 \ge 0$ (Non-negativity constraint)

Example 30. A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and given below is the required processing time in minutes for each product on each machine

Machines		Product A	Product B	Product C
	C	4	3	5
	D	3	2	4

Machine C and D have 2000 and 2500 machines minutes respectively. The firm must manufacture 100 A's 200 B's and 50 C's but no more than 150 A's. Setup an LP problem to maximize the profit.

at least

less than equal to

Formulation :

Let x_1 = the number of units of the product A

x_2 = the number of units of the product B

x_3 = the number of units of the product C

Since the profits are Rs. 3, Rs. 2 and Rs. 4 respectively, the total profit gained by the firm becomes :

$$Z = 3x_1 + 2x_2 + 4x_3$$

The total number of minutes required in producing these three products at machine C is given by $4x_1 + 3x_2 + 5x_3$ and at machine D is given by $3x_1 + 2x_2 + 4x_3$. The restrictions on the machine C and D are given by 2000 minutes and 2500 minutes respectively. The constraints becomes

$$4x_1 + 3x_2 + 5x_3 \le 2000$$
$$3x_1 + 2x_2 + 4x_3 \le 2500$$

Also since the firm manufactures 100 A's 200 B's and 50 C's but not more than 150 A's the further restriction becomes

$$100 \le x_1$$
$$150 \ge x_1$$
$$200 \le x_2$$
$$50 \le x_3$$

Finally we have

Maximize $Z = 3x_1 + 2x_2 + 4x_3$

Subject to the constraints

$$4x_1 + 3x_2 + 5x_3 \le 2000$$
$$2x_1 + 2x_2 + 4x_3 \le 2500$$
$$100 \le x_1,\ 150 \ge x_1,\ 200 \le x_2,\ 50 \le x_3$$

where $x_1, x_2, x_3 \ge 0$ (Non-negativity constraints)

Example 31. An electric appliance company produces two products : refrigerators and ranges. Production takes place in two separate departments I and II. Refrigerators are produced in department I and ranges in department II. The company's two product are sold on a weekly basis. The weekly production can not exceed 25 refrigerators and 35 ranges. The company regularly employs a total of 60 workers in two departments. A refrigerators requires 2 man-weeks labour while a range requires 1 man-week labour. A refrigerator contributes a profit of Rs. 60 and a range contributes a profit of Rs. 40. How many units of refrigerators and ranges should the company produce to realize the maximum profit ? Formulate the above as a LPP.

$x_1 \le 25;\ x_2 \le 35$

$2x_1 + 1x_2 \le 60$

Formulation :

Let x_1 = the number of units of refrigerator to be produced.

Max $Z = 60x_1 + 40x_2$

x_2 = the number of units of range to be produced.

where x_1, x_2

Each refrigerator and range contributes a profit of Rs. 60 and Rs. 40.

∴ The objective function is to maximize $Z = 60x_1 + 40x_2$

Since the weekly production cannot exceed 25 refrigerators and 35 ranges.

$$x_1 \le 25$$
$$x_2 \le 35$$

A refrigerator requires 2 man-weeks of labour and a range requires 1 man week of labour and the total number of workers is 60.

$$2x_1 + x_2 \le 60$$

Hence the production of refrigerator and ranges problem can be stated as below :

Maximise $Z = 60x_1 + 40x_2$

Subject to constraints

$$x_1 \le 25$$
$$x_2 \le 35$$
$$2x_1 + x_2 \le 60$$
and $x_1, x_2 \ge 0$

Example 32. A leading C.A. is attempting to determine a 'best' investment portfolio and is considering six alternative investment proposals. The following table indicates point estimates for the price per share, the annual growth rate in the price per share, the annual dividend per share and a measure of the risk associated with each investment.

Portfolio Data

Shares under consideration :	A	B	C	D	E	F
Current price per share (Rs.)	80	100	160	120	150	200
Projected annual growth rate	0·08	0·07	0·10	0·12	0·09	0·15
Projected annual dividend per share (Rs.)	4·00	4·50	7·50	5·50	5·75	0·00
Projected risk in return	0·05	0·03	0·10	0·20	0·06	0·08

The total amount available for investment is Rs. 25 lakhs and the following conditions are required to be satisfied.

(i) The maximum rupee amount to be invested in alternative F is Rs. 2,50,000.

(ii) No more than Rs. 5,00,000 should be invested in alternatives A and B combined.

(iii) Total weighted risk should not be greater than 0.10, where

$$\text{Total weighted risk} = \frac{(\text{Amount invested in alternative}) \cdot (\text{Risk of alternative } j)}{\text{Total amount invested in all the alternatives}}$$

(iv) For the sake of diversity, at least 100 shares of each stock should be purchased.

(v) At least 10 per cent of the total investment should be in alternatives A and B combined.

(vi) Dividends for the year should be at least 10,000.

Rupees return per share of stock is defined as price per share one year hence less current price per share PLUS dividend per share. If the objective is to maximize total rupee return, formulate the linear programming model for determing the optimal number of shares to be purchased in each of the shares

under consideration. You may assume that the time horizon for the investment is one year. The formulated LPP is not required to be solved.

Formulation :

Let x_1 = the number of shares to be purchased in investment proposal A

x_2 = the number of shares to be purchased in investment proposal B

x_3 = the number of shares to be purchased in investment proposal C

x_4 = the number of shares to be purchased in investment proposal D

x_5 = the number of shares to be purchased in investment proposal E

x_6 = the number of shares to be purchased in investment proposal F

Note : Rupee return per share

= Price per share one year hence – current price per share + dividend per share

= Current price per share × Projected annual growth rate (*i.e.* Project growth each year) + dividend per share

Thus, we compute the following data :

Investment Alternatives	:	A	B	C	D	E	F
No. of shares purchased		x_1	x_2	x_3	x_4	x_5	x_6
Projected growth for each share (Rs.)		6.40	7.00	16.00	14.40	13.50	30.00
Projected annual dividend per share (Rs.)		4.00	4.50	7.50	5.50	5.75	0.00
Rupee return per share		10.40	11.50	23.50	19.90	19.25	30.00

The Chartered Accountant wishes to maximize the total rupee return, thus the objective function of the linear programming problem is given by :

Maximum $Z = 10.40x_1 + 11.50x_2 + 23.50x_3 + 19.90x_4 + 19.25x_5 + 30.00x_6$.

Subject to the restrictions.

(1) $80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \le 25{,}00{,}000$

(2) $200x_6 \le 25{,}0000$ [from condition (1)]

(3) $80x_1 + 100x_2 \le 5{,}00{,}000$ [from condition (2)]

(4) According to condition (3) of the problem

$$\left[\frac{80x_1 \times 0.05 + 100x_2 \times 0.03 + 160x_3 \times 0.10 + 120x_4 \times 0.20 + 150x_5 \times 0.06 + 200x_6 \times 0.08}{80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6}\right] \le 0.$$

By Cross multiplication, we get

or $4x_1 + 3x_2 + 16x_3 + 24x_4 + 9x_5 + 16x_6 \le 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $-4x_1 - 7x_2 + 0x_3 + 12x_4 - 6x_5 - 4x_6 \le 0$

(5) $x_1 \geq 100,\ x_2 \geq 100,\ x_3 \geq 100,\ x_4 \geq 100,\ x_5 \geq 100,\ x_6 \geq 100$ [from condition (4)]

(6) $80x_1 + 100x_2 \geq 0.10\,(80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6)$ [from condition (5)]

or $\quad 80x_1 + 100x_2 \geq 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $\quad 72x_1 + 90x_2 - 16x_3 - 12x_4 - 15x_5 - 20x_6 \geq 0$

(7) $4x_1 + 4\cdot 5x_2 + 7\cdot 5x_3 + 5\cdot 5x_4 + 5.75x_5 \geq 10{,}000$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0$

Finally, combining all the constraints from (1) to (7), the desired linear programming problem is formulated.

Maximize $Z = 10.40x_1 + 11.50x_2 + 23.50x_3 + 19.90x_4 + 19.25x_5 + 30.00x_6$

Subject to restrictions

$$80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \leq 25{,}00000$$
$$200x_6 \leq 25{,}0000$$
$$80x_1 + 100x_2 \leq 5{,}00{,}000$$
$$-4x_1 - 7x_2 + 0x_3 + 12x_4 - 6x_5 - 4x_6 \leq 0$$
$$x_1 \geq 100,\ x_2 \geq 100,\ x_3 \geq 100,\ x_4 \geq 100,\ x_5 \geq 100,\ x_6 \geq 100$$
$$72x_1 + 90x_2 - 16x_3 - 12x_4 - 15x_5 - 20x_6 \geq 0$$
$$4x_1 + 4\cdot 5x_2 + 7\cdot 5x_3 + 5\cdot 5x_4 + 5.75x_5 \geq 10{,}000$$

where $\quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Example 33. From the following particulars, formulate LPP to find the most profitable product mix.

	Product A	*Product B*	*Product C*
Unit budgeted to be produced and sold (at least)	1,800	3,000	1,200
Selling price per unit (Rs.)	60	55	50
Requirements per unit :			
Direct Materials	5 kg	3 kg	4 kg
Direct Labour	4 hrs	3 hrs	2 hrs
Variable overheads	Rs. 7	Rs. 13	Rs. 8
Fixed Overheads	Rs. 10	Rs. 10	Rs. 10
Cost of Direct Materials per kg.	Rs. 4	Rs. 4	Rs. 4
Direct Labour Hour Rate	Rs. 2	Rs. 2	Rs. 2
Maximum Possible Units of Sales	4,000	5,000	1,500

All the three products are from the same direct material using the same type of machines and labour. Direct labour, which is the key factor, is limited to 18,600 hours. Formulate the LPP.

Formulation :

Statement Showing the Contribution per Unit

Note : Contribution = Sales – Variable Cost

	Particulars	**Products A**	**Products B**
P.	Selling price per unit (Rs.)	60	55
Less Q.	Variable costs per unit (Rs.)		
	Direct Material	20	12
	Direct Labour	8	6
	Variable Overhead	7	13
	Total Variable Cost per unit (Rs.)	35	31
	Contribution per unit (Rs.) [P – Q]	25	24

Let x_1 = Number of units to be produced of Product A

x_2 = Number of units to be produced of Product B

x_3 = Number of units to be produced of Product C

S − V = C

Since the objective is to maximise profit, the objective function is given by–

Maximize $\quad Z = 25x_1 + 24x_2 + 22x_3$

Subject to constraints :

$4x_1 + 3x_2 + 2x_3 \leq 18{,}600$ (Maximum Labour Hours Constraint)

$x_1 \leq 4{,}000$ (Maximum Sales of Product A constraint)

$x_2 \leq 5{,}000$ (Maximum Sales of Product B constraint)

$x_3 \leq 1{,}500$ (Maximum Sales of Product C constraint)

$x_1 \geq 1{,}800$ (Minimum Sales and production of Product A constrain

$x_2 \geq 3{,}000$ (Minimum Sales and production of Product B constrain

$x_3 \geq 1{,}200$ (Minimum Sales and production of Product C constrain

where, $\quad x_1, x_2, x_3 \geq 0.$ (Non-Negativity Constraint)

Example 34. The most recent audited summarised Balance Sheet of Malhotra Financial ser below.

Balance Sheet
as on March 31, 1994

Liabilities	Rs. (in lakhs)	Assets
Equity Share Capital	65	**Fixed Assets :**
Reserves and Surplus	110	Assets on Lease (original cost : Rs. 550 lakhs)
Term Loan from IFCI	80	Other Fixed Assets
Public Deposits	150	Investments (on wholly owned subsidiaries)

Bank Borrowings	147	**Current Assets :**	
Other Current Liabilities	50	Stock on Hire	80
		Receivables	30
		Other Current Assets	35
		Miscellaneous expenditure (not written off)	12
Total	602	Total	602

The company intends to enhance its investment in the lease portfolio by another Rs. 1,000 lakhs. For this purpose it would like to raise a mix of debt and equity in such a way that the overall cost of raising additional funds is minimized. The following constraints apply to the way the funds can be mobilized :

(1) Total debt dividend by net owned funds, cannot exceed 10.

(2) Amount borrowed from financial institutions cannot exceed 25% of the new worth.

(3) Maximum amount of bank borrowings cannot exceed three times the net owned funds.

(4) The company would like to keep the total public deposit limited to 40% of the total debt.

The post-tax costs of the different sources of finance are as follows :

Equity	**Term Loans**	**Public Deposits**	**Bank Borrowings**
2.5%	8.5%	7%	10%

Formulate the funding problem as LPP.

Note : (*a*) Total Debt = Term Loans from Financial Institutions + Public Deposits + Bank Borrowings.

(*b*) New Worth = Equity Share Capital + Reserves & Surplus

(*c*) New Owned Funds = Net Worth – Miscellaneous Expenditures.

Formulation :

Let x_1, = quantity of additional funds (in lakhs) raised on account of addition equity

x_2, = quantity of additional funds (in lakhs) raised on account of additional term loans

x_3, = quantity of additional funds (in lakhs) raised on account of additional public deposits

x_4, = quantity of additional funds (in lakhs) raised on account of additional bank borrowings

The objective function is to minimize the cost of additional funds raised by the company. That is,

Minimize $Z = 0\cdot 25x_1 + 0\cdot 85x_2 + 0\cdot 07x_3 + 0.1x_4$, subject to the following constraints :

(1) $$\left[\frac{\text{Total Debt}}{\text{Net owned funds}} \le 10\right]$$

Total Debt = Existing Debt + Additional

Net Owned funds = Equity share capital + Reserve and Surplus + Additional Equity – Misc. Exp.

or $$\left[\frac{\text{(Existing debt + Additional)}}{\text{(Equity share capital + Reserve \& Surplus + Additional Equity - Misc. Exp.)}}\right] \le 10$$

or $$\frac{80+150+147+x_2+x_3+x_4}{(65+110+x_1)-12} \le 10$$

or $$\frac{x_2+x_3+x_4+377}{x_1+163} \le 10$$

or $$x_2 + x_3 + x_4 + 377 \le 10x_1 + 1630$$

or $$-10x_1 + x_2 + x_3 + x_4 \le 1253.$$

(2) Amount borrowed (financial institutions) ≤ 25% of net worth

Amount borrowed = Existing long term loan from financial institutions + Additional loan

or (Existing long term loan from financial institutions + Additional loan)

≤ 25% (Existing Equity Capital + Reserve & Surplus + Addl. Equity Capital)

$$80 + x_2 \le 0{\cdot}25\,(175 + x_1)$$

or $$80 + x_2 \le \frac{1}{4}(175 + x_1)$$

or $$320 + 4x_2 \le 175 + x_1$$

or $$-x_1 + 4x_2 \le -145$$

or $$x_1 - 4x_2 \ge 145$$

or

(3) Maximum bank borrowings ≤ 3 (Net owned funds)

Maximum bank borrowings = Existing bank borrowings + Addl. bank borrowings

Net own fund = Existing Equity Capital + Reserve & Surplus + Addl. Equity Capital

or (Existing bank borrowings + Addl. bank borrowings) ≤ 3 (Existing Equity Capital + Reserve & Surplus + Addl. Equity Capital – Misc. Exp

or $$(147 + x_4) \le 3\,(65 + 110 + x_1 - 12)$$

or $$x_4 - 3x_1 \le 525 - 36 - 147$$

or $$-3x_1 + x_4 \le 342.$$

(4) Total public deposit ≤ 40% of total debt.

Total public deposit = Existing public deposit + addl. public deposits

Total Debts = Existing total debt + Additional total debt.)

or (Existing public deposit + addl. public deposits) ≤ 0·40 (Existing total debt + Addl. total

or $$150 + x_3 \le 0{\cdot}40\,(80 + 150 + 147 + x_2 + x_3 + x_4)$$

or $$150 + x_3 \le 0{\cdot}40\,(x_2 + x_3 + x_4 + 377)$$

or $$1500 + 10x_3 \le 4x_2 + 4x_3 + 4x_4 + 1508$$

or $$-4x_2 + 6x_3 - 4x_3 \le 8.$$

(5) Addl. equity capital + Addl. term loan + Addl. public deposits + Addl. bank borrowin

= 1000 (since the company wants to enhance the investment by Rs.1,000 lakhs)

or $$x_1 + x_2 + x_3 + x_4 = 1000$$

Finally we get

Maximize $Z = 0\cdot 25x_1 + 0\cdot 85x_2 + 0\cdot 07x_3 + 0.1x_4$,

Subject to constraints

$$-10x_1 + x_2 + x_3 + x_4 \le 1253.$$
$$x_1 - 4x_2 \ge 145$$
$$-3x_1 + x_4 \le 342$$
$$-4x_2 + 6x_3 - 4x_3 \le 8$$
$$x_1 + x_2 + x_3 + x_4 = 1000$$

where $x_1, x_2, x_3 \ge 0$

Imp

Example 35. An advertising company wishes to plan and advertisement campaign in three different media : T.V., radio and magazine. The purpose of the advertising is to reach as many potential customers as possible. Result of market survey are as follows :

Includes both Men & Women

	TV		Radio	Magazine
	Prime day (Rs.)	Prime Time (Rs.)	(Rs.)	(Rs.)
Cost of an Ad/unit	40,000	75,000	30,000	15,000
(Total) No. of Potential customer reach per unit	4,00,000	9,00,000	5,00,000	2,00,000
No. of Women customer reach per unit	3,00,000	4,00,000	2,00,000	1,00,000

The company does not want to spend more than Rs. 8,00,000 on advertisement. It is further required that :

means six zeros.

(i) At least 2 million exposures take place among women.

(ii) Ad. on T.V. be limited to Rs. 5,00,000.

(iii) At least 3 ad. units be bought on Prime day and two units during prime time and

(iv) No. of ad. units on Radio and Magazine should each be between 5 and 10.

[G.N.D.U. B.Com. III (Prof.) 2003]

Formulation :

1 Million has 6 zeros

Let X_1 = Number of potential customer attached with T.V. on prime day

X_2 = Number of potential customer attached with T.V. on prime time

X_3 = Number of potential customer attached with Radio

X_4 = Number of potential customer attached with Magazine

Main purpose of the advertisement is to reach as many potential customers as possible. Hence, objective function becomes

$$\text{Maximize } Z = 400000\,X_1 + 900000\,X_2 + 500000X_3 + 200000X_4$$

As the company does not want to spend more than Rs. 8,00,000 on advertisement and on the basis of cost of advertisement, the constraints becomes :

$$40000X_1 + 75000X_2 + 30000X_3 + 15000X_4 \le 800000 \quad ...(1)$$

Number of women customer reach per unit is given in the problem as it is known that at least 2 million exposures take place among women, the constraints becomes :

$$300000X_1 + 400000X_2 + 200000X_3 + 100000X_4 \ge 2000000 \quad ...(2)$$

1 Million = 6 zero

According to third condition advertisement on T.V. (prime day) is at least 3 and on T.V. at least 2, the constraints becomes :

$$X_1 \ge 3, X_2 \ge 2$$

According to condition 4th number of advertisement units on Radio and Magazine sh between 5 and 10, constraints are as :

$$X_3 \ge 5,\ X_3 \le 10,\ X_4 \ge 5,\ X_4 \le 10$$

According to condition 2nd, advertisement on T.V. be limited to Rs. 5,00,000. Henc becomes :

$$40000X_1 + 75000X_2 \le 500000$$

Finally we get

$$\text{Maximize } Z = 400000\,X_1 + 900000\,X_2 + 500000X_3 + 200000X_4$$

Subject to constraints

$$40000X_1 + 75000X_2 + 30000X_3 + 15000X_4 \le 800000$$
$$300000X_1 + 400000X_2 + 200000X_3 + 100000X_4 \ge 2000000$$
$$40000X_1 + 75000X_2 \le 500000.$$
$$X_1 \ge 3$$
$$X_2 \ge 2$$
$$X_3 \ge 5.$$
$$X_3 \le 10$$
$$X_4 \ge 5$$
$$X_4 \le 10$$

Between

Bet

where $X_1, X_2, X_3, X_4 \ge 0$

Integer
Quala
Obj
Complex
Price Ch

LIMITATIONS OF LINEAR PROGRAMMING

1. Integer Values are Ignored : Linear programming gives result in fractional val 77.12, 112.13 etc. But sometimes decisions can be taken in integer values only *i.e.* profit m Rs. 20,000 etc., cost may be Rs. 40,000, Rs. 12220 etc. In such particular cases, we c programming but we have to use integer programming.

2. Sole Objective :. Linear programming can be used when there is only one management. Objective may be either maximisation of profit or minimisation of cos management has multiple goals, this technique of operation research fails and we programming in such cases.

with more gets complicated

3. Limited Variables : In the application of linear programming variable are assu but in some real life problems where huge number variables are present and inter-rel them is very complicated the problem is reduced to meaningful decisions to achieve some but such reductions create doubt about the optimality of the results of the problem.

change with passage of time

4. Certainty is blindly followed : The coefficients of objective function and constraints are assumed to be known with certainty in linear programming and assumed they do not change during the period of study which may not happen in real life situation.

5. Quadratic Equations are ignored : A linear programming can be used only in the situations in which the objective function and the constraints can be expressed in terms of linear equations only. Non-linearly quadratic, cubic or higher equations are not considered in LPP.

QUESTIONS

1. What do you mean by a linear programming problem ? What are its major limitations ?
2. What are major assumptions of a L.P. model ?
3. What are the advantages and limitations of a L.P. model ?
4. Discuss briefly the application of linear programming in various functional areas of management.
5. Discuss briefly the steps to formulate a linear programming problem. Explain with an example.
6. Give mathematical statement of linear programming.

EXERCISE 2.3

1. A firm produces 3 products A, B, C. It uses 2 types of raw materials I and II of which 500 and 750 units respective are available. The raw material requirements per unit of the products are given below :

Raw Material	Requirements per unit of Product		
	A	B	C
I	30	40	50
II	50	30	50

The labour time for each unit of product A is twice that of product B and 3 times that of product C. The entire labour force of the firm can produce the equivalent of 3000 units of product A. The minimum demand of the 3 products is 600, 650 and 500 units respectively. Also the ratios of the number of units produced must he equal to 2 : 3 : 4. Assuming the profits per units of A, B and C as Rs. 150, 150 and 180 respectively. Formulate the problem as a linear programming model in order to determine the number of units of each product which will maximize the profits ?

2. A manufacturer of three products tries to follow a policy of producing those which contribute most to fixed cost and profit. However there is also a policy of recognising certain minimum sales requirements currently these are :

Product	Unit per week
X	120
Y	230
Z	360

There are 3 producing departments. The product time in hour per unit in each department and the total times available for each week in each department are :

		Time required per product, (hr)			Total hours available
Product		X	Y	Z	
Department	1	2	1	3	420
	2	1	4	6	1,048
	3	3	6	2	529

The contribution per unit of product X, Y, Z is Rs. 10.50, Rs. 9 and Rs. 8 respectively. The company has scheduled 20 units of X, 30 units of Y and 60 units of Z for production in the following week.

Formulate the problem.

3. A mutual fund company has Rs. 20 lakhs available for investment in govt. bonds blue chips stocks, speculative stocks and short term deposits. The annual expected return and risk factor are given below :

Type of Investment	Annual Expected return (%)	Risk Factor (0–100)
Govt. Bonds	14	12
Blue chip stocks	19	24
Speculative stocks	23	48
Shot-term Deposits	12	6

Mutual fund is required to keep at least Rs. 2 lakhs in short-term deposits and not to exceed average risk factor 42. Speculative stocks must be at the most 20% of the total amount invested. How should mutual fund invest the funds so as to maximize its total expected annual return ? Formulate this as a linear programming problem. Do not solve it.

4. A company makes 3 products X, Y and Z which flow through 3 department Drill, lathe and Assembly. The hours of department time required by each of the products, the hours available in each of the departments, the marginal contribution of each of the products and the estimated incremental cost of idle time per hour are given in the following table :

	Time Required (hrs./unit) Department			Marginal contribution per unit (Rs.)
Products	Drill	Lathe	Assembly	
X	3	3	8	90
Y	6	5	10	150
Z	7	4	12	200
Hours available	180	240	860	
Incremental cost of idle, Time per hr. in (Rs.)	85	115	130	

Determine the optimum product mix.

5. A medical scientist claims to have found a cure for cold that consists of 3 drugs A, B and C. The minimum daily adult dosage for effective treatment is 10 mg of A, 6 mg of B and 8 mg of C. 2 substances are readily available for preparing pills. Each unit of substance of X contains 6 mg, 1 mg and 2 mg of drugs A, B and C respectively. Each unit of substance Y contains 2 mg, 3 mg, 2 mg of the same drug. Substance of X costs Rs. 3 per unit and substance Y costs Rs. 5 per unit.

Find the least cost combination of the 2 substance that will yield a pill designed to contain the minimum daily recommended adult dosage.

6. A rubber company is engaged in producing 3 different kinds of tyres A, B and C. There are 3 different plants with different production capacities. In a normal 8 hours working day, plant 1 produces 150, 200 and 200 tyres of type A, B and C respectively. Plant 2 produces 160, 160 and 200 tyres of type A, B and C respectively. The monthly demand for type A, B and C is 12500, 30000 and 27000 units respectively. The daily cost of operation of plant 1 and plant 2 is 2500 and 3500 respectively. Form LP Model to determine the minimum number of days of operation per month at 2 different plants to minimise the total cost while meeting the demand.

7. A farmer owns an archard which has an area of 350 acre on which he grows apples, apricots, cherries and plums. Of the total areas, 250 acres of land are unstainable for growing apples and plums and are suitable only for apricots and cherries. On the remaining 100 acres of land any of the four fruits can be grown. The marketing policy requires that in each season all the four types of fruits must be produced and quantity of any one type should not be less than 12000 boxes. It is also essential that the area devoted to any one should be in terms of complete acres and not in fraction of an acre. There are no physical or marketing limitations and there is an adequate supply of all types of labour. The details regarding the selling price, production and costs are given below :

	Apples	Apricots	Cherries	Plums
Selling price per box (in Rs.)	10	10	20	30
Seasons yield per acre (boxes)	500	150	100	200
Weight per box (kgs)	30	30	40	20
Material per acres (Rs.)	180	70	60	100
Labour growing per acre (Rs.)	200	150	100	130
Harvesting and packing per box (Rs.)	1	1	2	3
Transport per box (Rs.)	2	2	1	3

Fixed overheads each season :

Cultivation and Growing Rs. 58,000, Harvesting Rs. 68000 Transport Rs. 5000 Administration is Rs. 42000, land revenue Rs. 9000.

8. A manufacturer has 3 products A, B and C. Current Sales, cost and selling price details and processing time requirements are as follows :

Machine	Product A	Product B	Product C
Annual Sales (per unit)	6000	6000	6000
Selling price per unit (Rs.)	200	310	390

Variable cost (Rs.)	180	240	300
Processing time per unit (hrs.)	2	2	4

The time is working at full capacity (13,500 processing hrs. per year) fixed manufacturing overheads are absorbed into unit cost by a charge of 200% of variable cost. This procedure fully absorbs the fixed manufacturing overhead. Assuming that :

(*i*) Processing time can be switched from 1 product line to another.

(*ii*) The demand at current selling price is :

Product A	Product B	Product C
11600	8000	2000

(*iii*) The selling prices are not to be altered. Formulate the problem.

9. A company manufacturers four products. The cost data per unit are as under :

	A	B	C	D
Selling Price (Rs.)	900	710	1000	860
Direct Materials	300	200	400	400
Direct Labour	240	180	300	120
Variable Overhead	120	90	150	60

The fixed costs are estimated at Rs. 200000 per month. The company employs 250 direct workers, who work eight hours a day for 25 days a month. The direct wage rate is Rs. 6 per hour. It is not possible for the company to increase its operatives in the short run nor it is practicable to work overtime. The company's policy does not allow subcontracting of work.

The Marketing Director has forecast the following demand for a month :

Product	Units	Product	Units
A	8500	C	6250
B	5000	D	8250

The management desires you to revise the product mix in order to yield the maximum profit for the month. Formulate the problem.

10. Jindal Furnishers Ltd. manufactures one type of sofa set exclusively. The set contains the following seven components. One sofa, 2 centre tables and four chairs. These components can be either manufactured by the company or sub-contracted and the following are the relevant data.

	Sofa	Table	Chair
Direct material cost per component Rs.	1000	500	550
Direct labour hours per component Rs.	100	50	10
Sub contract price per component Rs.	2500	1000	750

Sales of sofa sets are currently 8000 per period, each set selling for Rs. 7500. A capacity constraint of 500000 direct labour hours obliges the company to sub-contract some components.

The variable overheads vary with direct labour hours worked and are incurred at a rate of Rs. 2 per hour. Fixed cost is Rs. 17,80,000 per period and labour costs Rs. 5.50 per hour.

11. A big hospital has the following minimal daily requirements of doctors.

Period	Clock time (24 hrs.)	No. of doctors required
1	6 am – 10 am	72
2	10 am – 2 pm	77
3	2 pm – 6 pm	85
4	6 pm – 10 pm	68
5	10 pm – 2 am	25
6	2 am – 6 am	23

Doctors report to the hospital at the beginning of each period and work for 6 consecutive hours. Formulate this problem as a linear programming problem to minimise the total no. doctors to meet the needs of the hospital throughout the day.

12. The postmaster of a local post office wishes to hire extra helpers during the Deepawali season, because of a large increase in the volume of mail handling and delivery. Because of the limited office space and budgetary condition, the number of temporary helpers must not exceed 10. According to the past experience, man can handle 300 letters and 80 packages per day, on the average, and women can handle 400 letters and 50 packages per day. The postmaster believes that the daily volume of extra mail and packages will be no less than 3400 and 680 respectively. A man receives Rs. 25 a day and a woman receives Rs. 22 a day. How many men and women helpers should be hired to keep the pay roll at a minimum ? *(Meerut L.P. 1983)*

3

LINEAR PROGRAMMING–II

(GRAPHICAL METHOD)

- ***Procedure of Graphical Method***
- ***Important Definitions***
- ***Graphical Solution by Corner Method***
  - ***Maximization Problem With All ≤ Constraints***
  - ***Minimization Problem With All ≥ Constraints***
  - ***Maximization and Minimization Problem With Mixed Constraints***
- ***Exceptional Cases***
  - ***Infeasible Solution***
  - ***Unbounded Solution***
  - ***Multiple Optimal Solution***
  - ***Redundancy***
- ***ISO-Profit or ISO-Cost Approach***
  - ***Maximization Problem***
  - ***Minimization Problem***

Linear Programming Problems involving two decision variables can be solved by graphical m This method also provides an insight into the concepts of simplex method – a powerful and e method to solve the linear programming problems involving any number of decision variables. H we will deal with graphical method only, in this chapter.

GRAPHIC METHOD

To use the graphic method for solving LPP, the following steps are required.

Step 1. Consider each inequality constraint as equation.

Step 2. Plot a graph showing all the constraints of the problem graph

Step 3. Identify the feasible region which is the intersection of all the regions *i.e.* the a satisfies all the constraints simultaneously.

Step 4. The feasible region or solution space may be bounded or unbounded. Determin ordinates of all the corner points of the feasible region.

Step 5. Find the maximum or minimum value (as the case may be) of the Z (objective func

SOME IMPORTANT DEFINITIONS

1. **Solution** : For a problem in n variables and m constraints, any set $\{x_1, x_2, ..., x_n, s_1, s_2, s_3, ..., s_m\}$ of variables is called a **solution** if it is satisfies all the constraints (except non-negativity constraints).
2. **Feasible Solution** : For linear programming problem in n variables and m constraints, any set $\{x_1, x_2, ..., x_n, s_1, s_2, s_3, ..., s_m\}$ of variables is called **feasible solution** if it satisfies all the constraints including non-negativity restrictions also.
3. **Feasible Region** : The collection of all feasible solutions is called feasible region or the common region is called feasible region.
4. **Basic Solution** : A basic solution to a set of constraints (m constraints involving n variables) is obtained by setting n variables (among $m + n$ variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero. Such m variables are called **basic variables** and remaining n zero-valued variables are called **non-basic variables.**

m = basic
n = non basic

5. **Basic Feasible Solution** : A basic feasible solution is a basic, feasible solution which also satisfies the non-negativity restrictions.
   Basic feasible solutions are of two types.
6. **Non-Degenerate Basic Feasible Solution** : A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive x_i $(i = 1, 2, ..., m)$. Equivalently, all m basic variables are positive and the remaining n variables will be all zero.
7. **Degenerate Basic Feasible Solution** : A basic feasible solution is called degenerate, if one or more basic variables are zero-valued. $\rightarrow s_1, s_2, s_3 = 0$
8. **Optimum Basic Feasible Solution** : A basic feasible solution is said to be optimum if it also optimizes (maximizes or minimizes) the objective function.
9. **Corner Points** : In the feasible region the points of intersection on boundary lines is called corner points.
   intersection of boundary lines
10. **Infeasible Solutions** : If there is no solution, satisfying all the constraints of LPP, the given situation depicts infeasibility. It basically represents a state of inconsistency in the set of constraints.
11. **Unbounded Solutions** : A Linear Programming Problem is said to have unbounded solution when the common feasible region is not bounded in any respect. In simple words, it shows that the feasible region is open ended.
12. **Redundant Constraint** : It is one that does not affect the feasible solution area or feasible region.

is not the feasible region

Graphs may be represented in two forms :

1. Corner Method 2. ISO-profit or ISO-cost Method

GRAPHICAL SOLUTION BY CORNER METHOD

MAXIMISATION PROBLEM WITH ALL ≤ CONSTRAINTS

Example 1. Solve the following LPP by search approach method *i.e.* (graphic method)

Maximize $Z = 6\,X_1 + 10\,X_2$

Subject to constraints :

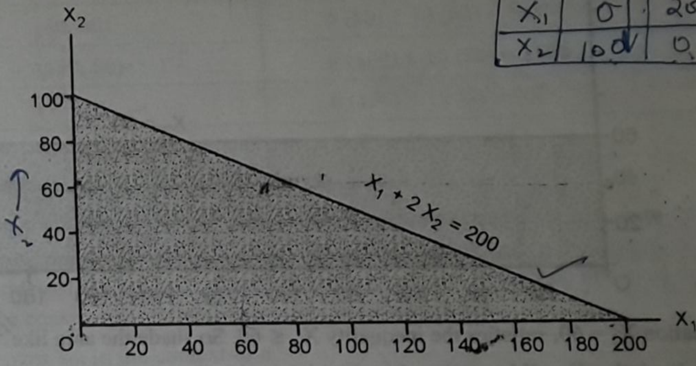
$$X_1 + 2\,X_2 \le 200$$

$$X_1 + X_2 \le 150$$
$$X_2 \le 60$$
and $X_1 \ge 0,\ X_2 \ge 0$

Graphical Solution

Step 1. Taking equation $X_1 + 2X_2 \le 200$, the inequalities are graphed by taking them as equation. The equation is re-written as $X_1 + 2X_2 = 200$. To plot the line $X_1 + 2X_2 = 200$, Put $X_1 = 0$ and find $X_2 = 100$, then put $X_2 = 0$, $X_1 = 200$. Now join the points ($X_1 = 200$, $X_2 = 100$) with the straight line given below :

X_1	0	200
X_2	100	0



Clearly, any point lying on or below the line $X_1 + 2X_2 = 200$ will satisfy the inequality $X_1 + 2X_2 \le 200$.

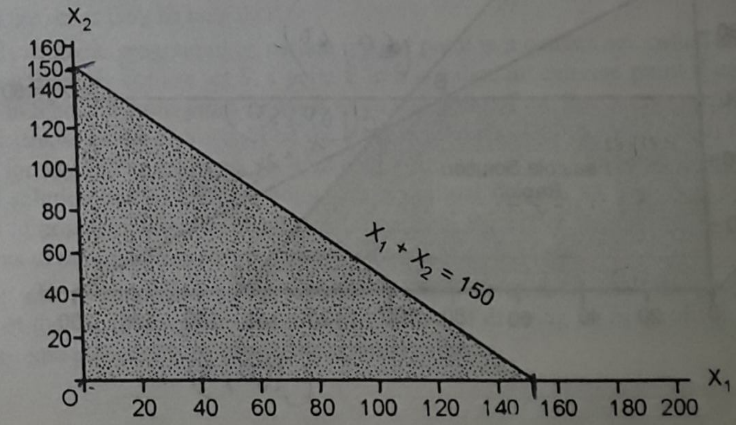
[**Check :** If we take (40, 40) *i.e.* $X_1 = 40$, $X_2 = 40$ then we have

$$40 + 2 \times 40 < 200$$
$$40 + 80 < 200$$
$$120 < 200, \quad \text{which is true}]$$

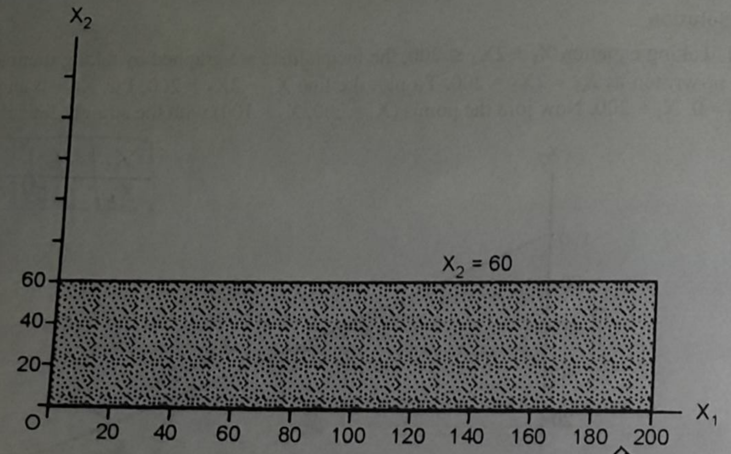
X_1	0	150
X_2	150	0

because the condition is satisfied than shade the area like ▨ *i.e.* towards origin.

Similar procedure is now adopted to plot the other line.

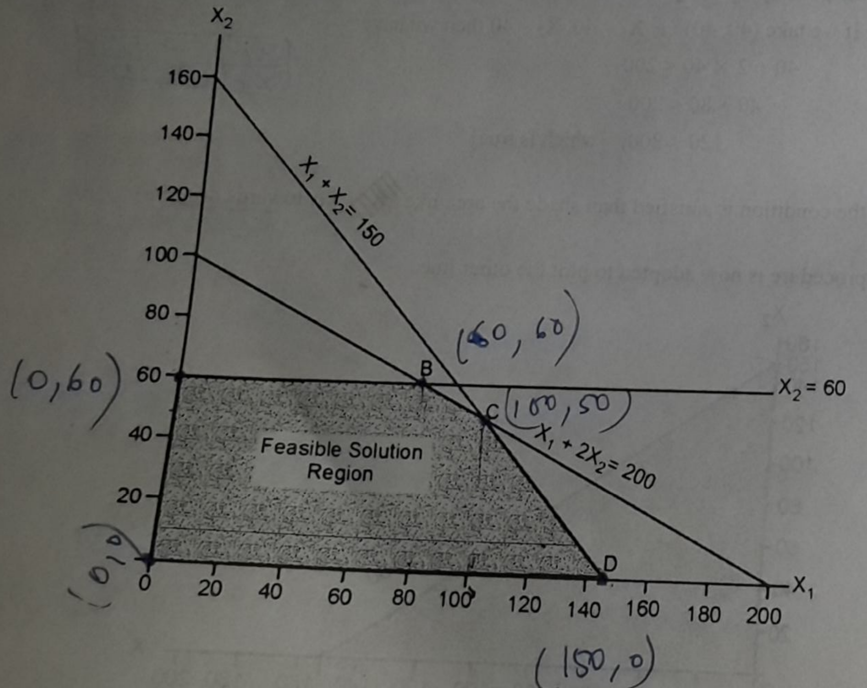


Any point on or below the equation $X_1 + X_2 = 150$ will also satisfy the inequality $X_1 + X_2 \leq 150$ so, again shade the area like *i.e.* towards origin. Now take the last constraints $X_2 = 60$. Here X_1 is absent so consider it as zero.



The equation $X_2 = 60$, satisfies the inequality $X_2 \leq 60$. So shade the area like *i.e.* towards origin.

Step 2. Find the **Feasible region** by combining the figures first, second and third given above together. Obtain the common shaded area *i.e.* A B C D O.



Note : Any point in the shaded area provides a feasible solution to the given LPP (including boundaries).

Step 3. Find the co-ordinates of the corner points ABCDO (feasible region)

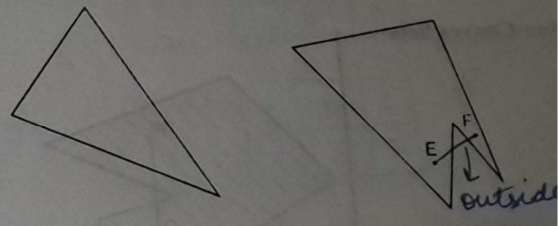
Step 4. Reading the co-ordinates of the shaded feasible region and putting the co-ordinates in the objective function

Corner Points	Co-ordinates of Corner Points	Value of Z $6X_1 + 10X_2$
A	(0, 60)	6 (0) + 10 (60) = 600
B	(80, 60)	6 (80) + 10 (60) = 1080
C	**(100, 50)**	**6 (100) + 10 (50) = 1100**
D	(150, 0)	6 (150) + 10 (0) = 900
O	(0, 0)	6 (0) + 10 (0) = 0

Optimal Solution : C, gives maximum value of Z where $X_1 = 100$, $X_2 = 50$.

Convex Sets and LPPs : Observe the feasible region from Example 1. It is seen to be formed by a five-sided polygon, represented by OABCD. Besides the two sides provided by non-negativity restrictions, each of the other three sides of the polygon is provided by a constraint. It may be observed that the feasible region determined by the constraints of the given system is a **Convex Set.**

The concept of convex set in the context of a two-variable problem can be understood as follows. If *any* two points are selected in the region and the line segment formed by joining these two points lies completely in this region, including on its boundary, then this represents a convex set. Thus, for the feasible region to be convex, no part of any line obtainable by joining a pair of points in that region should lie outside it. Figures given below illustrate the difference between a convex and a non-convex set.



CONVEX SET (a) NON-CONVEX SET (b)

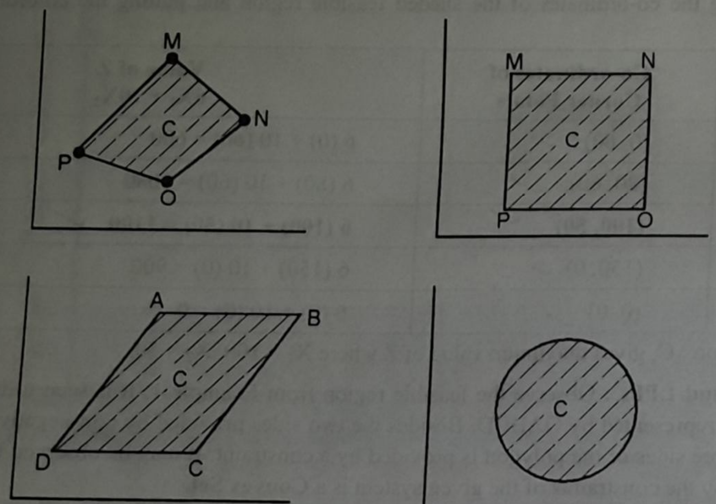
The set shown in part (*a*) of the figure is convex but the one shown in part (*b*) is not. For the set shown in part (*b*), there are many points, like E and F, for which the connecting line segment contains points that are not a part of the set as they lie outside it.

In our study of linear programming, certain type of point in a convex set, called an *extreme point*, is of particular interest. For any convex set S, a point E in S is called an extreme point if each line segment that lies completely in S and contains point E, has E as an end point of the line segment. And, if S happens to be a polygon, the extreme points of S would be the corners or vertices of the polygon. It may be noted that the feasible region for any LPP is a convex set with only a finite number of extreme points and that an LPP that has an optimal solution has an extreme point that is optimal. This is an important observation because it reduces the set of points yielding an optimal solution from the entire feasible region containing an infinite number of points to the set of extreme points which are few in number.

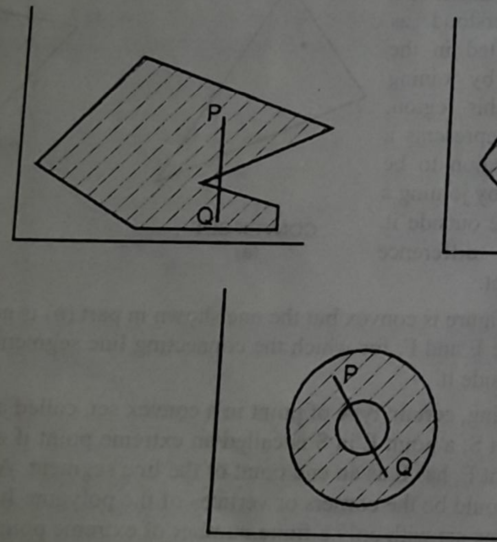
Thus, our task is simplified as the optimal solution to an LPP shall be given by some corner or extreme point of the feasible region. Accordingly, in lieu of drawing an iso-profit line, we may proceed to evaluate the extreme points only to obtain the optimal solution.

A set of points is said to be **convex** if for any two points in the set, the line segment joining these two points is also in the set. In other words, a set is convex if the convex combination of any two points in the set, is also in the set.

Convex Sets



Non-Convex Sets



Example 2. Solve graphically

Maximize $Z = 1.4X_1 + X_2$

Subject to constraints

$X_1 \leq 3$

$2X_1 + X_2 \leq 8$

$3X_1 + 4X_2 \leq 24$

$X_1, X_2 \geq 0$

X1	0	4
X2	8	0

X1	0	8
X2	6	0

[G.N.D.U. (April) 19

Solution :

First convert the inequalities into equations :

$X_1 = 3$

$2X_1 + X_2 = 8$

$3X_1 + 4X_2 = 24$

In equation (*ii*),

$2X_1 + X_2 = 8$

If $X_1 = 0, \quad X_2 = 8$ (0, 8)

$X_2 = 0, \quad X_1 = 4$ (4, 0)

In equation (*iii*),

$3X_1 + 4X_2 = 24$

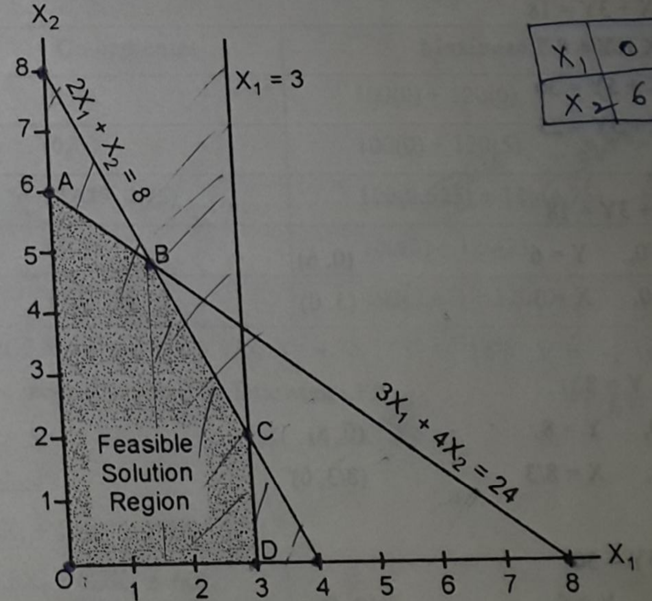
If $X_1 = 0, \quad X_2 = 6$ (0, 6)

$X_2 = 0, \quad X_1 = 8$ (8, 0)

X1	0	4
X2	8	0

X1	0	8
X2	6	0

Put the above constraints on the graph



Corner Points	Co-ordinates	Maximize $Z = 1\cdot4X_1 + X_2$	
O	(0, 0)	$1\cdot4(0) + 1(0)$	= 0
A	(0, 6)	$1\cdot4(0) + 1(6)$	= 6
B	**(1·6, 4·8)**	**$1\cdot4(1\cdot6) + 1(4\cdot8)$**	**= 7·04**
C	(3, 2)	$1\cdot4(3) + 1(2)$	= 6·2
D	(3, 0)	$1\cdot4(3) + 1(0)$	= 4·2

Maximum $Z = 7\cdot04$, $X_1 = 1\cdot6$, $X_2 = 4\cdot8$.

Example 3. Solve the following LPP, graphically

Maximize $Z = 100X + 120Y$

Subject to constraints

$$6X + 3Y \le 18$$
$$3X + Y \le 8$$
$$4X + 5Y \le 30$$
$$2X + 5Y \le 25$$

where $X, Y \ge 0$ [*G.N.D.U. (April) 2002*]

Solution :

Convert the above inequalities into equations first

$$6X + 3Y = 18 \quad ...(i)$$
$$3X + Y = 8 \quad ...(ii)$$
$$4X + 5Y = 30 \quad ...(iii)$$
$$2X + 5Y = 25 \quad ...(iv)$$

Taking equation (*i*),

$6X + 3Y = 18$

Put $X = 0$, $Y = 6$ (0, 6)

$Y = 0$, $X = 3$ (3, 0)

Taking equation (*ii*),

$3X + Y = 8$

Put $X = 0$, $Y = 8$ (0, 8)

$Y = 0$, $X = 8/3$ (8/3, 0)

Taking equation (*iii*),

$4X + 5Y = 30$

Put $X = 0$, $Y = 6$ (0, 6)

$Y = 0$, $X = 15/2$ (15/2, 0)

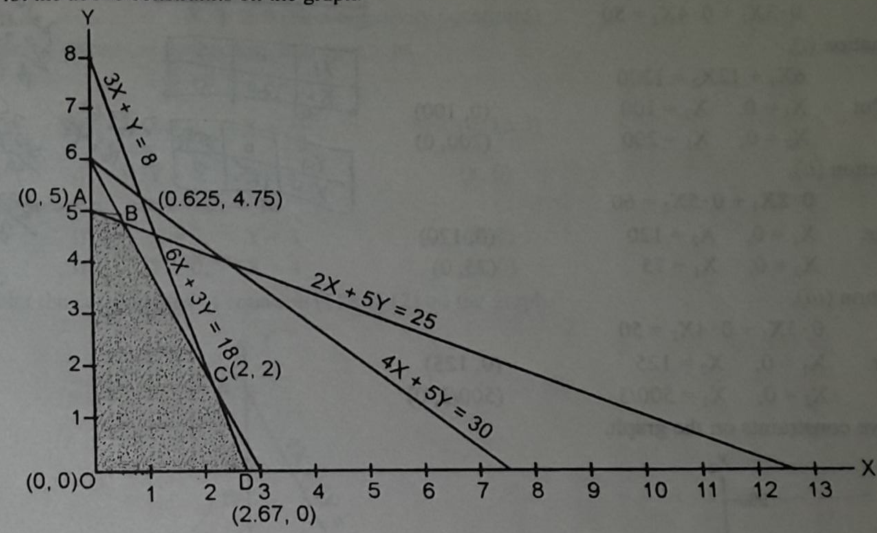
Taking equation (*iv*),

$2X + 5Y = 25$

Put $X = 0$, $Y = 5$ (0, 5)

$Y = 0$, $X = 25/2$ (25/2, 0)

Plot the above constraints on the graph.



Corner Points	Co-ordinates	Maximize Z = 100X + 120Y	
O	(0, 0)	100(0) + 120(0)	= 0
A	(0, 5)	100(0) + 120(5)	= 600
B	**(0.625, 4.75)**	**100(0.625) + 120(4.75)**	**= 632.50**
C	(2, 2)	100(2) + 120(2)	= 440
D	(2.67, 0)	100(2.67) + 120(0)	= 267

Hence, Max. Z = 632.50, where X = 0.625, Y= 4.75.

100 × 6/10 + 120 × 48/10 = 60 + 576 =

Example 4. Use graphic method to solve the following LPP

Maximize $Z = 3X_1 + 4X_2$

Subject to constraints

$$6X_1 + 12X_2 \le 1200$$
$$0.8X_1 + 0.5X_2 \le 60$$
$$0.3X_1 + 0.4X_2 \le 50$$
$$X_1, X_2 \ge 0$$

(*G.N.D.U.* (...

Solution :

Convert the inequalities into equations first

$6X_1 + 12X_2 = 1200$ ...(*i*)

$0{\cdot}8X_1 + 0{\cdot}5X_2 = 60$ ...(*ii*)

$0{\cdot}3X_1 + 0{\cdot}4X_2 = 50$ ...(*iii*)

X_1	0	200
X_2	100	0

X_1	0	75
X_2	120	0

X_1	0	500/3
X_2	125	0

Take equation (*i*),

$6X_1 + 12X_2 = 1200$

Put $X_1 = 0, \quad X_2 = 100$ (0, 100)

$X_2 = 0, \quad X_1 = 200$ (200, 0)

Take equation (*ii*),

$0{\cdot}8X_1 + 0{\cdot}5X_2 = 60$

Put $X_1 = 0, \quad X_2 = 120$ (0, 120)

$X_2 = 0, \quad X_1 = 75$ (75, 0)

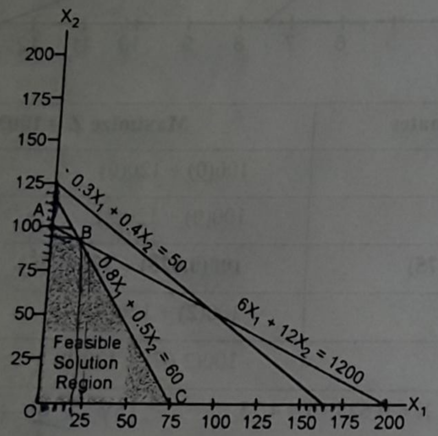
Take equation (*iii*),

$0{\cdot}3X_1 + 0{\cdot}4X_2 = 50$

Put $X_1 = 0, \quad X_2 = 125$ (0, 125)

$X_2 = 0, \quad X_1 = 500/3$ (500/3, 0)

Put the above constraints on the graph.



Corner Points	Co-ordinates	Maximize $Z = 3X_1 + 4X_2$	
O	(0, 0)	3(0) + 4(0)	= 0
A	(0, 100)	3(0) + 4(100)	= 400
B	**(200/11, 1000/11)**	**3(200/11) + 4(1000/11)**	**= 4600/11**
C	(75, 0)	3(75) + 4(0)	= 225

ence, Max $Z = 4600/11$, where $X_1 = 200/11$, $X_2 = 1000/11$.

Example 5. Maximize $Z = 6X + 8Y$

Subject to constraints :

$2X + 3Y \le 16$

$4X + 2Y \le 16$

$X, Y \ge 0$ (Non-negativity constraint)

Solution : Convert the inequalities into equations

$2X + 3Y = 16$ ...(1)

If $X = 0, \quad Y = \frac{16}{3}$ (0, 16/3)

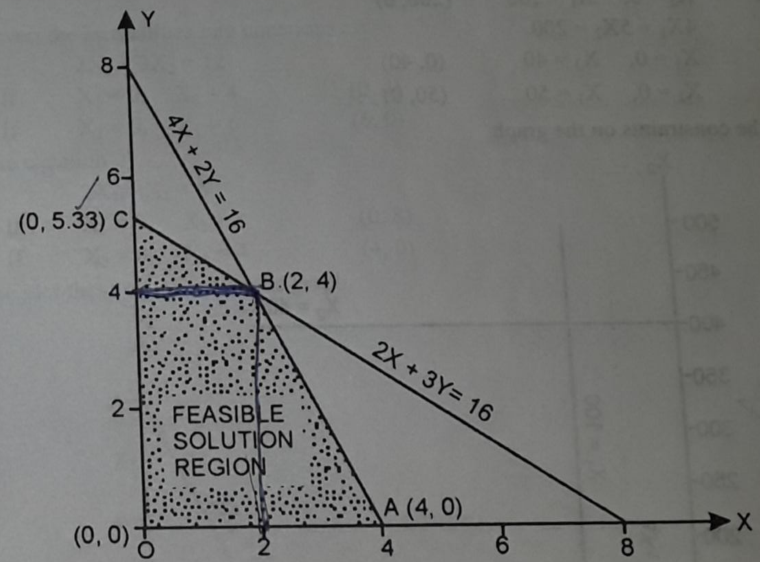
If $Y = 0, \quad X = 8$ (8, 0)

In equation $4X + 2Y = 16$ ...(2)

If $X = 0, \quad Y = 8$ (0, 8)

If $Y = 0, \quad X = 4$ (4, 0)

Plot the co-ordinates of equation (1) and (2) on the graph.



Corner Points	Co-ordinates of Common Shaded Feasible Region	Values of Z $6X + 8Y$
O	(0, 0)	$(6 \times 0) + (8 \times 0) = 0$
A	(4, 0)	$(6 \times 4) + (8 \times 0) = 24$
B	**(2, 4)**	$\mathbf{(6 \times 2) + (8 \times 4) = 44}$
C	(0, 16/3)	$(6 \times 0) + (8 \times 16/3) = 128/3$ *i.e.* 42.67

The maximum value of Z = 44 is at point B where X = 2 and Y = 4.

Example 6. Solve Graphically

Maximize $Z = 8X_1 + 2X_2$

Subject to constraints :

$$X_1 \le 100$$
$$X_2 \le 400$$
$$2X_1 + 4X_2 \le 400$$
$$4X_1 + 5X_2 \le 200$$

where $X_1 \ge 0, X_2 \ge 0$

Solution : Convert the inequalities into equalities

$X_1 = 100$ (100, 0) ...(1)

$X_2 = 400$ (0, 400) ...(2)

$2X_1 + 4X_2 = 400$ ...(3)

If $X_1 = 0, \; X_2 = 100$ (0, 100)

If $X_2 = 0, \; X_1 = 200$ (200, 0)

$4X_1 + 5X_2 = 200$ ...(4)

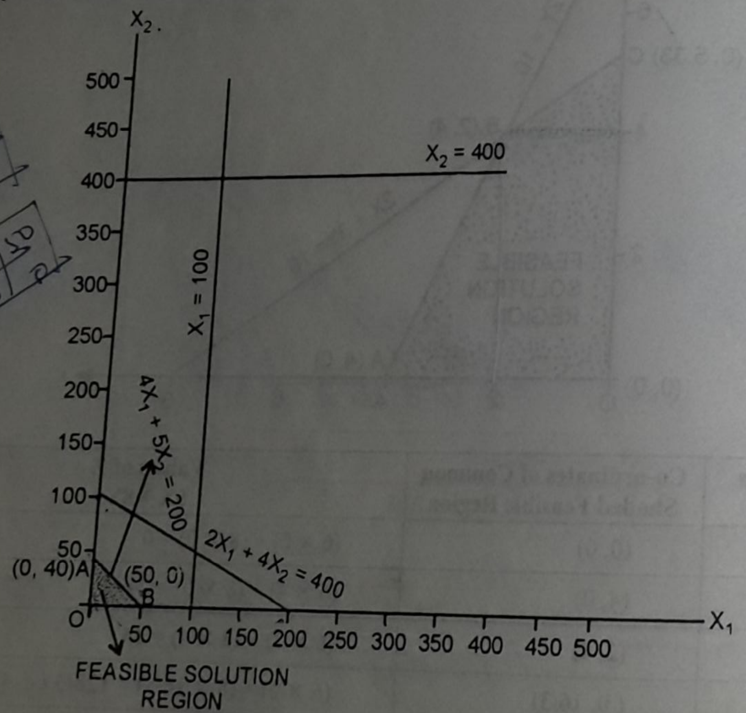
If $X_1 = 0, \; X_2 = 40$ (0, 40)

If $X_2 = 0, \; X_1 = 50$ (50, 0)

Now plot the constraints on the graph

X_1	0	200
X_2	100	0

X_1	0	50
X_2	40	0



The area OAB (shaded area) satisfied by the constraints and called the feasible solutions region.

Corner Point	Coordinates of Corner Point	Max. $Z = 8X_1 + 2X_2$	Value
O	(0, 0)	8 (0) + 2 (0)	0
A	(0, 40)	8 (0) + 2 (40)	80
B	**(50, 0)**	**8 (50) + 2 (0)**	**400**

Max. $Z = 400$, where $X_1 = 50, X_2 = 0$

Example 7: Solve the LPP by Graphic Method

Maximize $Z = 6X_1 + 7X_2$

Subject to constraints,

$$2X_1 + 3X_2 \le 12$$
$$2X_1 + X_2 \le 8$$

and $X_1 \ge 0, \; X_2 \ge 0$

Solution :

First convert the inequalities into equations *i.e.*

$2X_1 + 3X_2 = 12$

If $X_1 = 0, \; X_2 = 4$ (0, 4)

If $X_2 = 0, \; X_1 = 6$ (6, 0)

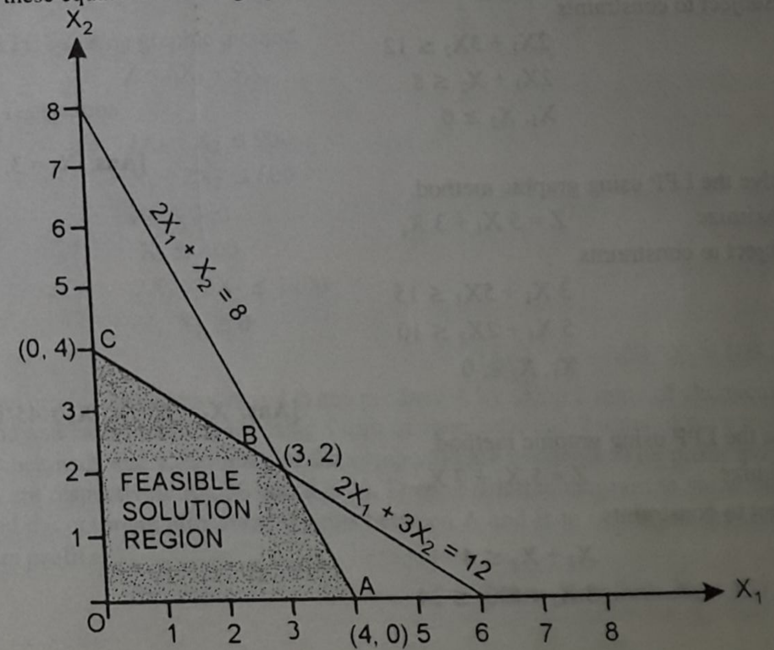
Now take equation

$2X_1 + X_2 = 8$

If $X_1 = 0, \; X_2 = 8$ (0, 8)

If $X_2 = 0, \; X_1 = 4$ (4, 0)

Now, we plot these equations on the graph.



The area ABCO satisfied by the constraints is shown as shaded area.

Corner Points	Co-ordinates of Common Shaded Feasible Region	Values of Z $6X_1 + 7X_2$
A	(4, 0)	6 (4) + 7 (0) = 24
B	(3, 2)	6 (3) + 7 (2) = 32
C	(0, 4)	6 (0) + 7 (4) = 28
O	(0, 0)	6 (0) + 7 (0) = 0

Hence Max Z = 32, where $X_1 = 3$, $X_2 = 2$.

EXERCISE 3.1

1. Solve the LPP using graphic method.

Maximize $Z = 2X_1 + 3X_2$

Subject to constraints

$$X_1 + X_2 \le 1$$
$$3X_1 + 2X_2 \le 4$$
$$X_1, X_2 \ge 0$$

[Ans. $X_1 = 0$, $X_2 = 1$, Max. Z = 3]

2. Solve the LPP using graphic method.

Maximize $Z = 6X_1 + 7X_2$

Subject to constraints

$$2X_1 + 3X_2 \le 12$$
$$2X_1 + X_2 \le 8$$
$$X_1, X_2 \ge 0$$

[Ans. $X_1 = 3$, $X_2 = 2$, Max. Z = 32]

3. Solve the LPP using graphic method.

Maximize $Z = 5X_1 + 3X_2$

Subject to constraints

$$3X_1 + 5X_2 \le 15$$
$$5X_1 + 2X_2 \le 10$$
$$X_1, X_2 \ge 0$$

[Ans. $X_1 = 20/19$, $X_2 = 45/19$, Max. Z = 235/19]

4. Solve the LPP using graphic method.

Maximize $Z = 5X_1 + 7X_2$

Subject to constraints

$$X_1 + X_2 \le 4$$
$$3X_1 + 8X_2 \le 24$$

$$10X_1 + 7X_2 \le 35$$
$$X_1, X_2 \ge 0$$

[Ans. $X_1 = 8/5$, $X_2 = 12/5$, Max. Z = Rs. 24·

5. Solve the LPP using graphic method.

Maximize $Z = 50X_1 + 60X_2$

Subject to constraints

$$2X_1 + 3X_2 \le 1500$$
$$3X_1 + 2X_2 \le 1500$$
$$X_1 \le 400$$
$$X_2 \le 400$$
$$X_1, X_2 \ge 0$$

[Ans. $X_1 = 300$, $X_2 = 300$, Max. Z = 33

6. Solve the LPP by using graphic method.

Maximize $Z = 200X_1 + 120X_2$

Subject to constraints

$$40X_1 + 80X_2 \le 800$$
$$10X_1 + 4X_2 \le 80$$
$$X_1 \le 6$$
$$X_2 \le 9$$
$$X_1, X_2 \ge 0$$

[Ans. $X_1 = 5$, $X_2 = 7{\cdot}5$, Max. Z

7. Solve the LPP by using graphic method.

Maximize $Z = 6X_1 + 9X_2$

Subject to restrictions

$$2X_1 + X_2 \le 900$$
$$X_1 + 2X_2 \le 600$$
$$X_2 \le 300$$
$$X_2 \le 400$$
$$2X_1 + 2X_2 \le 1200$$
$$X_1, X_2 \ge 0$$

[Ans. $X_1 = 400$, $X_2 = 100$, Max

8. A machine producing product A and B can produce A by using 2 units of chemicals and compound and can produce B by using 1 unit of chemicals and 2 units of the compoun units of chemicals and 1,000 units of the compound are available. The profit available A and B are respectively Rs. 30 and Rs. 20. Draw a suitable diagram to show the feas Also, find the optimum allocation of units between A and B to maximize the total pro maximum profit.

[Ans. $X_1 = 200$, $X_2 = 400$, Max

9. A firm makes two types of furniture chairs and tables. The contribution for each product as calculated by the accounting department is Rs. 20 per chair and Rs. 30 per table. Both products are processed on three machines M_1, M_2 and M_3. The time required in hours by each product and total time available in hours per week on each machine are as follows :

Machine	Chair	Table	Available Time
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximise contribution ?

[*C.A., May 1979 ; Madurai Univ. M.B.A., 1988*]

[**Ans.** $X_1 = 3$, $X_2 = 9$, Max. $Z = 330$]

10. XYZ chemical company is producing two products A and B. The processing times are 3 hours and 4 hours per unit for A on operations one and two respectively and 4 hours and 5 hours per unit for B on operations one and two respectively. The available time is 18 hours and 21 hours for operation one and two respectively. The product A can be sold at Rs. 3 profit per unit and B at Rs. 8 profit per unit. Solve for maximum profit programme.

[*Rajasthan University M.Com., Oct., 1988*]

[**Ans.** $X_1 = 0$, $X_2 = 4 \cdot 2$, Max. $Z = 33 \cdot 6$]

11. A manufacturer can produce two different products, A and B during a given time period. Each of these products requires four different manufacturing operations : Grinding, Turning, Assembling and Testing. The manufacturing requirements in hours per unit of product are given below for A and B :

	A	B
Grinding	1	2
Turning	3	1
Assembling	6	3
Testing	5	4

The available capacities of these operations in hours for the given time period are : Grinding, 30 ; Turning, 60 ; Assembly, 200 ; Testing 200.

The contribution to profit is Rs. 2 for each unit of A and Rs. 3 for each unit of B. The firm can sell all that it produces at the prevailing market price. Formulate the problem as a linear programming model to maximise profit and solve it by graphic method.

[*Patna Uni. M.B.A., 1988*]

[**Ans.** $X_1 = 18$, $X_2 = 6$, Max. $Z = 54$]

12. A company sells two different products A and B. The company makes a profit of Rs. 40 and Rs. 30 per unit respectively on both products. The two products are produced by a common production process and are sold in two different markets. The production process has a capacity of 30,000 man-hours. It takes 3 man hours to produce one unit of A and one man hour to produce one unit of B. The market has been surveyed and the company officials found out that the maximum units sold for product A and B are 8,000 and 12,000 units respectively. Subject to these limitations, the products can be sold in any combinations. Formulate the above as a linear programming problem and solve it by graphic method.

[**Ans.** $X_1 = 6000$, $X_2 = 12000$, Max. $Z = 600000$]

13. The ABC company has been a producer of picture tubes for television sets and certain printed circuits for radios. The company has just expanded into full scale production and marketing of AM and AM-FM radios. It has built a new plant that can operate 48 hours per week. Production of an AM radio in the new plant will require 2 hours and contribute Rs. 40 to profits while an AM-FM radio will require 3 hours and contribute Rs. 80 to profits. The marketing department, after extensive research, has determined that a maximum of 15 AM radios and 10 AM-FM radios can be sold each week.

(*i*) Formulate a linear programming model to determine the optimal production mix of AM-FM radios that with maximize profits.

(*ii*) Solve the above problem using the graphical method.

[*Delhi Univ. M.B.A. 1982*]

[**Ans.** $X_1 = 9$, $X_2 = 10$, Max. $Z = 1160$]

14. Upon completing the construction of his house Mr. Natragen discovers that 100 square feet of plywood scarp and 80 square feet of white pine scarp are in unsaleable form for the construction of tables and book cases. It takes 16 square feet of plywood and 8 square feet of white pine to make a table ; 12 square feet of plywood and 16 square feet of white pine are required to construct a book case. By selling the finished products to a local furniture store Mr. Natragen can realise a profit of Rs. 25 on each table and Rs. 20 on each book case. How may he most profitably use the left-over wood ? Use Graphical Method to solve the problem.

[*I.C.W.A., Dec., 1984*]

[**Ans.** $X_1 = 4$, $X_2 = 3$, Max. $Z = 160$]

MINIMIZATION PROBLEM WITH ALL ≥ CONSTRAINTS

Now, we shall consider the graphical solution to the LPP of the minimisation nature.

Example 8. Minimise $Z = 40 X_1 + 24 X_2$ (Cost)

Subject to constraints

$$20 X_1 + 50 X_2 \geq 4800$$

$$80 X_1 + 50 X_2 \geq 7200$$

$$X_1 \geq 0, X_2 \geq 0$$

x_1	0	240
x_2	96	0

x_1	0	90
x_2	144	0

Solution :

Convert the inequalities into equations first,

$$20 X_1 + 50 X_2 = 4800$$

If $X_1 = 0$, $X_2 = 96$ (0, 96)

If $X_2 = 0$, $X_1 = 240$ (240, 0)

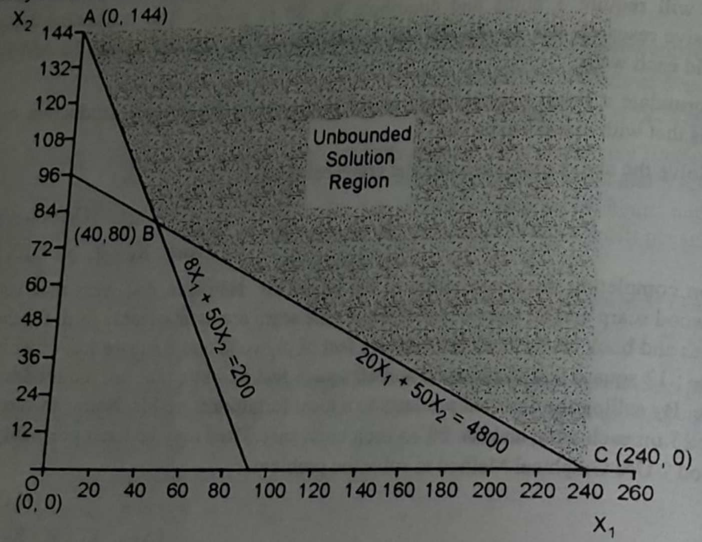
Similarly take the equation

$$80\,X_1 + 50\,X_2 = 7200$$

If $X_1 = 0, X_2 = 144$ (0, 144)

If $X_2 = 0, X_1 = 90$ (90, 0)

Since both the requirements are to be met, the feasible region falls above the common lines (represented by the shaded area).



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 40\,X_1 + 24\,X_2$
A	**(0, 144)**	**40 (0) + 24 (144) = 3456** Min. value
B	(40, 80)	40 (40) + 24 (80) = 3520
C	(240, 0)	40 (240) + 24 (0) = 9600

Total minimum cost = Rs. 3456 where $X_1 = 0,\ X_2 = 144$.

Example 9. Solve the following LPP by Graphic Method :

Minimize $Z = 12\,X_1 + 20\,X_2$

Subject to constraints

$$5\,X_1 + 4\,X_2 \geq 100$$

$$6\,X_1 + 10\,X_2 \geq 120$$

$$X_1 \geq 0,\ X_2 \geq 0$$

Solution : Convert the inequalities into equations

$5\,X_1 + 4\,X_2 = 100$ ...(1)

Put $X_1 = 0,\ X_2 = 25$ (0, 25)

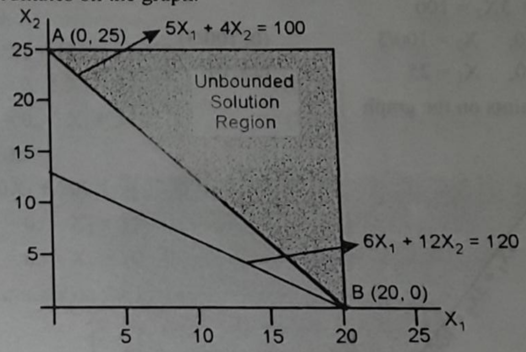
Put $X_2 = 0,\ X_1 = 20$ (20, 0)

$\Rightarrow$ $6X_1 + 10\,X_2 = 120$ ...(2)

Put $X_1 = 0,\ X_2 = 12$ (0, 12)

Put $X_2 = 0,\ X_1 = 20$ (20, 0)

Plot the above coordinates on the graph.



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 12\,X_1 + 20\,X_2$
A	(0, 25)	12 (0) + 20 (25) = 500
B	**(20, 0)**	**12 (20) + 20 (0) = 240**

Cost is minimum at point B *i.e.* Rs. 240, where $X_1 = 20\ X_2 = 0$

Example 10. Vitamin A and B found in foods F_1 and F_2. One unit of food F_1 contains 3 units of vitamin A and 4 units of vitamin B and that of F_2 contains 6 units of vitamin A and 3 units of vitamin B. One unit of food F_1 and F_2 cost Rs. 4 and Rs. 5 respectively. The minimum daily need per person of vitamin A and B is 80 and 100 units respectively. Assuming that anything in excess of daily minimum requirement is harmful, find out the optimum mixture of F_1 and F_2 at the minimum cost which meets the minimum requirement of vitamin A and B.

Formulate this as a linear programming problem. [G.N.D.U. April 1996]

Min $Z = 4X_1 + 5X_2$

$3X_1 + 6X_2 \geq 80$

$4X_1 + 3X_2 \geq 100$

Solution :

Minimize $Z = 4X_1 + 5X_2$

Subject to constraints :

$$3X_1 + 6X_2 \geq 80$$

$$4X_1 + 3X_2 \geq 100$$

$X_1, X_2 \geq 0$ Non negativity constraint

Convert the inequalities into equations first

$$3X_1 + 6X_2 = 80 \quad ...(i)$$
$$4X_1 + 3X_2 = 100 \quad ...(ii)$$

Take equation (*i*),

$$3X_1 + 6X_2 = 80$$

Put $X_1 = 0$, $X_2 = 80/6$ (0, 80/6)

$X_2 = 0$, $X_1 = 80/3$ (80/3, 0)

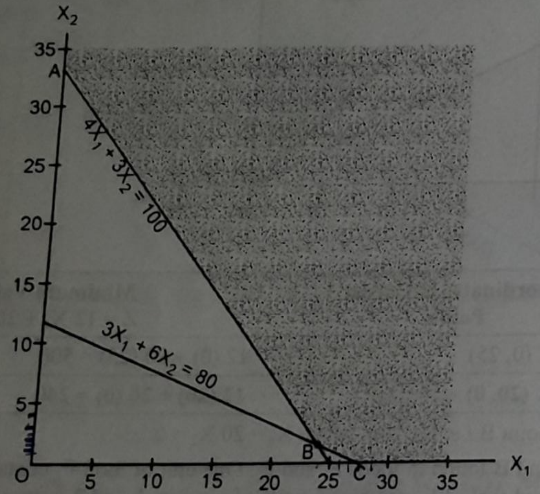
Take equation (*ii*),

$$4X_1 + 3X_2 = 100$$

Put $X_1 = 0$, $X_2 = 100/3$ (0, 100/3)

$X_2 = 0$, $X_1 = 25$ (25, 0)

Plot the above constraints on the graph



Corner Points	Co-ordinates of Corner Points	Minimize $Z = 4X_1 + 5X_2$
A	(0, 100/3)	4(0) + 5(100/3) = 500/3
B	**(24, 4/3)** → or 1·5	**4(24) + 5(4/3) = 308/3**
C	(26·67, 0)	4(26·67) + 5(0) = 106·68

96 + 7.5 = 103.5

80/3

Hence, Min. $Z = 308/3$, where $X_1 = 24$, $X_2 = 4/3$.

Example 11. Solve graphically :

Minimize $Z = 2500X_1 + 3500X_2$

Subject to constraints

$$50X_1 + 60X_2 \geq 2500$$
$$100X_1 + 60X_2 \geq 3000$$
$$100X_1 + 200X_2 \geq 7000$$
$$X_1, X_2 \geq 0$$

[*GNDU April 1996*]

Solution : Convert the inequalities into equations

$$50X_1 + 60X_2 = 2500 \quad ...(i)$$
$$100X_1 + 60X_2 = 3000 \quad ...(ii)$$
$$100X_1 + 200X_2 = 7000 \quad ...(iii)$$

Taking equation (*i*),

$$50X_1 + 60X_2 = 2500$$

Put $X_1 = 0$, $X_2 = 250/6$ (0, 250/6)

$X_2 = 0$, $X_1 = 50$ (50, 0)

Taking equation (*ii*),

$$100X_1 + 60X_2 = 3000$$

Put $X_1 = 0$, $X_2 = 50$ (0, 50)

$X_2 = 0$, $X_1 = 30$ (30, 0)

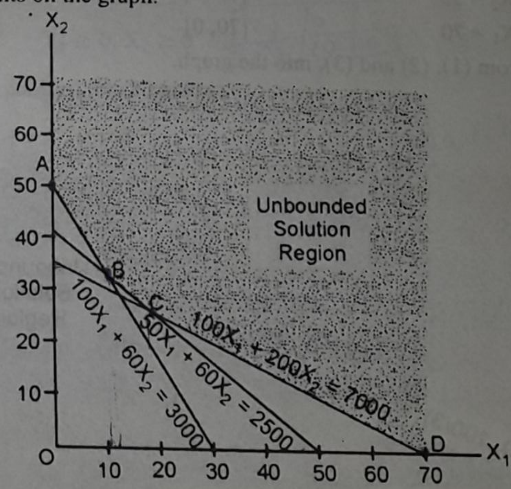
Taking equation (*iii*),

$$100X_1 + 200X_2 = 7000$$

Put $X_1 = 0$, $X_2 = 35$ (0, 35)

$X_2 = 0$, $X_1 = 70$ (70, 0)

Plot the above constraints on the graph.



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 2500X_1 + 3500X_2$
A	(0, 50) ✓	2500(0) + 3500(50) = 175000
B	(10, 100/3) → or 34	2500(10) + 3500(100/3) = 425000/3 = 144…
C	**(20, 25)**	**2500(20) + 3500(25) = 1,37,500** ✓ Min.
D	(70, 0) ✓	2500(70) + 3500(0) = 1,75,000

Hence Min. $Z = 1{,}37{,}500$ where $X_1 = 20$, $X_2 = 25$.

Example 12. Minimize $Z = 5000\,X_1 + 7000\,X_2$

Subject to constraints

$$100\,X_1 + 120\,X_2 \geq 5000$$
$$200\,X_1 + 120\,X_2 \geq 6000$$
$$200\,X_1 + 400\,X_2 \geq 14000$$
$$X_1 \geq 0 \text{ and } X_2 \geq 0$$

Solution :

Convert the inequalities into equations

$100\,X_1 + 120\,X_2 = 5000$ ...(1)

Put $X_1 = 0,\quad X_2 = \frac{5000}{120}$ *i.e.* $\frac{125}{3}$ $\left[0, \frac{125}{3}\right]$

$X_2 = 0,\ X_1 = 50$ [50, 0]

$200\,X_1 + 120\,X_2 = 6000$ ...(2)

Put $X_1 = 0,\ X_2 = 50$ [0, 50]

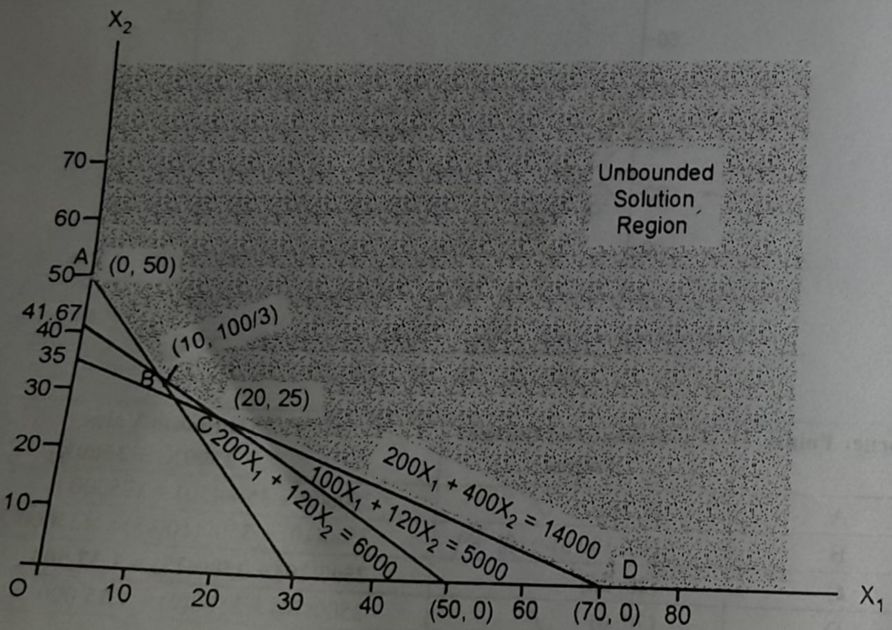
$X_2 = 0,\ X_1 = 30$ [30, 0]

$200\,X_1 + 400\,X_2 = 14000$ ...(3)

Put $X_1 = 0,\ X_2 = 35$ [0, 35]

Put $X_2 = 0,\ X_1 = 70$ [70, 0]

Put the co-ordinates from (1), (2) and (3), into the graph.



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 5000\,X_1 + 7000\,X_2$
A	(0, 50)	5000 (0) + 7000 (50) = 350000
B	(10, 100/3)	$5000\,(10) + 7000\left(\frac{100}{3}\right) = 283333.33$
C	**(20, 25)**	**5000 (20) + 7000 (25) = 275000**
D	(70, 0)	5000 (70) + 7000 (0) = 350000

So, Min. cost is at point C = 2,75,000, where $X_1 = 20$ and $X_2 = 25$

Example 13. Minimize $Z = 20\,X_1 + 40\,X_2$

see min. value

Subject to constraints

$$36X_1 + 6X_2 \geq 108 \quad \text{(I)}$$
$$3X_1 + 12X_2 \geq 36 \quad \text{(II)}$$
$$20X_1 + 10X_2 \geq 100 \quad \text{(III)}$$
$$X_1 \geq 0,\ X_2 \geq 0$$

x_1	0	3
x_2	18	0

x_1	0	12
x_2	3	0

x_1	0	5
x_2	10	0

Solution : Convert the inequalities into equations

⇒ $36\,X_1 + 6\,X_2 = 108$

Yani Max. Point

Put $X_1 = 0,\quad X_2 = 18$ (0, $\boxed{18}$)

Put $X_2 = 0,\quad X_1 = 3$ (3, 0)

⇒ $3\,X_1 + 12\,X_2 = 36$

Put $X_1 = 0,\quad X_2 = 3$ (0, 3)

Yani Max. Point

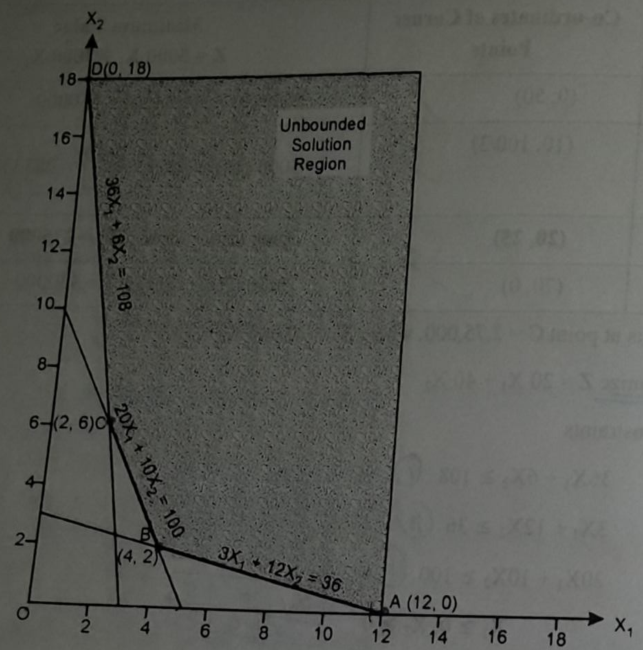
Put $X_2 = 0,\quad X_1 = 12$ ($\boxed{12}$, 0)

⇒ $20\,X_1 + 10\,X_2 = 100$

Put $X_1 = 0,\quad X_2 = 10$ (0, 10)

Put $X_2 = 0,\quad X_1 = 5$ (5, 0)

Now plot the above constraints on the graph.



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 20X_1 + 40X_2$
A	(12, 0)	20 (12) + 40 (0) = 240
B	(4, 2)	**20 (4) + 40 (2) = 160** Min Value
C	(2, 6)	20 (2) + 40 (6) = 280
D	(0, 18)	20 (0) + 40 (18) = 720

So, Minimise cost is at point B = Rs. 160 where $X_1 = 4, X_2 = 2$

EXERCISE 3.2

Use graphic method to solve

Min. $Z = 9X_1 + 10X_2$

Subject to constraints

$$2X_1 + 4X_2 \geq 50$$
$$4X_1 + 3X_2 \geq 24$$

$$3X_1 + 2X_2 \geq 60$$
$$X_1, X_2 \geq 0$$

[Ans. $X_1 = 35/2, X_2 = 15/4$, Min. $Z = 195$]

2. Min $Z = -X_1 + 2X_2$

Subject to constraints

$$-X_1 + 3X_2 \leq 10$$
$$X_1 + X_2 \leq 6$$
$$X_1 - X_2 \leq 2$$
$$X_1, X_2 \geq 0$$

[Ans. $X_1 = 2, X_2 = 0$, Min. $Z = -2$]

3. Min $Z = 2X_1 - 10X_2$

Subject to constraints

$$X_1 - X_2 \geq 0$$
$$X_1 - 5X_2 \geq -5$$
$$X_1, X_2 \geq 0$$

[Ans. $X_1 = 5/4, X_2 = 5/4$, Min. $Z = -10$]

4. Min $Z = 600X_1 + 400X_2$

Subject to constraints

$$150X_1 + 150X_2 \geq 2000$$
$$300X_1 + 100X_2 \geq 4000$$
$$200X_1 + 500X_2 \geq 4400$$
$$X_1, X_2 \geq 0$$

[Ans. $X_1 = 12, X_2 = 4$, Min. $Z = 8800$]

5. Arsh Ltd. has two products A and B. To produce one unit of A, 2 units of material X and 4 units of material Y are required. To produce one unit of B, 3 units of material X and 2 units of material Y are required. At least 16 units of each material must be used in order to meet the committed sales of A and B cost per unit of material X and material Y are Rs. 6 per unit and Rs. 8 per unit respectively.

You are required :

(*i*) to formulate mathematical model

(*ii*) to solve it for the minimum cost (Graphically).

[Ans. $X_1 = 2, X_2 = 4$, Min. Z = 4

6. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 14 calories. Two foods A and B are available at a cost of Rs. 4 and Rs. 3 per unit respectively. If unit of A contains 200 units of vitamins, 1 unit of mineral and 40 calories and one unit of foo contains 100 units of vitamins, 2 units of minerals and 40 calories, find by graphic method combination of foods be used to have least cost ?

[Ans. $X_1 = 5, X_2 = 30$, Min. Z =

7. The vitamins A and B are found in two different foods F_1 and F_2. One unit of food F_1 contains units of vitamins A and four units of vitamin B. One unit of food F_2 contains five units of vitamin A and two units of vitamin B. One unit of food F_1 and F_2 cost Rs. 3 and Rs. 2.50, respectively. The minimum daily requirement for a person of vitamin A and B is 40 and 50 units, respectively. Find the optimum mix of food F_1 and F_2 at the minimum cost which meets the daily minimum requirement of vitamin A and B. Assume that anything in excess of daily minimum requirement of vitamin A and B is not harmful, formulate the given problem as a linear programming problem and solve it by graphic method.

[**Ans.** $X_1 = 85/8$, $X_2 = 15/4$, Min. $Z = 41.25$]

8. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. Because of the need to ensure nutrient constituents, it is necessary to buy additional one or two products, which we shall call A and B. The nutrient constituents (vitamins and proteins) in each unit of the product are given below :

Nutrient constituents	Nutrient in the product		Minimum requirement of nutrient constituents
	A	B	
X	36	6	108
Y	3	12	36
Z	20	10	100

Product A costs Rs. 20 per unit and product B costs Rs. 40 per unit. Determine how much of products A and B must be purchased so as to provide the pigs nutrients not less than the minimum required at the lowest possible cost. Solve graphically.

[*I.C.W.A., Dec., 1985*]

[**Ans.** $X_1 = 4$, $X_2 = 2$, Min. $Z = 160$]

9. G.J. Breveries Ltd. have two bottling plants, one located at 'G' and the other at 'J'. Each plant produces three drinks – whisky, beer and brandy named A, B and C respectively. The number of bottles produced per day are as follows :

Drink	Plant at	
	G	J
Whisky	1500	1500
Beer	3000	1000
Brandy	2000	5000

A market survey indicates that during the month of July, there will be a demand of 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at G and J are 600 and 400 monetary units. For how many days each plant be run in July so as to minimise the production cost, while still meeting the market demand ? Solve graphically.

[**Ans.** $X_1 = 12$, $X_2 = 4$, Min. $Z = 8800$]

MIXED CONSTRAINTS PROBLEM

Example 14. Solve graphically :

Minimise $Z = 6\,X_1 + 14\,X_2$

Subject to constraints

$$5\,X_1 + 4\,X_2 \geq 60$$
$$3\,X_1 + 7\,X_2 \leq 84$$
$$X_1 + 2\,X_2 \geq 18$$
$$X_1 \geq 0,\; X_2 \geq 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$ $5X_1 + 4\,X_2 = 60$ ...(*i*)

Put $X_1 = 0, X_2 = 15$ (0, 15)

Put $X_2 = 0, X_1 = 12$ (12, 0)

$3\,X_1 + 7\,X_2 = 84$ ...(*ii*)

Put $X_1 = 0,\; X_2 = 12$ (0, 12)

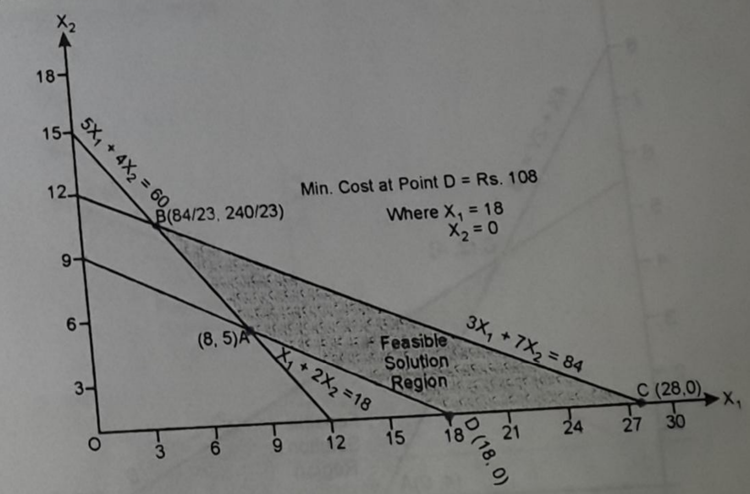
Put $X_2 = 0,\; X_2 = 28$ (28, 0)

$X_1 + 2\,X_2 = 18$ ...(*iii*)

Put $X_1 = 0,\; X_2 = 9$ (0, 9)

Put $X_2 = 0,\; X_1 = 18$ (18, 0)

Taking all points from (*i*), (*ii*) and (*iii*) and plot them on the graph.



Corner Point	Coordinates of Corner Point	Minimum Value $Z = 6X_1 + 14X_2$
A	(8, 5)	6 (8) + 14 (5) = 118
B	(84/23, 240/23)	6 (84/23) + 14 (240/23) = 168
C	(28, 0)	6 (28) + 14 (0) = 168
D	**(18, 0)**	**6 (18) + 14 (0) = 108**

Minimize $Z = 108$, $X_1 = 18$, $X_2 = 0$.

Example 15. Minimise $Z = 6X + 8Y$

Subject to constraints

$$2X + 3Y \le 16$$
$$4X + 2Y \ge 16$$
$$X \ge 0, Y \ge 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$ $2X + 3Y = 16$ ...(*i*)

Put $X = 0$, $Y = 16/3$ (0, 16/3)

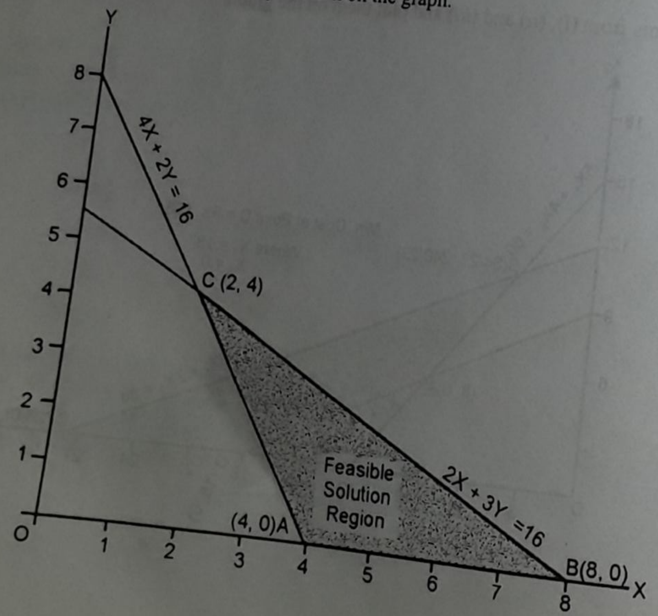
Put $Y = 0$, $X = 8$ (8, 0)

$\Rightarrow$ $4X + 2Y = 16$ ...(*ii*)

Put $X = 0$, $Y = 8$ (0, 8)

$Y = 0$, $X = 4$ (4, 0)

Take the values from (*i*) and (*ii*) and plot them on the graph.



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 6X + 8Y$
A	(4, 0)	**6 (4) + 8 (0) = 24**
B	(8, 0)	6 (8) + 8 (0) = 48
C	(2, 4)	6 (2) + 8 (4) = 44

Optimum solution is at point A with minimum value of 24 where $X = 4$ and $Y = 0$

Example 16. Minimize $Z = 2X_1 + X_2$

Subject to constraints

$$5X_1 + 10X_2 \le 50$$
$$X_1 + X_2 \ge 1$$
$$X_1 \le 4$$
$$X_1 \ge 0, X_2 \ge 0$$

Solution : Convert the inequalities into equations first

$5X_1 + 10X_2 = 50$

If $X_1 = 0$, $X_2 = 5$ (0, 5)

If $X_2 = 0$, $X_1 = 10$ (10, 0)

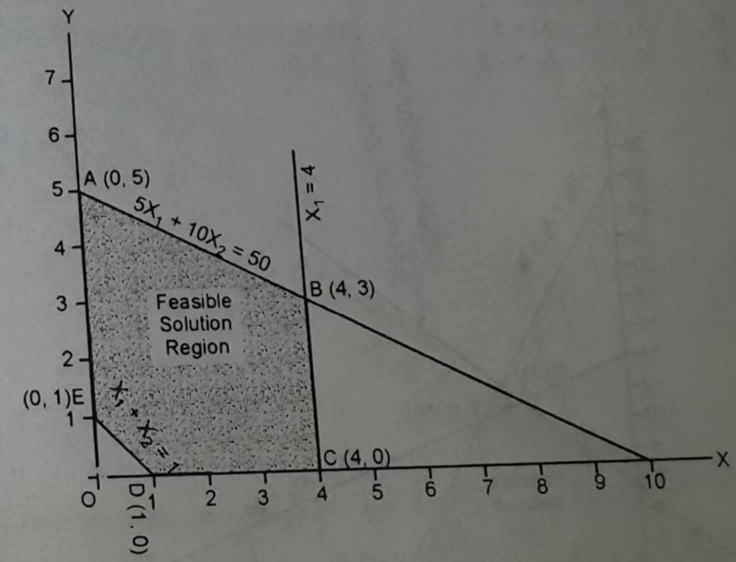
$\Rightarrow$ $X_1 + X_2 = 1$

If $X_1 = 0$, $X_2 = 1$ (0, 1)

If $X_2 = 0$, $X_1 = 1$ (1, 0)

$X_1 = 4$ (4, 0)

Plot these constraints on the graph.



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 2X_1 + X_2$
A	(0, 5)	2 (0) + 1(5) = 5
B	(4, 3)	2 (4) + 1(3) = 11
C	(4, 0)	2 (4) + 1(0) = 8
D	(1, 0)	2, (1) + 1(0) = 2
E	**(0, 1)**	**2 (0) + 1(1) = 1**

Optimum Solution : At point E with minimum value 1 where $X_1 = 0$, $X_2 = 1$

Example 17. Solve graphically :

Maximize $Z = 10X_1 + 15X_2$

Subject to constraints

$$2X_1 + X_2 \leq 26$$
$$2X_1 + 4X_2 \leq 56$$
$$X_1 - X_2 \boxed{\geq} -5 \quad \Rightarrow X_2 - X_1 \leq 5$$
$$X_1 \geq 0, X_2 \geq 0$$

[MBA (G.N.D.U.) MS Dec. 1999]

X_1	0	-5
X_2	5	0

Solution : Convert the inequalities into equations first

⇒ $2X_1 + X_2 = 26$

If $X_1 = 0$, $X_2 = 26$ (0, 26)

If $X_2 = 0$, $X_1 = 13$ (13, 0)

⇒ $2X_1 + 4X_2 = 56$

If $X_1 = 0$, $X_2 = 14$ (0, 14)

If $X_2 = 0$, $X_1 = 28$ (28, 0)

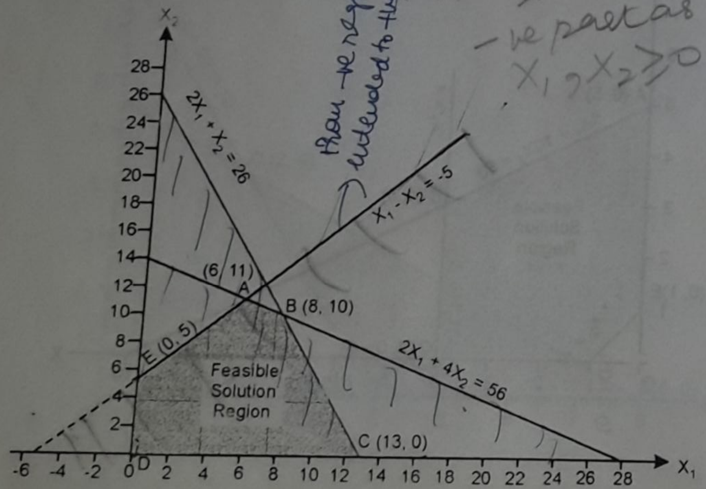
⇒ $X_1 - X_2 = -5$

If $X_1 = 0$, $X_2 = 5$ (0, 5)

If $X_2 = 0$, $X_1 = -5$ (– 5, 0)

than -ve region extended to the region

Do not take the -ve part as $X_1, X_2 \geq 0$



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 10X_1 + 15X_2$	
A	(6, 11)	10 (6) + 15 (11)	= 225
B	**(8, 10)**	**10 (8) + 15 (10)**	**= 230**
C	(13, 0)	10 (13) + 15 (0)	= 130
D	(0, 0)	10 (0) + 15 (0)	= 0
E	(0, 5)	10 (0) + 15 (5)	= 75

Optimum solution : At point B with maximum value *i.e.* 230, where $X_1 = 8$ and $X_2 = 10$.

Example 18. Solve by graphical method :

Minimise $Z = 5X_1 + 7X_2$

Subject to constraints

$$X_1 + 2X_2 \leq 20$$
$$5X_1 + X_2 \geq 15$$
$$4X_1 + 3X_2 \leq 60$$
$$X_1 \geq 0, X_2 \geq 0$$

(MA (EW) – I.G.N.D.U. 2001)

Solution : Convert the inequalities into equations first

⇒ $X_1 + 2X_2 = 20$ ...(*i*)

If $X_1 = 0$, $X_2 = 10$ (0, 10)

If $X_2 = 0$, $X_1 = 20$ (20, 0)

⇒ $5X_1 + X_2 = 15$ ...(*ii*)

If $X_1 = 0$, $X_2 = 15$ (0, 15)

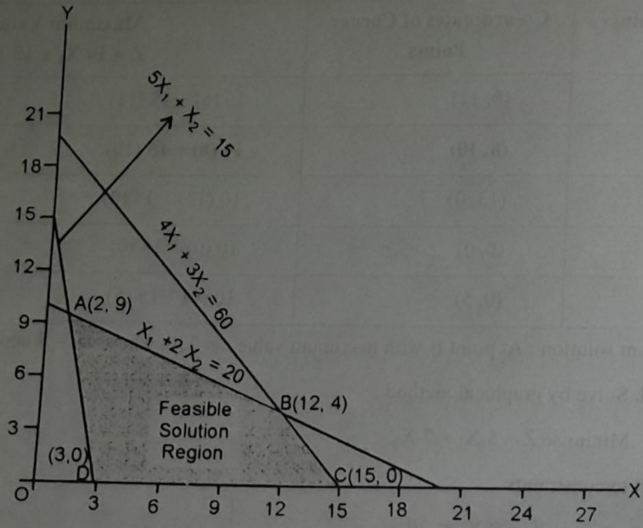
If $X_2 = 0$, $X_1 = 3$ (3, 0)

⇒ $4X_1 + 3X_2 = 60$ ...(*iii*)

If $X_1 = 0$, $X_2 = 20$ (0, 20)

If $X_2 = 0$, $X_1 = 15$ (15, 0)

Plot the above constraints on the graph



Corner Points	Co-ordinates of Corner Points	Minimum Value $Z = 5X_1 + 7X_2$
A	(1·11, 9·44)	5 (1·11) + 7(9·44) = 71·63
B	(12, 4)	5 (12) + 7 (4) = 88
C	(15, 0)	5 (15) + 7 (0) = 75
D	**(3, 0)**	**5 (3) + 7 (0) = 15**

From the above table, it is evident that solution lies at point D where $X_1 = 3, X_2 = 0$ and $Z = 15$

Example 19. Max. $Z = 500000\ X_1 + 100000\ X_2$

Subject to constraints

$$50{,}000\ X_1 + 20000\ X_2 \leq 200000$$
$$X_1 \geq 3$$
$$X_2 \leq 5$$
$$X_1 \geq 0,\ X_2 \geq 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$ $50000\ X_1 + 20000\ X_2 = 200000$ ...(*i*)

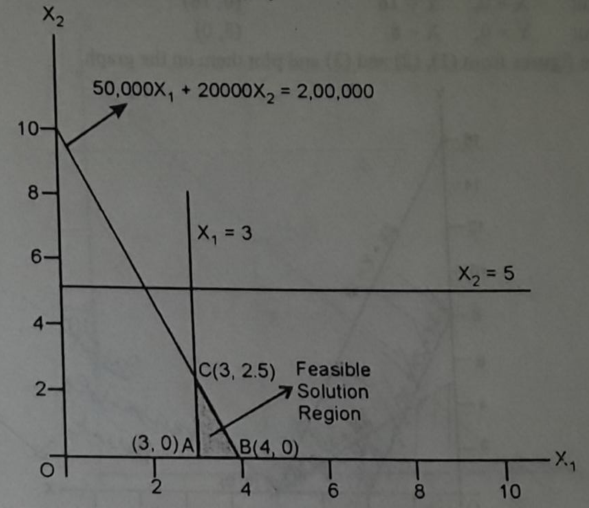
Put $X_1 = 0,\ X_2 = 10$ (0, 10)

Put $X_2 = 0,\ X_1 = 4$ (4, 0)

$\Rightarrow$ $X_1 = 3$ ...(*ii*)

$\Rightarrow$ $X_2 = 5$ ...(*iii*)

Plot all the figures from (*i*), (*ii*) and (*iii*) in the graph.



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 500000\ X_1 + 100000\ X_2$
A	(3, 0)	500000 (3) + 100000 (0) = 15,00,000
B	**(4, 0)**	**500000 (4) + 100000 (0) = 20,00,000**
C	(3, 2·5)	500000 (3) + 100000 (2·5) = 17,50,000

Optimal Solution lies at point B, where maximum value is 20,00,000 and where $X_1 = 4, X_2 = 0$.

Imp

Example 20. Solve Graphically :

Maximize $Z = 6\ X + 8\ Y$

Subject to constraints

$$2\ X + 3\ Y \leq 16$$
$$4\ X + 2\ Y \geq 16$$
$$2\ X + Y = 16$$
$$X \geq 0,\ Y \geq 0$$

Solution : Convert the inequalities into equalities

$\Rightarrow$ $2\ X + 3\ Y = 16$ ...(1)

Put $X = 0,\ Y = \frac{16}{3}$ = 5.33 (0, 16/3)

Put $Y = 0,\ X = 8$

$\Rightarrow$ $4\ X + 2\ Y = 16$ ...(2

Put $X = 0,\ Y = 8$ (0, 8)

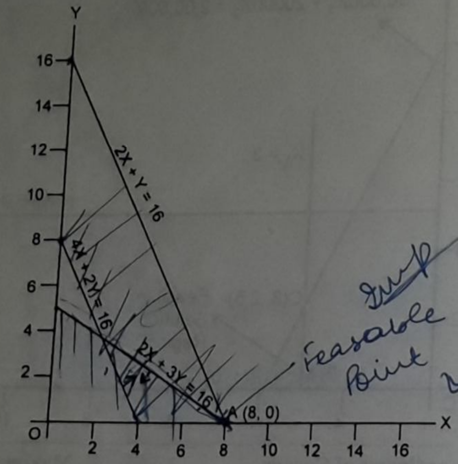
Put $Y = 0,\ X = 4$ (4, 0)

$\Rightarrow$ $2\ X + Y = 16$ ...(

Put $X = 0$, $Y = 16$ (0, 16)

Put $Y = 0$, $X = 8$ (8, 0)

Taking the figures from (1), (2) and (3) and plot them on the graph.



Imp Feasable Point

Corner Point	Co-ordinates of Corner Point	Maximum Value $Z = 6X + 8Y$
A	(8, 0)	6 (8) + 8 (0) = 48

Optimal solution : We obtain only one corner point *i.e.* A.

Hence value of $Z = 48$ where $X = 8$, $Y = 0$

Example 21. Solve the following LPP graphically.

Max. $Z = 4X_1 + 6X_2$

Subject to constraints

$$X_1 + X_2 = 5$$
$$X_1 \geq 2$$
$$X_2 \leq 4$$
$$X_1 \geq 0, X_2 \geq 0$$

Solution : Convert the inequalities into equalities

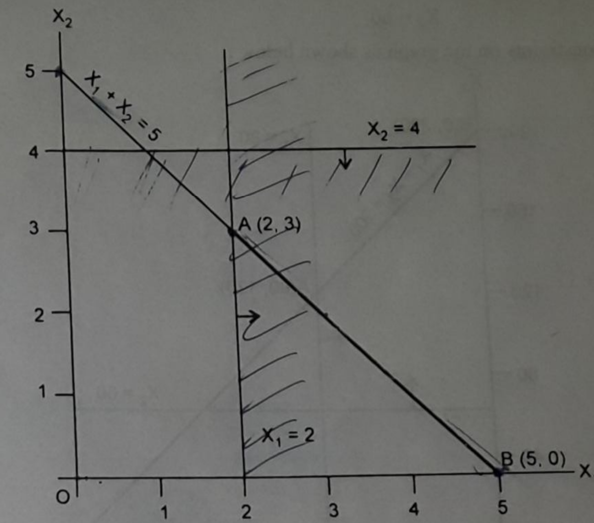
$$X_1 + X_2 = 5$$

Put $X_1 = 0$, $X_2 = 5$ (0, 5)

Put $X_2 = 0$, $X_1 = 5$ (5, 0)

$$X_1 = 2$$
$$X_2 = 4$$

Plot the constraints on the graph



Feasible region lies on line AB, (*i.e.* between point A and point B).

Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 4X_1 + 6X_2$
A	(2, 3)	4 (2) + 6 (3) = 26
B	(5, 0)	4 (5) + 6 (0) = 20

Maximum value of Z is 26, where $X_1 = 2$, $X_2 = 3$.

Example 22. Minimize $Z = 3X_1 + 5X_2$

Subject to constraints

$$X_1 + X_2 = 200$$
$$X_1 \leq 80$$
$$X_2 \geq 60$$
$$X_1 \geq 0, \quad X_2 \geq 0$$

Solution : Convert the inequalities into equations

$$X_1 + X_2 = 200$$

Put $X_1 = 0$, $X_2 = 200$ (0, 200)

Put $X_2 = 0$, $X_1 = 200$

Shading is not possible as the above one is already in equality form.

$$X_1 = 80$$

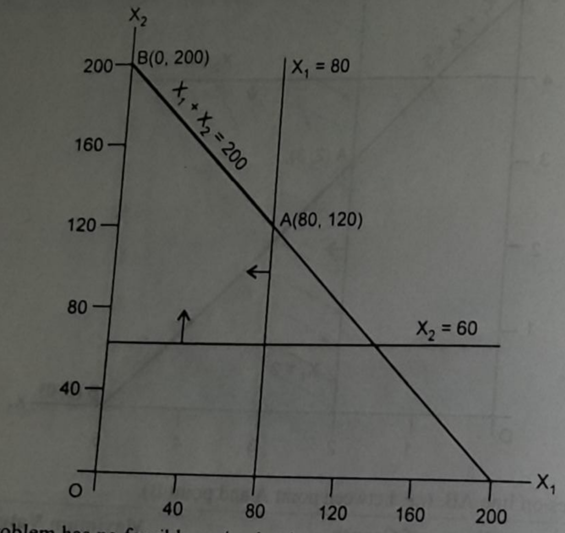
Condition :

Inequalities : $X_1 < 80$

Constraint Satisfaction : On and below the line *i.e.* towards origin.

$$X_2 = 60$$

Plot the constraints on the graph as shown below :



The given problem has no feasible region but has only as feasible points lying between A and B

Feasible Points	Co-ordinates of Points	Minimum Value $Z = 3X_1 + 5X_2$
A	**(80, 120)**	**3 (80) + 5 (120) = 840**
B	(0 , 200)	3 (0) + 5 (200) = 1000

Hence, optimum solution is at point A where $X_1 = 80$, $X_2 = 120$ Minimize Z = 840.

Example 23. Solve the following L.P.P. using graphic method :

Min. Z = 3X + 2Y

Subject to

$$3X + Y \geq 10$$
$$X + Y \geq 6$$
$$X + 4Y \geq 12$$
$$X, Y \geq 0$$

(*G.N.D.U. B.A./BSc.II, Q.T-A. Sept., 2006*)

Sol. To plot the constraints on the graph, we will convert the inequalities into equations temporarily

$$3X + Y = 10 \quad ...(1)$$
$$X + Y = 6 \quad ...(2)$$
$$X + 4Y = 12 \quad ...(3)$$

In equation (1) 3X + Y = 10

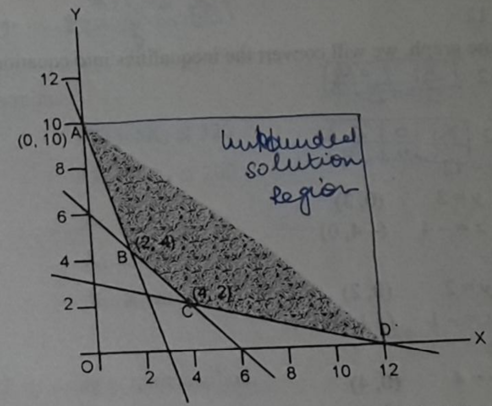
if	X = 0 ;	Y = 10	(0, 10)
if	Y = 0 ;	X = 10/3	(10/3, 0)

In equation (2) X + Y = 6

if	X = 0 ;	Y = 6	(0, 6)
if	Y = 0 ;	X = 6	(6, 0)

In equation (3) X + 4Y = 12

if	X = 0 ;	Y = 3	(0, 3)
if	Y = 0 ;	X = 12	(12, 0)



The corner points of the feasible region are A(0, 10), B(2, 4), C(4, 2) and D(12, 0).

The value of the objective function at corner points can be determined as follows :

Corner Points	Co-ordinates (X, Y)	Objective Function 3X + 2Y	Value
A	(0, 10)	3(0) + 2(10)	20
B	**(2, 4)**	**3(2) + 2(4)**	**14**
C	(4, 2)	3(4) + 2(2)	16
D	(12, 0)	3 (12) + 2(0)	36

Maximum value of Z lies at point B.

Hence the required solution is X = 2, Y = 4, Z = 14.

Example 24. Solve the following LPP, using graphic method :

Minimize Z = 3 x + 5 y

Subject to

$$3x - 4y + 12 \geq 0$$

$3x + 4y \geq -12$

$4y - 3x \leq 12$

$3x - 4y \geq -12$

$4y - 3x \leq 12$

☆ digits on right side of inequality sign should always be +ve.

$$2x - y + 2 \geq 0$$
$$2x + 3y - 12 \geq 0$$
$$x, y \geq 0$$

(G.N.D.U. B.A../BSc.II Q.T-A. April, 2004)

Sol. The given problem can be rewritten as below :

Minimize $Z = 3x + 5y$

Subject to

$-3x + 4y \leq 12$ ...(1) , or $3x - 4y = -12$, $y = 0$, $x = \frac{-12}{3} = -4$ $(-4, 0)$; $x = 0$, $4y = 12$, $y = 3$ $(0, 3)$

$-2x + y \leq 2$ ...(2)

$2x + 3y \geq 12$ ...(3)

To plot the constraints on the graph, we will convert the inequalities into equations temporarily

$-3x + 4y = 12$ ...(1)

$-2x + y = 2$ ...(2)

$2x + 3y = 12$ ...(3)

To constraint (1) $-3x + 4y = 12$

if $x = 0, y = 3$ $(0, 3)$

if $y = 0, x = -4$ $(-4, 0)$

In equation (2) $-2x + y = 2$

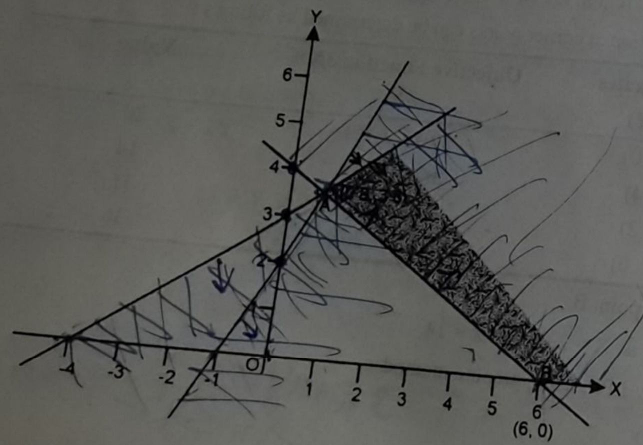
if $x = 0, y = 2$ $(0, 2)$

if $y = 0, x = -1$ $(-1, 0)$

In equation (3) $2x + 3y = 12$

if $x = 0, y = 4$ $(0, 4)$

if $y = 0, x = 6$ $(6, 0)$



The corner points of the solution space are $A(0 \cdot 8, 3 \cdot 6)$ and $B(6, 0)$.

The value of the objective function at the two corner points can be determined as follows :

Corner Points	Co-ordinates (X, Y)	Objective Function $3x + 5y$	Value
A	$(0 \cdot 8, 3 \cdot 6)$	$3(0 \cdot 8) + 5(3 \cdot 6)$	$20 \cdot 4$
B	$(6, 0)$	$3(6) + 5(0)$	**18**

Minimum value of Z lies at point B.

Hence the required solution is $X = 6$, $Y = 0$, $Z = 18$.

EXERCISE 3.3

1. Solve the LPP by using graphic method.

Maximize $Z = 1.75X_1 + 1.50X_2$

Subject to constraints

$$8X_1 + 5X_2 \leq 320$$
$$4X_1 + 5X_2 \leq 200$$
$$X_1 \geq 15$$
$$X_2 \leq 10$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 33 \cdot 75$, $X_2 = 10$, Max. $Z = 74 \cdot 06$]

2. Solve the LPP by using graphic method.

Minimum $Z = 20X_1 + 10X_2$

Subject to constraints

$$X_1 + 2X_2 \leq 40$$
$$3X_1 + X_2 \geq 30$$
$$4X_1 + 3X_2 \geq 60$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 6$, $X_2 = 12$, Min. $Z = 240$]

3. Solve the LPP by using graphic method.

Maximum $Z = 120X_1 + 100X_2$

Subject to constraints

$$10X_1 + 5X_2 \leq 80$$
$$6X_1 + 6X_2 \leq 66$$
$$4X_1 + 8X_2 \geq 24$$

$$5X_1 + 6X_2 \le 90$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 5$, $X_2 = 6$, Max. $Z = 1200$]

4. Solve the LPP by using graphic method.

Minimum $Z = 4X_1 - 2X_2$

Subject to constraints

$$X_1 + X_2 \le 14$$
$$3X_1 + 2X_2 \ge 36$$
$$2X_1 + X_2 \le 24$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 8$, $X_2 = 6$, Min. $Z = 20$]

5. Solve the LPP by using graphic method.

Maximum $Z = 30X_1 + 40X_2$

Subject to constraints

$$X_1 \le 20$$
$$X_2 \ge 10$$
$$4X_1 + 2X_2 \le 100$$
$$4X_1 + 6X_2 \le 180$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 15$, $X_2 = 20$, Max. $Z = 1250$]

6. The manager of an oil refinery must decided on the optimal mix of two possible blending processes of which tae inputs and outputs per production run are as follows :

Process	Input (units) Crude A	Crude B	Output (units) Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amount available of crudes A and B is 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Solve the LPP by graphical method.

[**Ans.** $X_1 = 400/13$, $X_2 = 150/13$, Max. $Z = 180000/13$]

7. A manufacturer produces two different models, X and Y of the same product. The raw materials r_1 and r_2 are required for production. At least 18 kg of r_1 and 12 kgs of r_2 must be used daily. Also at most 34 hours of labour are to be utilised 2 kg of r_1 are needed for each model X and 1 kg of r_1 for each model Y. For each model of X and Y, 1 kg of r_2 is required. It takes 3 hours to manufacture a model X and 2 hours to manufacture a model Y. The profit is Rs. 50 for each model X and Rs. 30 for each model Y. How many units of each model should be produced to maximize the profit.

[**Ans.** $X_1 = 10$, $X_2 = 2$, Max. $Z - 560$]

8. An advertising agency wishes to reach two types of audiences – Customers with annual income greater than Rs. 15,000 (target audience A) and customers with annual income of less than Rs. 15,000 target audience B). The total advertising budget is Rs. 2,00,000. One programme of TV advertising costs 50,000 ; one programme of radio advertising costs Rs. 20,000. For contract reasons, at least 3 programmes ought to be on TV and the number of radio programme must be limited to 5. Surveys indicate that a single TV programme reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to maximize the total reach.

[*Delhi Univ. M.B.A., April 1985*]

[**Ans.** $X_1 = 4$, $X_2 = 0$, Max. $Z = 2000000$]

9. Solve the LPP by using graphic method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$-2X_1 + X_2 = 1$$
$$X_1 \le 2$$
$$X_1 + X_2 \le 3$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 2/3$, $X_2 = 7/3$, Max. $Z = 20/3$]

10. Solve the LPP by using graphic method.

Minimize $Z = 3X_1 + 8X_2$

Subject to constraints

$$X_1 + X_2 = 200$$
$$X_1 \ge 80$$
$$X_2 \ge 60$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 140$, $X_2 = 60$, Min. $Z = 900$]

11. Solve the LPP by using graphic method.

Minimize $Z = 8X_1 + 2X_2$

Subject to constraints

$$X_1 = 80$$
$$X_1 + X_2 = 140$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 80$, $X_2 = 60$, Min. $Z = 7$

12. Solve the LPP by using graphic method.

Minimize $Z = 2X_1 + 8X_2$

Subject to constraints

$$5X_1 + 10X_2 = 150$$
$$X_1 \le 20$$
$$X_2 \ge 14$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 2$, $X_2 = 14$, Min. Z

13. Solve the LPP by using graphic method.

Maximize $Z = 6X_1 + 8X_2$

Subject to constraints

$$2X_1 + 3X_2 \leq 16$$
$$4X_1 + 2X_2 \geq 16$$
$$2X_1 + X_2 = 16$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 8$, $X_2 = 0$, Max. $Z = 48$]

14. Solve the LPP by using graphic method.

Minimize $Z = 8X_1 + 4X_2$

Subject to constraints

$$3X_1 + X_2 \geq 27$$
$$X_1 + X_2 = 21$$
$$X_1 + 2X_2 \leq 40$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 3$, $X_2 = 18$, Min. $Z = 96$]

15. A firm has an advertising budget of Rs. 7,20,000. It wishes to allocate this budget to media : magazines and television, so that total exposure is maximised. Each page of magazine advertising is estimated to result in 60,000 exposures, whereas each spot on television is estimated to result in 1,20,000 exposures. Each page of magazine advertising costs Rs. 9,000 and each spot on television costs Rs. 12,000. An additional condition is that the firm has specified that at least two pages of magazine advertising be used and at least 3 spots on television. Determine the optimal media-mix for this firm.

[**Ans.** $X_1 = 2$, $X_2 = 58 \cdot 5$, Max. $Z = 7140000$]

Anita Electric Company produces two products P_1 and P_2. Products are produced and sold on a weekly basis. The weekly production cannot exceed 25 for product P_1 and 35 for product P_2 because of limited available facilities. The company employs total of 60 workers. Product P_1 requires 2 man-weeks of labour, while P_2 requires one man-week of labour. Profit margin on P_1 is Rs. 60 and on P_2 is Rs. 40. Formulate it a linear programming problem and solve for maximum profit.

[**Ans.** $X_1 = 12 \cdot 5$, $X_2 = 35$, Max. $Z = 2150$]

EXCEPTIONAL CASES

I. INFEASIBLE SOLUTION

It has already been stated that a solution is said to be feasible if it satisfies all the constraints and the non-negativity conditions. Sometimes the given linear programming problem does not possess a solution that satisfies all the constraints simultaneously. Graphically, the problem is formulated with conflicting constraints and no feasible solution region exists and problem is said to have infeasible solution.

Example 25. Maximise $Z = 4X + 3Y$

Subject to constraints

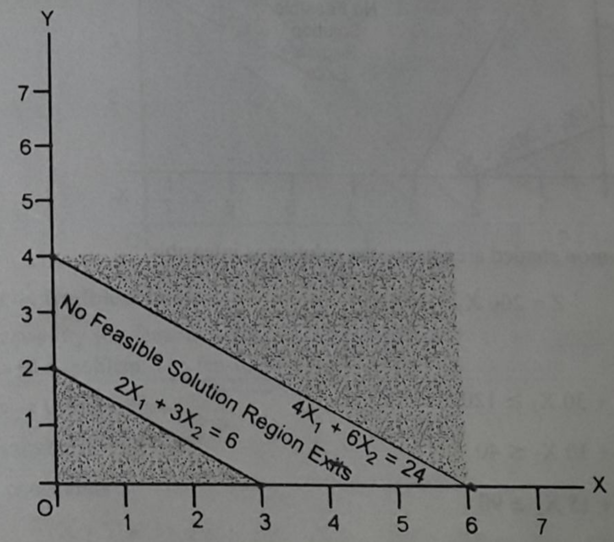
$$2X + 3Y \leq 6$$
$$4X + 6Y \geq 24$$
$$X \geq 0, Y \geq 0$$

Solution : Convert the inequalities into equations

$2X + 3Y = 6$

Put	$X = 0$,	$Y = 2$	[3, 2]	...(*i*)
Put	$Y = 0$,	$X = 3$		
⇒	$4X + 6Y = 24$			
Put	$X = 0$,	$Y = 4$	[6, 4]	...(*ii*)
Put	$Y = 0$,	$X = 6$		

Taking points from (*i*) and (*ii*), plot the graph.



These exists no feasible solution to this problem as there does not exist common shaded area.

Example 26. Maximize $30 X_1 + 20 X_2$

Subject to constraints

$$10 X_1 + 20 X_2 \le 20$$
$$20 X_1 + 10 X_2 \ge 60$$
$$X_1 \ge 0 \; X_2 \ge 0$$

Solution : Convert the above inequalities into equations

$\Rightarrow$ $10 X_1 + 20 X_2 = 20$ ...(1)

Put $X_1 = 0, X_2 = 1$ [2, 1]

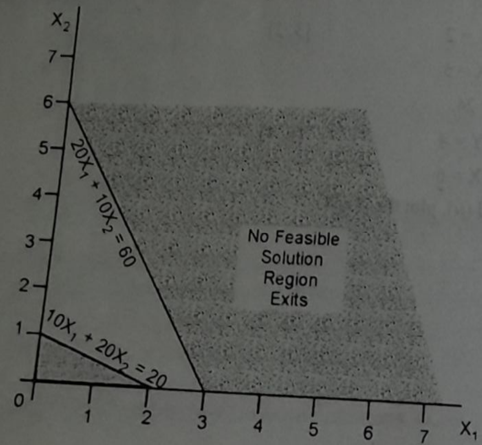
Put $X_2 = 0, X_1 = 2$

$\Rightarrow$ $20 X_1 + 10 X_2 = 60$ ...(2)

Put $X_1 = 0, X_2 = 6$ [3, 6]

Put $X_2 = 0, X_1 = 3$

Plot the coordinates from (1) and (2) on the graph.



Since there is no common shaded area hence the solution is infeasible.

Example 27. Minimise $Z = 200 X_1 + 300 X_2$

Subject to constraints

$$20 X_1 + 30 X_2 \ge 120$$
$$10 X_1 + 10 X_2 \le 40$$
$$20 X_1 + 15 X_2 \ge 90$$
$$X_1 \ge 0, \; X_2 \ge 0$$

No common point

Solution : Convert the inequalities into equations

$\Rightarrow$ $20 X_1 + 30 X_2 = 120$

Put $X_1 = 0, \; X_2 = 4$ (0, 4)

Put $X_2 = 0, \; X_1 = 6$ (6, 0)

$\Rightarrow$ $10 X_1 + 10 X_2 = 40$

Put $X_1 = 0, \; X_2 = 4$ (0, 4)

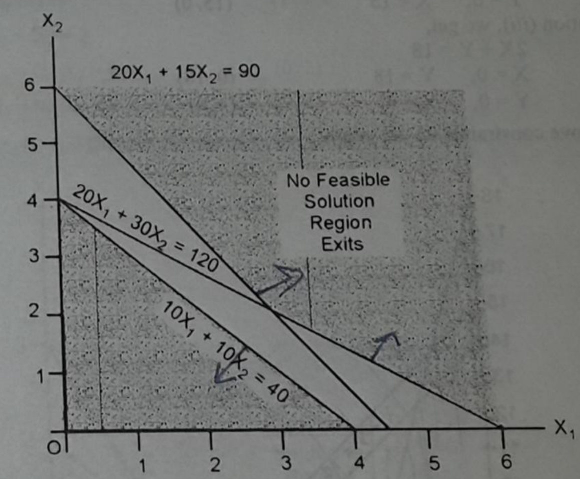
Put $X_2 = 0, \; X_1 = 4$ (4, 0)

$\Rightarrow$ $20 X_1 + 15 X_2 = 90$

Put $X_1 = 0, \; X_2 = 6$ (0, 6)

Put $X_2 = 0, \; X_1 = 4 \cdot 5$ (4·5, 0)

Plot the above constraints on the graph as shown below :



There exist **no feasible** solution.

Note : Infeasibility (no feasible solution) exists when there is no common point in the feasible areas for the constraint of a problem. The feasible region is empty in such a case.

Example 28. Solve Graphically :

Minimize $Z = 2X + 4Y$

Subject to constraints

$$X + Y \le 14$$
$$2X + 2Y \ge 30$$

$2X + Y \le 18$

$X, Y \ge 0$ [G.N.D.U. April 2000]

Solution : Convert the above inequalities into equations first

$X + Y = 14$ ...(i)

$2X + 2Y = 30$ ...(ii)

$2X + Y = 18$ ...(iii)

Take equation (i),

$X + Y = 14$

If	$X = 0$,	$Y = 14$	(0, 14)
If	$Y = 0$,	$X = 14$	(14, 0)

Take equation (ii), we get,

$2X + 2Y = 30$

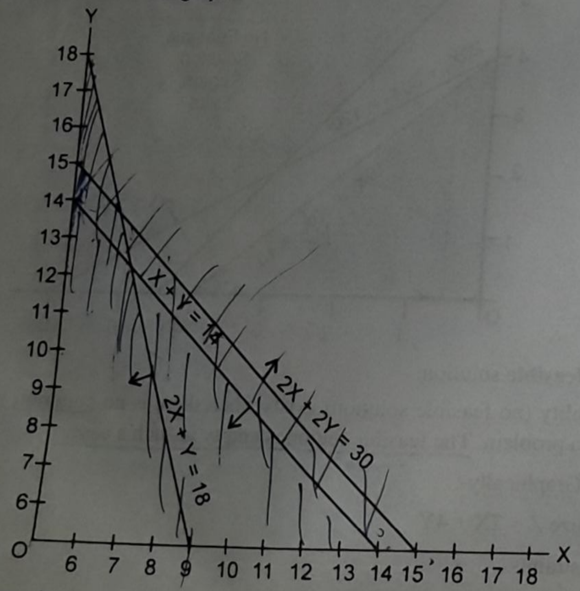
If	$X = 0$,	$Y = 15$	(0, 15)
If	$Y = 0$,	$X = 15$	(15, 0)

Take equation (iii), we get,

$2X + Y = 18$

If	$X = 0$,	$Y = 18$	(0, 18)
If	$Y = 0$,	$X = 9$	(9, 0)

Plot the above constraints on the graph.



As there is no common shaded area, hence the solution is infeasible.

Notes copy

Max + Min

II. UNBOUNDED SOLUTION

A linear programming problem is said to have unbounded solution when the common feasible region is not bounded in any respect. In such a case the value of the objective function can be increased infinitely and thus there is no limit on constraints. Unboundedness occurs in case of a maximization type of linear programming problem.

Example 29. Use graphical method to solve the following LPP.

Maximise $Z = 6X_1 + X_2$

Subject to constraints

$$2X_1 + X_2 \ge 3$$
$$X_1 + X_2 \ge 2$$
$$X_1 \ge 0, X_2 \ge 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$ $2X_1 + X_2 = 3$ ...(i)

Put $X_1 = 0$, $X_2 = 3$ (0, 3)

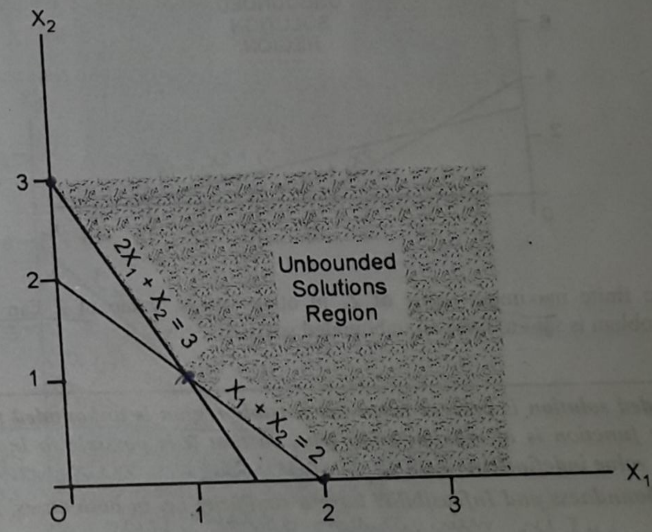
Put $X_2 = 0$, $X_1 = 1 \cdot 5$ (1·5, 0)

$\Rightarrow$ $X_1 + X_2 = 2$ ...(ii)

Put $X_1 = 0$, $X_2 = 2$ (0, 2)

Put $X_2 = 0$, $X_1 = 2$ (2, 0)

Taking points from (i) and (ii) and plot them on the graph.



We observed that the feasible solution is unbounded.

Note : An unbounded solution means that there exists an infinite number of solutions to the given pro

Example 30. Maximise $Z = 10X_1 + 20X_2$

Subject to constraints

$$2X_1 + 4X_2 \geq 16$$
$$X_1 + 5X_2 \geq 15$$
$$X_1 \geq 0, X_2 \geq 0$$

Solution : Convert the inequalities into equalities

In equation $2X_1 + 4X_2 = 16$

If $X_1 = 0,\ X_2 = 4$ (0, 4)

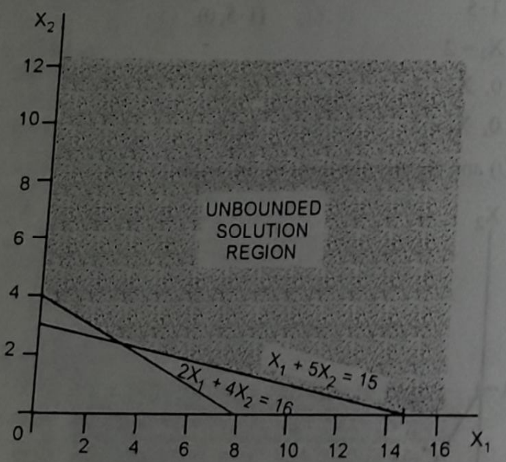
$X_2 = 0,\ X_1 = 8$ (8, 0)

In equation $X_1 + 5X_2 = 15$

If $X_1 = 0,\ X_2 = 3$ (0, 3)

$X_2 = 0,\ X_1 = 15$ (15, 0)

Plot the above constraints as shown in the graph below :



There is no finite maximum value of Z, in other words, value of Z can be increased infinitely. Therefore, the problem is said to have an unbounded solution.

Important

1. ***Unbounded solution is present when the feasible region is unbounded from above and the objective function is of maximisation type, so that it is possible to increase the objective function value indefinitely.*** but if unbounded in min problem, obj. function can be determined
2. ***Both unboundness and Infeasibility have a similarity i.e. in both cases, there is no optimal solution.*** but in minimisation problem, there is an optimal so-
3. ***Difference between unboundness and Infeasibility is that in Infeasibility there is no single feasible solution exits, while in unboundness there are infinite feasible solutions but none of them can be termed as the optimal.***

III. MULTIPLE OPTIMAL SOLUTIONS

A linear programming problem may have two or more than two optimal solutions. In other words, here, the objective function gives the same optimum values at more than one solution points. Graphically, for getting optimal solutions, the following two conditions should be satisfied.

(*a*) The objective function should be parallel to a binding (not redundant) constraint ; and

(*b*) The constraint should be an active constraint.

Example 31. Maximize $Z = 2X_1 + 4X_2$

Subject to constraints

$$X_1 + 2X_2 \leq 5$$
$$X_1 + X_2 \leq 4$$
$$X_1 \geq 0, X_2 \geq 0$$

Solution : Convert the inequalities into equations

$$X_1 + 2X_2 = 5 \quad ...(i)$$
$$X_1 + X_2 = 4 \quad ...(ii)$$

In equation (*i*)

If $X_1 = 0, X_2 = 2\cdot5$ (0, 2·5)

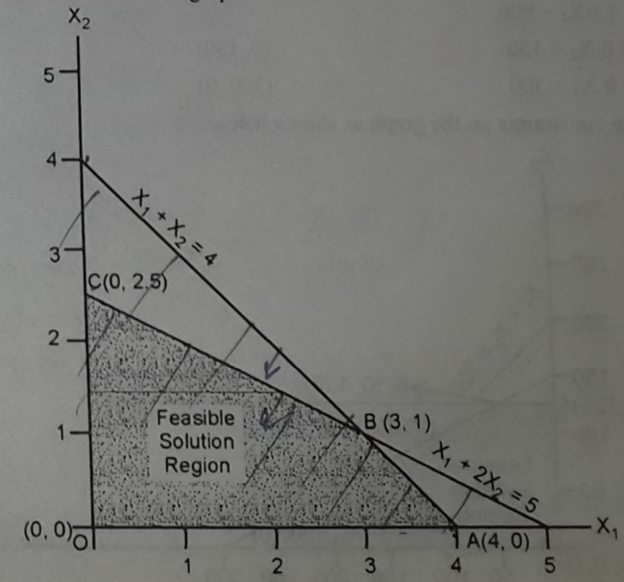
If $X_2 = 0,\ X_1 = 5$ (5, 0)

In equation (*ii*)

If $X_1 = 0, X_2 = 4$ (0, 4)

If $X_2 = 0,\ X_1 = 4$ (4, 0)

Now plot the above constraints on the graph.



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 2X_1 + 4X_2$
O	(0, 0)	2 (0) + 4 (0) = 0
A	(4, 0)	2 (4) + 4 (0) = 8
B	**(3, 1)**	**2 (3) + 4 (1) = 10 – Max.**
C	**(0, 2·5)**	**2 (0) + 4 (2·5) = 10 – Max.**

Since any point on the line segment CB gives maximum values. There exits alternative optimum solutions.

Example 32. Solve the LPP, graphically

Maximize $Z = 8X_1 + 16X_2$

Subject to constraints

$$X_1 + X_2 \le 200$$
$$X_2 \le 125$$
$$3X_1 + 6X_2 \le 900$$
$$X_1 \ge 0, X_2 \ge 0$$

Solution : Convert the inequalities into equations first

$\Rightarrow$ $X_1 + X_2 = 200$ ...(*i*)

Put $X_1 = 0,\ X_2 = 200$ (0, 200)

Put $X_2 = 0,\ X_1 = 200$ (200, 0)

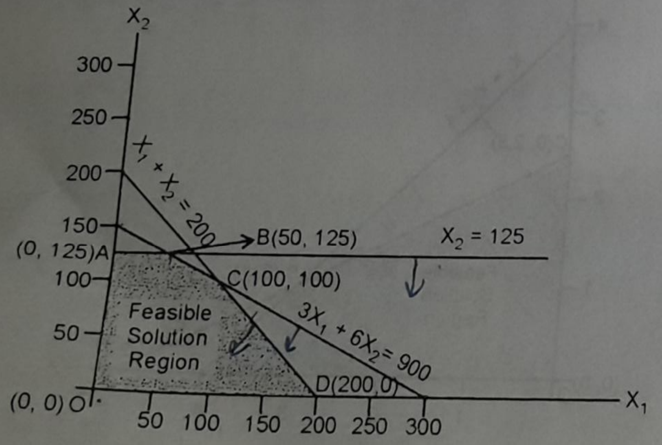
$\Rightarrow$ $X_2 = 125$ ...(*ii*)

$\Rightarrow$ $3X_1 + 6X_2 = 900$ ...(*iii*)

Put $X_1 = 0\ X_2 = 150$ (0, 150)

Put $X_2 = 0\ X_1 = 300$ (300, 0)

Plot the above constraints on the graph as shown below :



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 8X_1 + 16X_2$
O	(0, 0)	8 (0) + 16 (0) = 0
A	(0, 125)	8 (0) + 16 (125) = 2000
B	**(50, 125)**	**8 (50) + 16 (125) = 2400**
C	**(100, 100)**	**8 (100) + 16 (100) = 2400**
D	(200, 0)	8 (200) + 16 (0) = 1600

The points B and C represent the alternative solutions.

Note : Multiple optimal solutions are obtained when the objective function is parallel to a constraints, which is binding and which forms an edge or boundary on the feasible region.

IV. REDUNDANCY

A constraint which is less restrictive or loose as compared to another constraint is said to be redundant constraint. This constraint, when plotted, does not form part of the boundary marking the feasible region of the problem. This constraint is unnecessary in the formulation and solution of the problem because it does not affect the optimal solution to the problem. Redundancy is a condition that arises when redundant constraint is present in the problem.

* It is unnecessary in formulation as it does not effect the optimal sol

Example 33. Maximize $Z = X_1 + 2X_2$

Subject to constraints

$$X_1 + X_2 \le 20$$
$$2X_1 + X_2 \le 30$$
$$X_1 \le 25$$
$$X_1 \ge 0, X_2 \ge 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$ $X_1 + X_2 = 20$ ...(1)

Put $X_1 = 0,\ X_2 = 20$ (0, 20)

Put $X_2 = 0,\ X_1 = 20$ (20, 0)

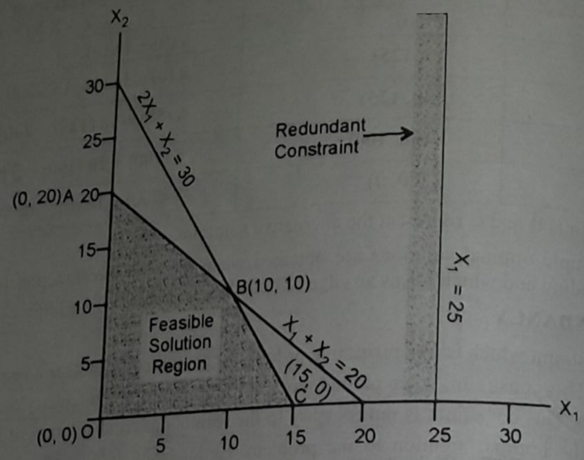
$\Rightarrow$ $2X_1 + X_2 = 30$ ...(2)

Put $X_1 = 0, X_2 = 30$ (0, 30)

Put $X_2 = 0, X_1 = 15$ (15, 0)

$\Rightarrow$ $X_1 = 25$ ...(3)

Taking coordinates from (1), (2) and (3). Plot them on the graph.



Constraint $X_1 \le 25$ is unnecessary, as it has no effect on the feasible area. Hence it is redundant constraint.

The value of the objective function at different corner points can be determined as follows :

Corner Points	Co-ordinates (X_1, X_2)	Objective Function $X_1 + 2X_2$	Value
O	(0, 0)	1 (0) + 2 (0)	0
A	**(0, 20)**	**1 (0) + 20 (20)**	**40**
B	(10, 10)	1 (10) + 2 (10)	30
C	(15, 0)	1 (15) + 1 (0)	15

Maximum value of Z is 40, where $X_1 = 0$, $X_2 = 20$ **Ans.**

Example 34. Maximize $Z = 40\,X_1 + 35\,X_2$

Subject to constraints

$$2\,X_1 + 3\,X_2 \le 60$$
$$4\,X_1 + 3\,X_2 \le 96$$
$$4\,X_1 + 3.5\,X_2 \le 105$$
$$X_1 \ge 0,\ X_2 \ge 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$ $2X_1 + 3\,X_2 = 60$ ...(1)

Put $X_1 = 0,\ X_2 = 20$ (0, 20)

Put $X_2 = 0,\ X_1 = 30$ (30, 0)

$\Rightarrow$ $4\,X_1 + 3\,X_2 = 96$ ...(2)

Put $X_1 = 0,\ X_2 = 32$ (0, 32)

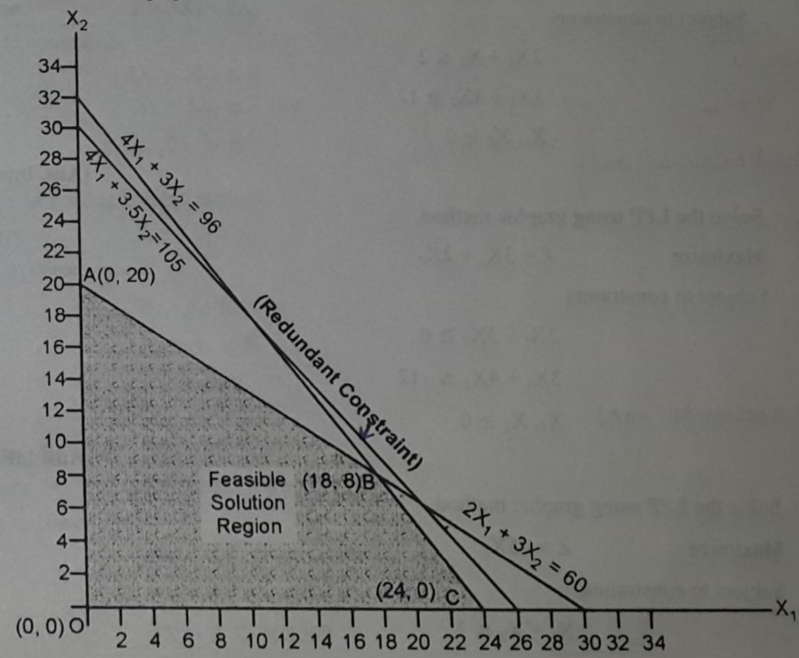
Put $X_2 = 0,\ X_1 = 24$ (24, 0)

$\Rightarrow$ $4\,X_1 + 3\cdot 5\,X_2 = 105$ ...(3)

Put $X_1 = 0,\ X_2 = 30$ (0, 30)

Put $X_2 = 0,\quad X_1 = \frac{105}{4}$ $\left(\frac{105}{4}, 0\right)$

Plot the coordinates on the graph.



We observe that inclusion of $4\,X_1 + 3\cdot 5\,X_2 \le 105$ constraint is unnecessary as this constraint is of no consequence or effect, hence it is **redundant**.

The value of the objective function at different corner points can be determined as follows :

Corner Points	Co-ordinates (X_1, X_2)	Objective Function $40X_1 + 35X_2$	Value
O	(0, 0)	40 (0) + 35 (0)	0
A	(0, 20)	40 (0) + 35 (20)	700
B	**(18, 8)**	**40 (18) + 35 (8)**	**1000**
C	(24, 0)	40 (24) + 35 (0)	960

Maximum value of Z is 1000, where $X_1 = 18$, $X_2 = 8$ **Ans.**

EXERCISE 3.4

1. Solve the LPP using graphic method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$2X_1 + X_2 \le 2$$
$$3X_1 + 4X_2 \ge 12$$
$$X_1, X_2 \ge 0$$

[**Ans.** Infeasible Solution]

2. Solve the LPP using graphic method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$2X_1 - 3X_2 \ge 0$$
$$3X_1 + 4X_2 \le -12$$
$$X_1, X_2 \ge 0$$

[**Ans.** Infeasible Solution]

3. Solve the LPP using graphic method.

Maximize $Z = -5X_2$

Subject to constraints

$$X_1 + X_2 \le 1$$
$$-0.5X_1 - 5X_2 \le -10$$
$$X_1, X_2 \ge 0$$

[Meerut M.Sc. (Math.) 1980]

[**Ans.** Infeasible Solution]

4. Solve the LPP using graphic method.

Maximize $Z = 3X_1 + 4X_2$

Subject to constraints

$$X_1 + X_2 \le -1$$
$$-X_1 + X_2 \le 0$$
$$X_1, X_2 \ge 0$$

[**Ans.** Infeasible Solution]

5. Solve the LPP using graphic method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$2X_1 - X_2 \ge -2$$
$$X_1 + 2X_2 \ge 8$$
$$X_1, X_2 \ge 0$$

[**Ans.** Unbounded Solution]

6. Solve the LPP using graphic method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$-2X_1 + 3X_2 \le 9$$
$$X_1 - 5X_2 \ge -120$$
$$X_1, X_2 \ge 0$$

[**Ans.** Unbounded Solution]

7. Solve the LPP using graphic method.

Maximize $Z = X_1 + 2X_2$

Subject to constraints

$$X_1 - X_2 \le 1$$
$$X_1 + X_2 \ge 3$$
$$X_1, X_2 \ge 0$$

[**Ans.** Unbounded Solution]

8. Solve the LPP using graphic method.

Maximize $Z = 120X_1 + 160X_2$

Subject to constraints

$$3X_1 + 4X_2 \ge 2400$$
$$2X_1 + X_2 \ge 1000$$
$$5X_1 + 3X_2 \ge 1500$$
$$X_1, X_2 \ge 0$$

[**Ans.** Unbounded Solution]

9. Solve the LPP using graphic method.

Maximize $Z = 6X + 8Y$

Subject to constraints

$$2X + 3Y \geq 16$$
$$4X + 2Y \geq 16$$
$$Y \leq 8$$
$$X_1, X_2 \geq 0$$

[**Ans.** Unbounded Solution]

10. Solve the LPP using graphic method.

Maximize $Z = X_1 + X_2$

Subject to constraints

$$X_1 + X_2 \geq 1$$
$$-3X_1 + X_2 \geq 3$$
$$X_1, X_2 \geq 0$$

[**Ans.** Unbounded Solution]

11. Solve the LPP using graphic method.

Maximize $Z = 10X_1 + 10X_2$

Subject to constraints

$$-20X_1 + 10X_2 \leq 10$$
$$10X_1 \leq 20$$
$$10X_1 + 10X_2 \leq 30$$
$$X_1, X_2 \geq 0 \text{ (Non-negativity constraint)}$$

[**Ans.** $X_1 = 2/3, X_2 = 7/3$ Or $X_1 = 2, X_2 = 1$, Max. $Z = 30$]

12. Solve the LPP using graphic method.

Maximize $Z = 2X_1 + 3X_2$

Subject to constraints

$$6X_1 + 9X_2 \leq 100$$
$$2X_1 + X_2 \leq 20$$
$$X_1, X_2 \geq 0 \text{ (Non-negativity constraint)}$$

[**Ans.** $X_1 = 0, X_2 = 11 \cdot 11$ Or $X_1 = 40/6, X_2 = 40/6$, Max. $Z = 33 \cdot 33$]

13. Solve the LPP using graphic method.

Maximize $Z = 40X_1 + 40X_2$

Subject to constraints

$$10X_1 + 20X_2 \leq 100$$
$$60X_1 + 60X_2 \leq 360$$
$$X_1, X_2 \geq 0 \text{ (Non-negativity constraint)}$$

[**Ans.** $X_1 = 2, X_2 = 4$ Or $X_1 = 6, X_2 = 0$, Max. $Z = 240$]

14. A company buying scrap metal has two types of scrap metals available to him. The first type of scrap metal has 30% of metal A, 20% of metal B and 50% of metal C by weight. The second scrap has 40% of metal A, 10% of metal B and 30% of metal C. The company requires at least 240 kg of metal A, 100 kg of metal B and 290 kg of metal C. The price per kg of the two scraps are Rs. 120 and Rs. 160 respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the three metals are satisfied at a minimum cost.

[**Ans.** $X_1 = 400, X_2 = 300$ Or $X_1 = 800, X_2 = 0$, Max. $Z = 96000$]

15. A company is making two different types of radio. The profit per unit of the first radio is Rs. 60 while the profit per unit of the second radio is Rs. 25. The first radio requires 7 hours of assembly and 12 hours of body work. The second radio requires 9 hours of assembly work and 5 hours body work. The daily hours available for assembly work and body work are 63 hours respectively. Find the daily production schedule which maximises the profit. Give your comments about the solution.

[**Ans.** $X_1 = 3 \cdot 08, X_2 = 4 \cdot 6$ Or $X_1 = 5, X_2 = 0$, Max. $Z = 300$]

16. Min $Z = 10\,X_1 + 10\,X_2$

Subject to constraints

$$X_1 + X_2 \geq 10$$
$$3X_1 + 2X_2 \geq 24$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 4, X_2 = 6$ Or $X_1 = 10, X_2 = 0$, Min. $Z = 100$]

17. Solve the LPP using graphic method.

Minimum $Z = -6X_1 - 4X_2$

Subject to constraints

$$2X_1 + 3X_2 \geq 30$$
$$3X_1 + 2X_2 \leq 24$$
$$X_1 + X_2 \geq 3$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 2 \cdot 4, X_2 = 8 \cdot 4$ Or $X_1 = 0, X_2 = 12$, Min. $Z = -48$]

MISCELLANEOUS PROBLEMS

Example 35. Max. $Z = 300\,X_1 + 400\,X_2$ (After Mathematical Formulation)

Subject to constraints

$$5\,X_1 + 4\,X_2 \leq 200$$
$$3\,X_1 + 5\,X_2 \leq 150$$

$5X_1 + 4X_2 \geq 100$

$8X_1 + 4X_2 \geq 80$

$X_1 \geq 0, X_2 \geq 0$ *[G.NDU B.Com III (P), 2003]*

Solution : Convert the inequalities into equations

$\Rightarrow$ $5X_1 + 4X_2 = 200$

Put	$X_1 = 0, \quad X_2 = 50$	(0, 50)
Put	$X_2 = 0, \quad X_1 = 40$	(40, 0)

$\Rightarrow$ $3X_1 + 5X_2 = 150$

Put	$X_1 = 0, \quad X_2 = 30$	(0, 30)
Put	$X_2 = 0, \quad X_1 = 50$	(50, 0)

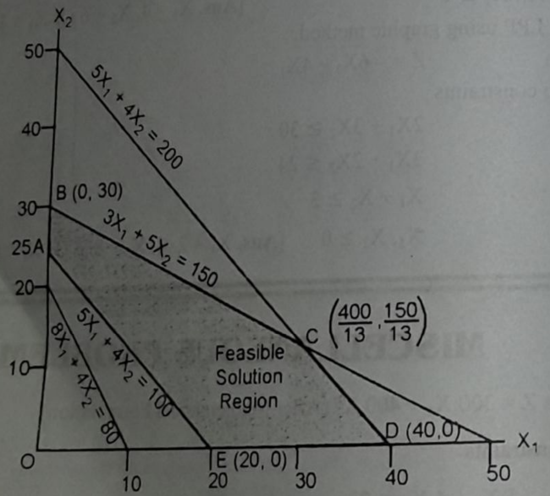
$\Rightarrow$ $5X_1 + 4X_2 = 100$

Put	$X_1 = 0, \quad X_2 = 25$	(0, 25)
Put	$X_2 = 0, \quad X_1 = 20$	(20, 0)

$\Rightarrow$ $8X_1 + 4X_2 = 80$

Put	$X_1 = 0, \quad X_2 = 20$	(0, 20)
Put	$X_2 = 0, \quad X_1 = 10$	(10, 0)

Now we plot these values on the graph.



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 300X_1 + 400X_2$
A	(0, 25)	300 (0) + 400 (25) = 10000
B	(0, 30)	300 (0) + 400 (30) = 12000
C	$\left(\frac{400}{13}, \frac{150}{13}\right)$	$300\left(\frac{400}{13}\right) + 400\left(\frac{150}{13}\right) = 180000/13$
D	(40, 0)	300 (40) + 400 (0) = 12000
E	(20, 0)	300 (20) + 400 (0) = 6000

Maximum Profit = Rs. 180000/13, where $X_1 = 400/13$, $X_2 = 150/13$.

Example 36. Solve the following problem graphically.

Maximize $Z = 2X_1 + 3X_2$

Subject the constraints

$X_1 + X_2 \leq 30$

$X_2 \geq 3$

$X_2 \leq 12$

$X_1 - X_2 \geq 0$

$X_1 \leq 20$

$X_1 \geq 0, X_2 \geq 0$

Solution : Convert the inequalities into equations

$\Rightarrow$ $X_1 + X_2 = 30$

Put	$X_1 = 0, \quad X_2 = 30$	(0, 30)
Put	$X_2 = 0, \quad X_1 = 30$	(30, 0)

$\Rightarrow$ $X_2 = 3$

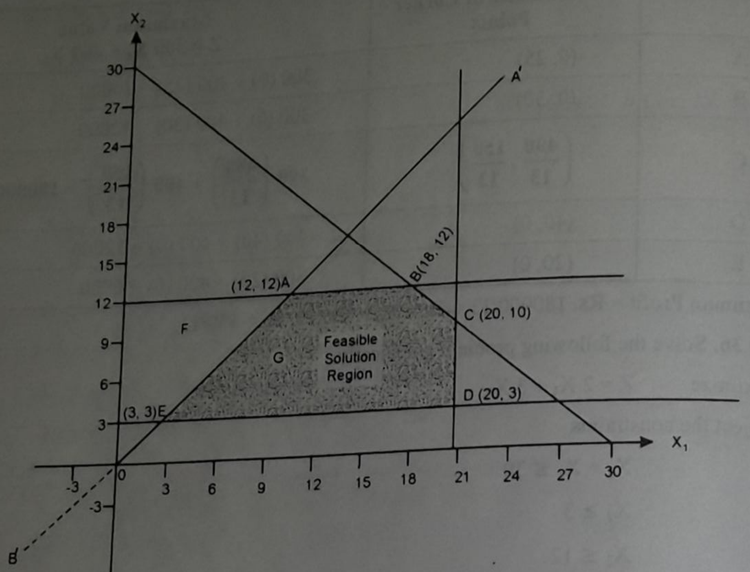
$\Rightarrow$ $X_2 = 12$

$\Rightarrow$ $X_1 = 20$

Now take the equation $X_1 - X_2 = 0$ or $X_1 = X_2$

Put	$X_2 = 0, \quad X_1 = 0$	(0, 0)
Put	$X_2 = 1, \quad X_1 = 1$	(1, 1)
Put	$X_2 = -1, \quad X_1 = -1$	(–1, –1)

Plot the figures on the graph.



Common area becomes ABCDE

Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 2X_1 + 3X_2$
A	(12, 12)	2 (12) + 3 (12) = 60
B	**(18, 12)**	**2 (18) + 3 (12) = 72**
C	(20, 10)	2 (20) + 3 (10) = 70
D	(20, 3)	2 (20) + 3 (3) = 49
E	(3, 3)	2 (3) + 3 (3) = 15

Maximum value of Z = 72, where $X_1 = 18$ and $X_2 = 12$.

Example 37. Maximize $Z = 26X_1 + 20X_2$

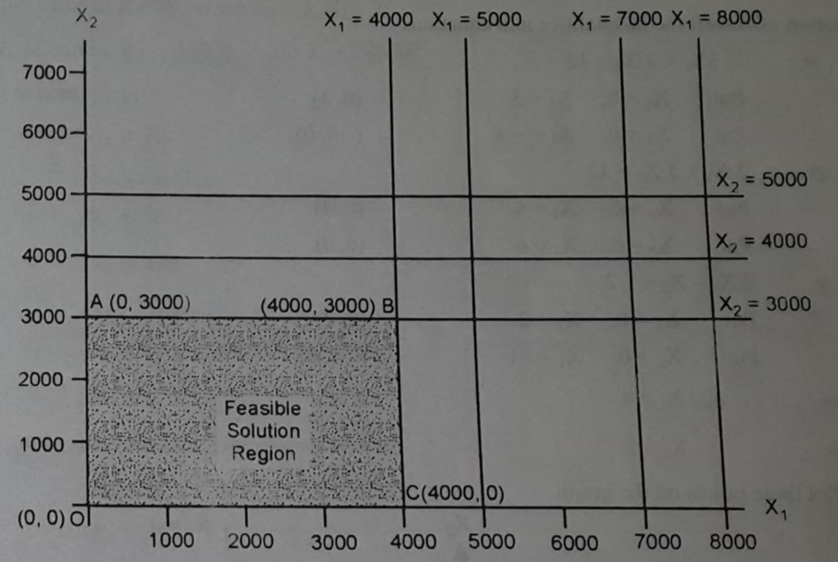
Subject to constraints

$$X_1 \le 4000$$
$$X_1 \le 5000$$
$$X_2 \le 5000$$
$$X_1 \le 7000$$
$$X_2 \le 4000$$
$$X_1 \le 8000$$
$$X_2 \le 3000$$
$$X_1 \ge 0, \quad X_2 \ge 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$

$$X_1 = 4000 \quad ...(i)$$
$$X_1 = 5000 \quad ...(ii)$$
$$X_2 = 5000 \quad ...(iii)$$
$$X_1 = 7000 \quad ...(iv)$$
$$X_2 = 4000 \quad ...(v)$$
$$X_1 = 8000 \quad ...(vi)$$
$$X_2 = 3000 \quad ...(vii)$$



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 26X_1 + 20X_2$
O	(0, 0)	26 (0) + 20 (0) = 0
A	(0, 3000)	26 (0) + 20 (3000) = 60000
B	**(4000, 3000)**	**26 (4000) + 20 (3000) = 164000**
C	(4000, 0)	26 (4000) + 20 (0) = 104000

Maximum value of Z = 1,64,000, where $X_1 = 4000$, $X_2 = 3000$

Example 38. Solve graphically the following linear programming problem

Minimise $\quad Z = 3X_1 + 5X_2$

Subject to constraints

$$-3X_1 + 4X_2 \le 12$$
$$2X_1 + 3X_2 \ge 12$$
$$2X_1 - X_2 \ge -2$$
$$X_1 \le 4$$
$$X_2 \ge 2$$
$$X_1 \ge 0, X_2 \ge 0$$

Solution : Convert the inequalities into equations

$\Rightarrow$ $-3X_1 + 4X_2 = 12$

Put $X_1 = 0$, $X_2 = 3$ (0, 3)

Put $X_2 = 0$, $X_1 = -4$ (– 4, 0)

$\Rightarrow$ $2X_1 + 3X_2 = 12$

Put $X_1 = 0$, $X_2 = 4$ (0, 4)

Put $X_2 = 0$, $X_1 = 6$ (6, 0)

$\Rightarrow$ $2X_1 - X_2 = -2$

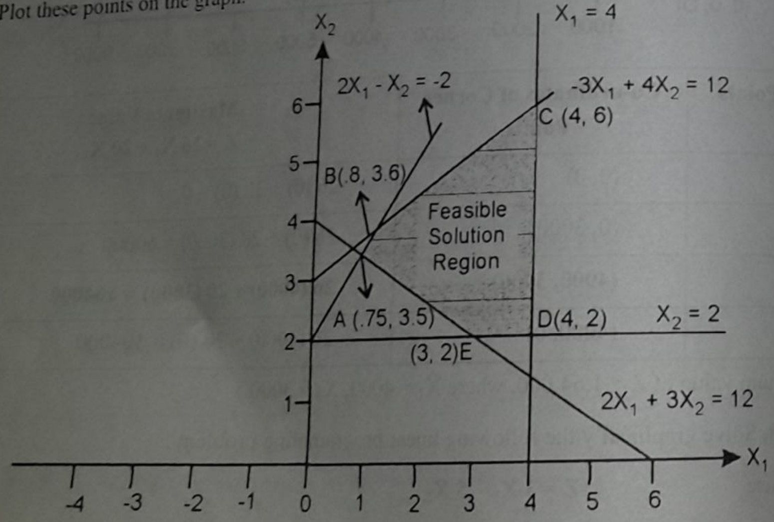
Put $X_1 = 0$, $X_2 = 2$ (0, 2)

Put $X_2 = 0$, $X_1 = -1$ (– 1, 0)

$\Rightarrow$ $X_1 = 4$

$\Rightarrow$ $X_2 = 2$

Plot these points on the graph.



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = 3X_1 + 5X_2$
A	(·75, 3·5)	3 (·75) + 5 (3·5) = 19·75
B	(·8, 3·6)	3 (·8) + 5 (3·6) = 20·40
C	(4, 6)	3 (4) + 5 (6) = 42
D	(4, 2)	3 (4) + 5 (2) = 22
E	**(3, 2)**	**3 (3) + 5 (2) = 19**

Minimum value of Z = 19, where $X_1 = 3$, $X_2 = 2$

Example 39. Maximise $P = -150X_1 - 100X_2 + 280000$

Subject to constraints

$$X_1 \ge 20$$
$$X_1 \le 60$$
$$X_2 \ge 70$$
$$X_2 \le 140$$
$$X_1 + X_2 \ge 120$$
$$X_1 + X_2 \le 140$$
$$X_1 \ge 0, X_2 \ge 0$$

[C.A. May, 1991 Formulated]

Solution : Convert the inequalities into equations

$X_1 = 20$

$X_2 = 70$

$X_1 + X_2 = 120$

Put $X_1 = 0$, $X_2 = 120$ (0, 120)

Put $X_2 = 0$, $X_1 = 120$ (120, 0)

$X_1 = 60$

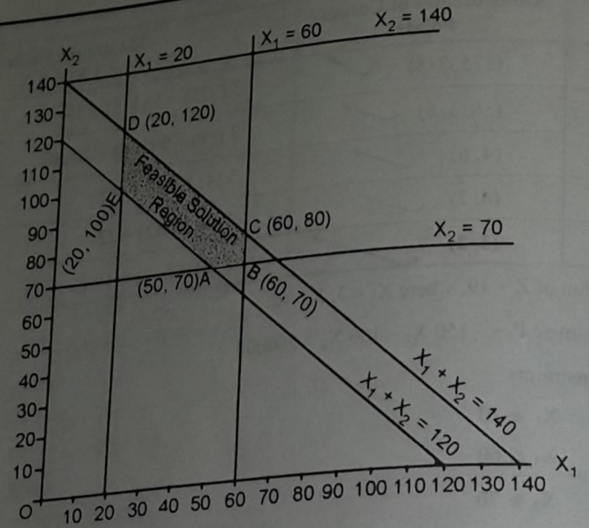
$X_2 = 140$

$X_1 + X_2 = 140$

Put $X_1 = 0$, $X_2 = 140$ (0, 140)

Put $X_2 = 0$, $X_1 = 140$ (140, 0)

Plot the above points on the graph.



Corner Points	Co-ordinates of Corner Points	Maximum Value $Z = -150X_1 - 100X_2 + 280000$
A	(50, 70)	$-150(50) - 100(70) + 280000 = 265500$
B	(60, 70)	$-150(60) - 100(70) + 280000 = 264000$
C	(60, 80)	$-150(60) - 100(80) + 280000 = 263000$
D	(20, 120)	$-150(20) - 100(120) + 280000 = 265000$
E	**(20, 100)**	$\mathbf{-150(20) - 100(100) + 280000 = 267000}$

Maximum value is at point E where $X_1 = 20, X_2 = 100$

Example 40. Maximize $Z = 200X_1 + 150X_2$

Subject to constraints

$$500X_1 + 350X_2 \le 60000$$
$$X_1 \le 100$$
$$X_2 \le 100$$
$$200X_1 + 150X_2 \ge 20000$$
$$X_1 = 60$$
$$X_1 \ge 0 \; X_2 \ge 0$$

Solution : Convert the above inequalities into equations

$\Rightarrow$ $500X_1 + 350X_2 = 60000$

Put $X_1 = 0$, $X_2 = \dfrac{60000}{350} = 171\cdot4$ (0, 171·4)

Put $X_2 = 0$, $X_1 = 120$ (120, 0)

$\Rightarrow$ $X_1 = 100$

$\Rightarrow$ $X_2 = 100$

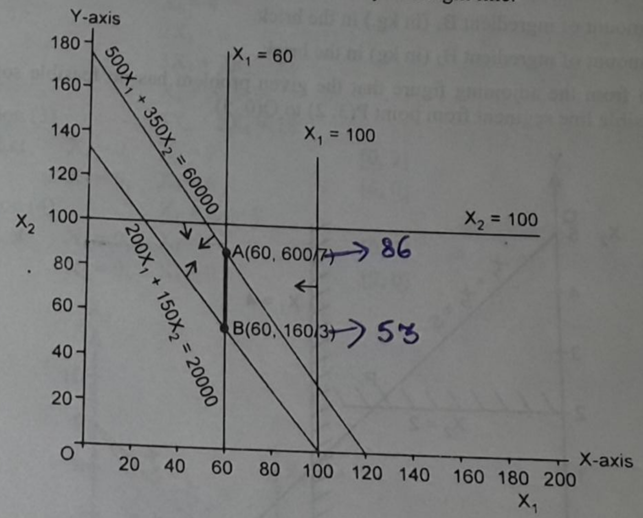
$\Rightarrow$ $200X_1 + 150X_2 = 20000$

Put $X_1 = 0$, $X_2 = \dfrac{20000}{150} = 133\cdot33$ (0, 133·33)

Put $X_2 = 0$, $X_1 = 100$ (100, 0)

$\Rightarrow$ $X_1 = 60$

Plot the coordinates on the graph and join them by a straight line.



Feasible solution region lies on the line between the point A and point B.

Corner Points	Co-ordinates of Corner Points	Maximum Value $P = 200X_1 + 150X_2$
A	$\left(60, \dfrac{600}{7}\right)$	$200(60) + 150\left(\dfrac{600}{7}\right) = 24857$
B	$\left(60, \dfrac{160}{3}\right)$	$200(60) + 150\left(\dfrac{160}{3}\right) = 20000$

Maximum $Z = 24857$, where $X_1 = 60$, $X_2 = \dfrac{600}{7}$.

Example 41. The standard weight of a special purpose brick is 5 kg and it contains two basic ingredients B_1 and B_2. B_1 costs Rs. 5 per kg and B_2 costs Rs. 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of B_1 and minimum of 2 kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions. [*I.C.W.A. (June), 1982, C.A. (Final) 2002*]

Solution :

The appropriate mathematical formulation of the given problem is :

Minimize (Total Cost) $Z = 5X_1 + 8X_2$

Subject to the constraints

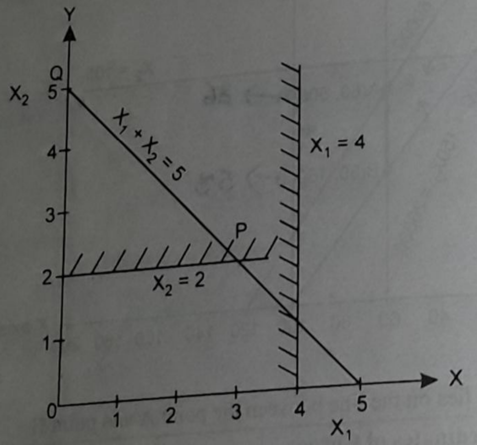
$$X_1 \leq 4$$
$$X_2 \geq 2$$
$$X_1 + X_2 = 5$$

and $X_1, X_2 \geq 0$ (Non-negativity constraint)

where X_1 = The amount of ingredient B_1 (in kg.) in the brick.

X_2 = The amount of ingredient B_2 (in kg) in the brick.

It may be observed from the adjoining figure that the given problem has no feasible solution space (shaded area) but has feasible line segment from point P(3, 2) to Q(0, 5).



The value of the objective function at these points can be determined as follows :

Corner Point	Coordinates of Corner Point	Objective Function $Z = 5X_1 + 8X_2$	Value
P	(3, 2)	5 (3) + 8 (2)	31
Q	(0, 5)	5 (0) + 8 (5)	40

The minimum value of Z (minimum cost) is found at the point P(3, 2), *i.e.*, $X_1 = 3$ and $X_2 = 2$. Hence, the optimal product mix is : 3 kg. of ingredient B_1 and 2 kg of ingredient B_2 of a special purpose brick so as to achieve the minimum cost of Rs. 31.

Example 42. Find the maximum value of $Z = 3X_1 + 5X_2$

Subject to constraints

$$X_1 \leq 4$$
$$2X_2 \leq 6$$
$$3X_1 + 2X_2 \leq 18$$
$$X_1 + X_2 \leq 9$$
$$X_1, X_2 \geq 0$$

Solution :

Firstly, convert the inequalities into equations

$$X_1 = 4 \quad ...(1)$$
$$2X_2 = 6 \quad ...(2)$$
$$3X_1 + 2X_2 = 18 \quad ...(3)$$
$$X_1 + X_2 = 9 \quad ...(4)$$

In equation (3) $3X_1 + 2X_2 = 18$

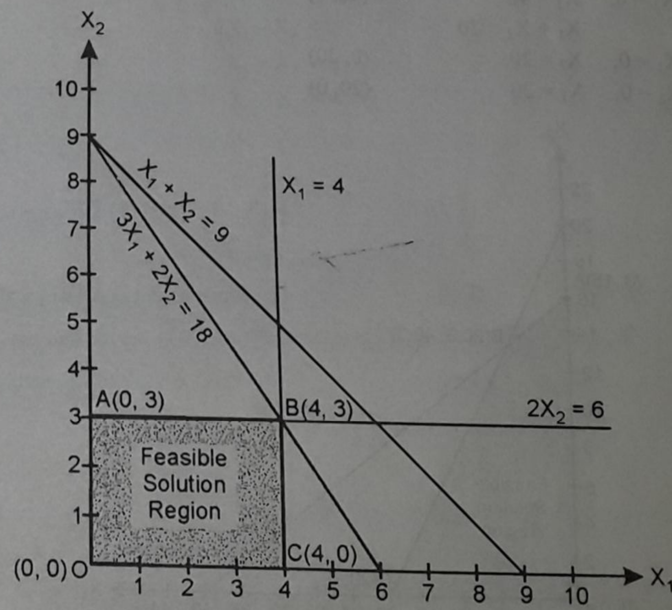
Let $X_1 = 0$, $X_2 = 9$ [0, 9]

$X_2 = 0$, $X_1 = 6$ [6, 0]

In equation (4) $X_1 + X_2 = 9$

Let $X_1 = 0$, $X_2 = 9$ [0, 9]

$X_2 = 0$, $X_1 = 9$ [9, 0]



Area OABC in the figure satisfied by the constraints is shown as shaded area and is called feasible solution region.

Corner Point	Coordinates of Corner Point	Max. $Z = 3X_1 + 5X_2$	Value
O	(0, 0)	3(0) + 5(0)	0
A	(0, 3)	3(0) + 5(3)	15
B	**(4, 3)**	**3(4) + 5(3)**	**27**
C	(4, 0)	3(4) + 5(0)	12

Hence, Max $Z = 27$, where $X_1 = 4$ and $X_2 = 3$.

Example 43. Maximize $Z = 0 \cdot 3\,(3X_1 + 5X_2) - (X_1 + X_2)$ $= 0{\cdot}9X_1 + 1{\cdot}5X_2 - X_1 - X_2$

$= -0 \cdot 1X_1 + 0 \cdot 5X_2$ $-0{\cdot}1X_2 + 0{\cdot}5X_2$

Subject to constraints

$$2X_1 + 5X_2 \le 80$$
$$X_1 + X_2 \le 20$$
$$X_1, X_2 \ge 0$$

Solution : Firstly, convert the inequalities into equations

$$2X_1 + 5X_2 = 80 \quad \ldots(1)$$
$$X_1 + X_2 = 20 \quad \ldots(2)$$

In equation (1) $2X_1 + 5X_2 = 80$

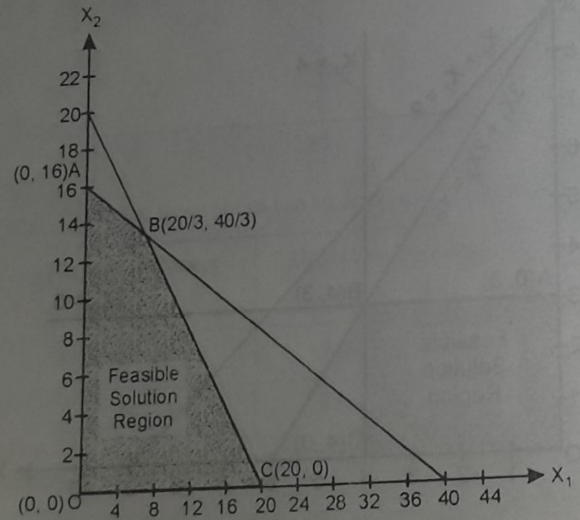
Let $X_1 = 0$, $X_2 = 16$ (0, 16)

$X_2 = 0$, $X_1 = 40$ (40, 0)

In equation (2) $X_1 + X_2 = 20$

Let $X_1 = 0$, $X_2 = 20$ (0, 20)

$X_2 = 0$, $X_1 = 20$ (20, 0)



Corner Point	Coordinates of Corner Point (X_1, X_2)	Objective Function $Z = -0 \cdot 1X_1 + 0 \cdot 5X_2$	Value
O	(0, 0)	$-0 \cdot 1(0) + 0 \cdot 5(0)$	0
A	**(0, 16)**	$\mathbf{-0 \cdot 1(0) + 0 \cdot 5(16)}$	**8**
B	(20/3, 40/3)	$-0 \cdot 1(20/3) + 0 \cdot 5(40/3)$	6
C	(20, 0)	$-0 \cdot 1(20) + 0 \cdot 5(0)$	–ve

Hence, Max. $Z = 8$, where $X_1 = 0$, $X_2 = 16$.

Example 44. Solve Graphically

Maximize $Z = 2X_1 + 3X_2$

Subject to constraints

$$X_1 + X_2 \ge 1$$
$$5X_1 - X_2 \ge 0$$
$$X_1 + X_2 \le 6$$
$$X_1 - X_2 \le 0$$
$$X_2 - X_1 \ge -1 \quad = -X_2 + X_1 \le 1$$
$$X_2 \le 3$$
$$X_1, X_2 \ge 0$$

Solution : First convert the constraints into equalities

$$X_1 + X_2 = 1 \quad \ldots(1)$$
$$5X_1 - X_2 = 0 \quad \ldots(2)$$
$$X_1 + X_2 = 6 \quad \ldots(3)$$
$$X_1 - X_2 = 0 \quad \ldots(4)$$
$$X_2 - X_1 = -1 \quad \ldots(5)$$
$$X_2 = 3 \quad \ldots(6)$$

In equation (1) if $X_1 = 0 : X_2 = 1$ (0, 1)

$X_2 = 0, X_1 = 1$ (1, 0)

In equation (2) $X_1 = 0, X_2 = 0$ (0, 0)

Now, put any values of X_1 and X_2 where equality is satisfied i.e. (1, 5)

In equation (3) if $X_1 = 0$; $X_2 = 6$ (0, 6)

$X_2 = 0, X_1 = 6$ (6, 0)

In equation (4) $X_1 = 0$; $X_2 = 0$ (0, 0)

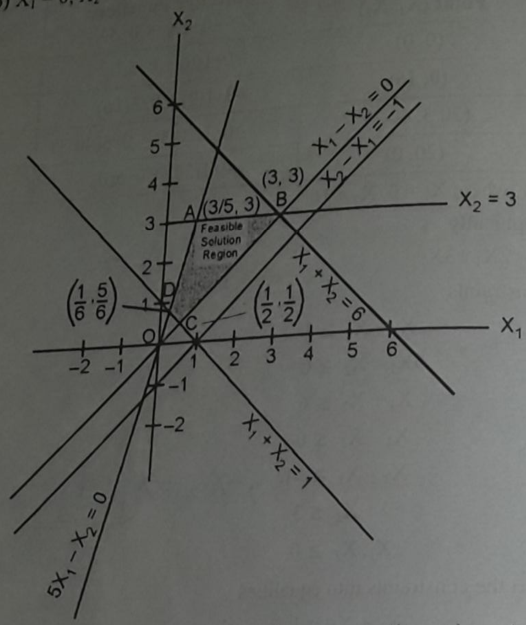
Now, put any values of X_1 and X_2 such that equality is satisfied e.g. (1, 1)

In equation (5) If $X_1 = 0, X_2 = -1$ (0, –1)

If $X_2 = 0, X_1 = 1$ (1, 0)

(0, 3)

In equation (6) $X_1 = 0, X_2 = 3$



The corner points of the feasible region are $A\left(\frac{3}{5}, 3\right)$, $B(3, 3)$, $C\left(\frac{1}{2}, \frac{1}{2}\right)$, $D\left(\frac{1}{6}, \frac{5}{6}\right)$

The value of the objective function at different corner points can be calculated as under :

Corner point	Co-ordinates (X_1, X_2)	Objective Function $2X_1 + 3X_2$	Value
A	(3/5, 3)	2(3/5) + 3(3)	51/5
B	**(3, 3)**	**2(3) + 3(3)**	**15**
C	(1/2, 1/2)	2(1/2) + 3(1/2)	5/2
D	(1/6, 5/6)	2 (1/6) + 3 (5/6)	17/6

Maximize $Z = 15, X_1 = 3, X_2 = 3$ **Ans.**

D

Example 45. Use the graphical method to solve the following LPP :

Maximize $Z = X_1 + X_2$

Subject to

$$X_1 + X_2 \le 1$$
$$-3X_1 + X_2 \ge 3$$
$$X_1, X_2 \ge 0$$

(G.N.D.U. BBA-III 2006)

$X_1, X_2 \ge 0$

Solution : To plot the constraints on the graph we will convert the inequalities into equations temporarily.

$$X_1 + X_2 = 1 \quad ...(1)$$
$$-3X_1 + X_2 = 3 \quad ...(2)$$

In equation (1) $X_1 + X_2 = 1$

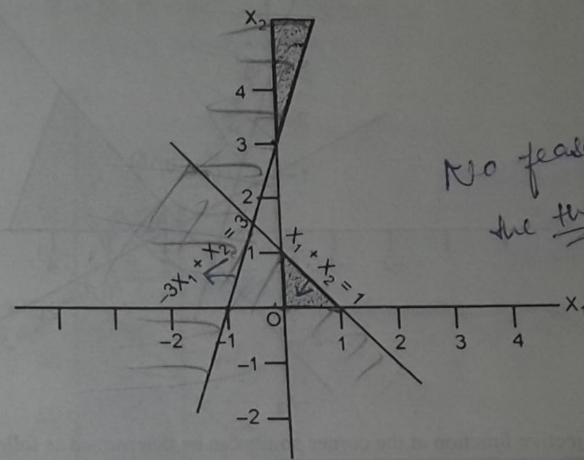
If $X_1 = 0 ; X_2 = 1$ (0, 1)

If $X_2 = 0 ; X_1 = 1$ (1, 0)

In equation (2) $-3X_1 + X_2 = 3$

If $X_1 = 0 ; X_2 = 3$ (0, 3)

If $X_2 = 0 ; X_1 = -1$ (–1, 0)



No feasable sol. in the the region

From the graph, it is clear that there is not even a single point which can lie in both areas, hence no feasible value of Z exist.

Example 46. Minimize $Z = 4X_1 + 2X_2$

Subject to constraints

$$X_1 + 2X_2 \ge 2$$
$$3X_1 + X_2 \ge 3$$
$$4X_1 - 3X_2 \ge 6$$
$$X_1, X_2 \ge 0$$

Using graphic method

[G.N.D.U. B.Com.(P) III, 2004]

Solution : First we consider the constraints as equalities

$$X_1 + 2X_2 = 2 \quad ...(1)$$
$$3X_1 + X_2 = 3 \quad ...(2)$$
$$4X_1 - 3X_2 = 6 \quad ...(3)$$

In equation (1) If $X_1 = 0 ; X_2 = 1$ (0, 1)

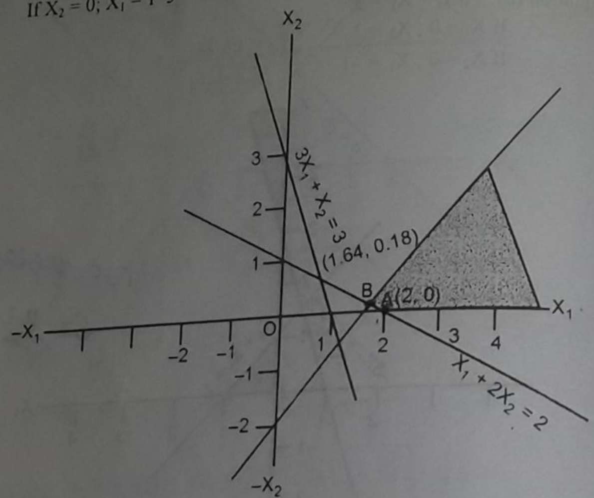
If $X_2 = 0$; $X_1 = 2$ (2, 0)

In equation (2) If $X_1 = 0$; $X_2 = 3$ (0, 3)

If $X_2 = 0$; $X = 1$ (1, 0)

In equation (3) If $X_1 = 0$; $X_2 = -2$ (0, –2)

If $X_2 = 0$; $X_1 = 1 \cdot 5$ (1·5, 0)



The value of the objective function at the corner points can be determined as follows :

Corner point	Co-ordinates (X_1, X_2)	Objective Function $4X_1 + 2X_2$	Value
A	(2, 0)	4(2) + 2(0)	8
B	(1·64, 0·18)	4 (1·64) + 2(0·18)	6·92

Minimum value of Z lies at point B, hence minimum $Z = 6{\cdot}92$, $X_1 = 1{\cdot}64$, $X_2 = 0{\cdot}18$

Example 47. Solve the following problem graphically :

Maximize $Z = 8{,}000\ X_1 + 7{,}000\ X_2$

Subject to

$$3X_1 + X_2 \le 66$$
$$X_1 + X_2 \le 45$$
$$X_1 \le 20$$
$$X_2 \le 40$$
$$X_1, X_2 \ge 0$$

[*G.N.D.U., B.Com. III. Sep. 2005*]

Solution : To plot the constraints on the graph, we will convert the inequalities into equations temporarily.

$$3X_1 + X_2 = 66 \quad ...(1)$$
$$X_1 + X_2 = 45 \quad ...(2)$$
$$X_1 = 20 \quad ...(3)$$
$$X_2 = 40 \quad ...(4)$$

In equation (1) if $X_1 = 0$; $X_2 = 66$ (0, 66)

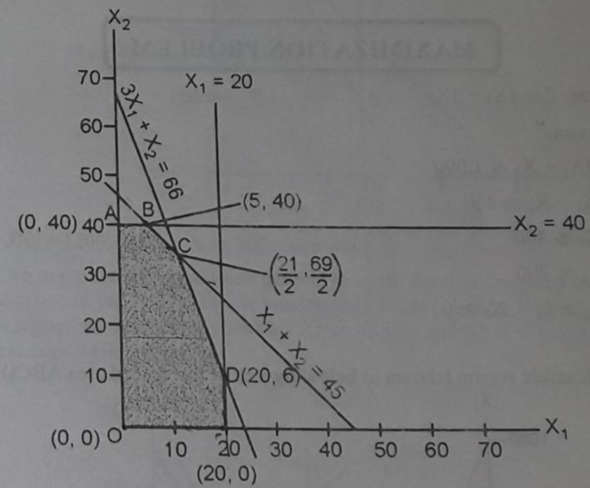
if $X_2 = 0$; $X_1 = 22$ (22, 0)

In equation (2) if $X_1 = 0$; $X_2 = 45$ (0, 45)

if $X_2 = 0$; $X_1 = 45$ (45, 0)

In equation (3) $X_1 = 20$; $X_2 = 0$ (20, 0)

In equation (4) $X_1 = 0$; $X_2 = 40$ (0, 40)



The value of the objective function at different corner points can be determined as follows :

Corner point	Co-ordinates (X_1, X_2)	Objective Function $8000X_1 + 7000X_2$	Value
O	(0, 0)	8000 (0) + 7,000 (0)	0
A	(0, 40)	8,000 (0) 7,000 (40)	2,80,000
B	(5, 40)	8,000 (5) + 7,000 (40)	3, 20,000
C	**(21/2, 69/2)**	**8,000(21/2) + 7,000 (69/2)**	**3,25,500**
D	(20, 6)	8,000(20) + 7,000(6)	2,02,000
E	(20, 0)	8,000(20) + 7,000(0)	1,60,000

Maximum value of Z lies at point C, where $X_1 = \dfrac{21}{2}$, $X_2 = \dfrac{69}{2}$ and $Z = 3{,}25{,}500$

ISO-PROFIT OR ISO-COST APPROACH

Another technique of getting LPP solution through graphs is called ISO profit (or cost) technique. his is an alternative approach which requires following steps :

1. Select a definite profit (or cost) figure and extract an ISO profit (cost) line so that it falls with in shaded arcs.
2. Change the position of this line parallel to itself and distant (closer) from (to) the beginning until further movement moved take this line fully outside the feasible region.
3. Identify the optimum solution as the coordinates of that point on the feasible region touched by the **highest** possible ISO profit line or **lower** possible ISO cost line.

MAXIMIZATION PROBLEM

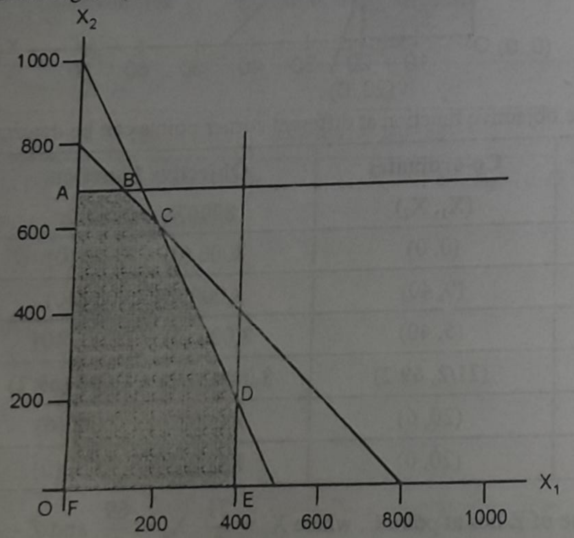
ample 48. Maximize $Z = 4X_1 + 3X_2$

Subject to constraints

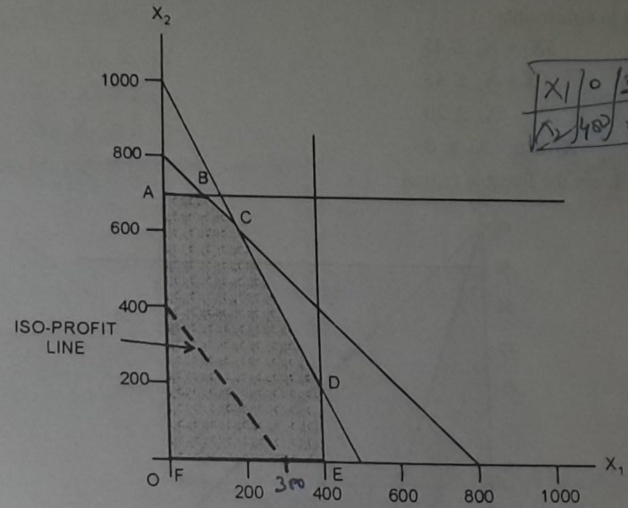
$$2X_1 + X_2 \leq 1{,}000$$
$$X_1 + X_2 \leq 800$$
$$X_1 \leq 400$$
$$X_2 \leq 700$$
$$X_1 \geq 0, \quad X_2 \geq 0$$

ion :

1 : Obtain the feasible region (shown in below figure) by the shaded area ABCDEF.



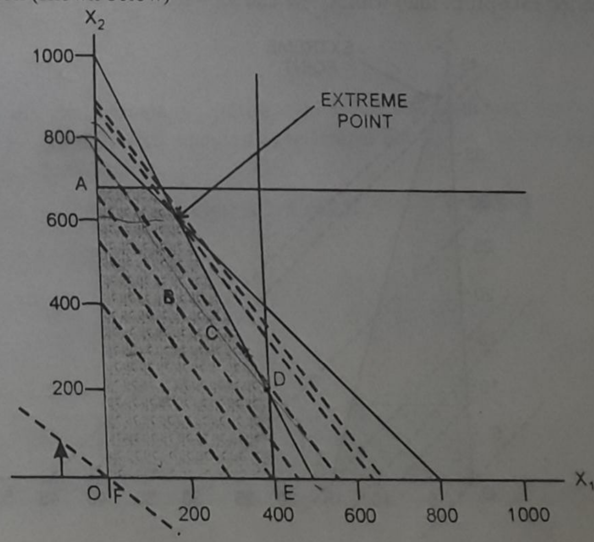
Step 2 : Let the profit of the company is Rs. 1200 (arbitrary). The objective function becomes $4X_1 + 3X_2 = 1200$, we draw this equation as a straight line in the feasible solution shown below. The line is called ISO-profit line.



X1	0	300
X2	400	0

$[4X_1 + 3X_2 = 1200, \text{ means } X_1 = 300, \; X_2 = 400]$

Step 3 : Now we move the ISO-profit line parallel to itself **farther** from the beginning (or origin). We find that one of the ISO profit line touches only Point C before leaving the feasible region. This line is termed as the highest possible ISO-profit line and point C gives the extreme point of the feasible region (shown below)



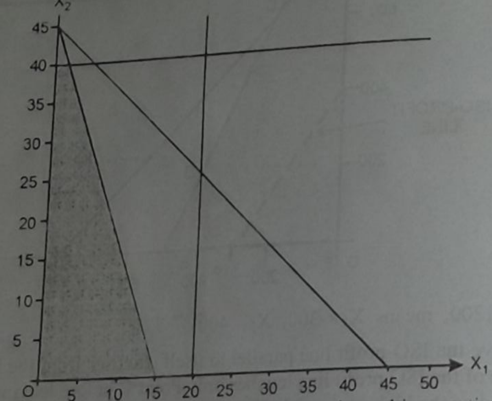
Step 4 : Hence the optimum feasible solution is Rs. 2600, were $X_1 = 200$ and $X_2 = 600$

Example 49. Max $Z = 8000\ X_1 + 7000\ X_2$

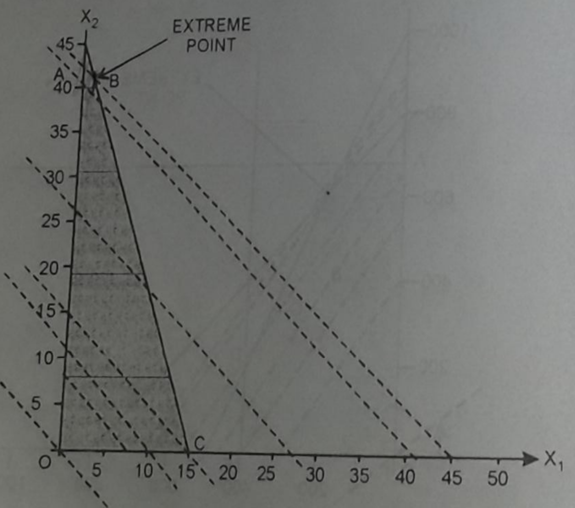
Subject to constraints

$$3X_1 + X_2 \leq 45$$
$$X + X_2 \leq 45$$
$$X_1 \leq 20$$
$$X_1 \geq 0,\ \ X_2 \geq 0$$

Solution : Obtain the feasible region



Let the profit to the company (arbitrary) is Rs. 56000 we draw this equation as a straight line in the feasible solution (called ISO-profit line) with $X_1 = 7$ and $X_2 = 8$



Optimal Solution is at Point B.

MINIMIZATION PROBLEM

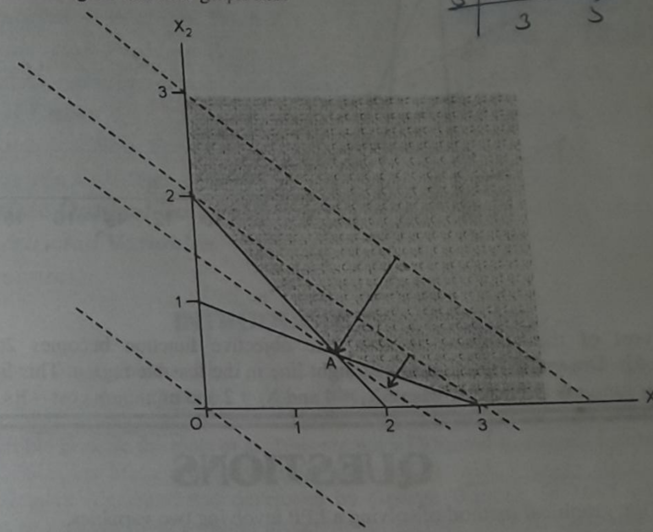
Example 50. Min $Z = 1 \cdot 5X_1 + 2 \cdot 5X_2$

Subject to constraints

$$X_1 + 3X_2 \geq 3$$
$$X_1 + X_2 \geq 2$$
where $$X_1,\ X_2 \geq 0$$

Solution : Obtain the feasible region from the graph first

5 | 15 25
3 5



Let the cost of the company (arbitrary) is 7·5. The objective function bec $1 \cdot 5\ X_1 + 2 \cdot 5X_2 = 7 \cdot 5$. We draw this equation as a straight line in the feasible region (This line ISO-cost line) with $X_1 = 5,\ X_2 = 3$.

Optimum solution at point A where Cost is 3·5 and $X_1 = 1 \cdot 5,\ X_2 = 0 \cdot 5$

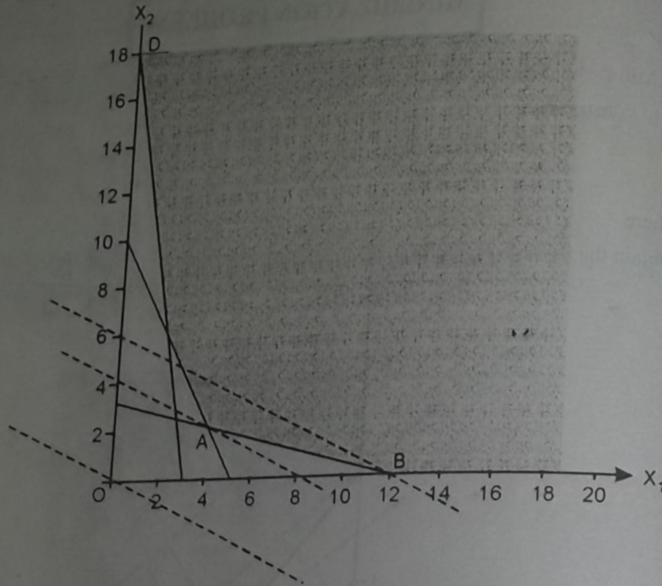
Example 51. Minimize $Z = 20X_1 + 40X_2$

20 | 20 −40
1 −2 = 40 or 200

Subject to constraints

$$36X_1 + 6X_2 \geq 108$$
$$3X_1 + 12X_2 \geq 36$$
$$20X_1 + 10X_2 \geq 100$$
$$X_1 \geq 0, X_2 \geq 0$$

Solution : Obtain feasible region on the graph



Let the cost of the company is 200. The objective function becomes $20X_1 + 40X_2 = 200$ $[X_1 = 10, X_2 = 5]$. Draw this equation in a straight line in the feasible region. This line is called ISO-cost line. Optimum solution is at Point A, where $X_1 = 4$ and $X_2 = 2$ and minimum cost = Rs. 160.

QUESTIONS

1. Explain the graphical method of solving a LPP involving two variables.
2. What steps are required in solving LPPs by graphic method ? Discuss in brief.
3. Explain the procedure of generating extreme point solutions to a LPP pointing out the assumptions made, if any.
4. What is feasibility region ? Is it necessary that is should always be a convex set ?
5. How would you know whether the solution to a linear programming problem is unique or not ? In this connection, state the condition that should be satisfied for more than one optimal solution to a problem to exist.
6. Write a short note on Extreme point solution to LPP.

   (G.N.D.U., B.Com III, Sept., 2006)
7. Explain the phenomenon of 'infeasibility' in an LP problem. What are the indicators of such a phenomenon ? How can it be handled ? Write a problem which you think will not have a feasible solution.

Page 157, 200 George Dantzig in 1947

4

Iterative

LINEAR PROGRAMMING–III

(Simplex Method)

- *Introduction*
- *Basic Terminology*
- *Computational Procedure of Simplex Method*
  - *Maximization Problem With All '≤' Constraints*
  - *Minimization Problem With All '≥' Constraints*
  - *Mixed Constraints Problems*
- *Exceptional Cases*
  - *Unbounded Solution*
  - *Infeasible Solution*
  - *Multiple Optimal Solution*
  - *Unrestricted Variables*
  - *Degeneracy*

★ Only slack & artificial variable can be BV.
★ Zero co-efficient always has a +ve sign
★ with artificial introduce M
In minimisation problem = +M.
In max. problem = −M.

INTRODUCTION

Iterative

The previous chapter on graphical method depicts that optimum (Linear Programming) solution is always ascertained at extreme points or corner points of the solution space and the graphical method is applicable in the case of two variables only. It means that the method is of limited application. The analytic solution is also not possible because the tools of analysis are not well suited to control inequalities. A more general method called 'Simplex Method' is suitable for solving LPP with a larger number of variables. Simplex method or Simplex Algorithm was developed by **George Dantzig** in 1947 which was made available in 1951. It is an iterative (Step by Step) procedure for solving LPP. At each step of the iteration, the value of the objective function is increased till no further improvement (*i.e.* maximum profits or minimum losses) is possible. Simplex Method always starts at the **zero solution point.** It provides that other steps in the solution finds the optimum product mix subject to the objective function and constraints.

Hence, this method provides an algorithm which consist of moving from one vertex of the region of feasible solution to another in such a way that the value of objective function at the succeeding vertex is less or more as the case may be than at the previous vertex. This procedure is repeated till the number of vertices is finite.

BASIC TERMS

1. Standard Form : A linear programme is said to be in standard form when all the constraints ar[e] expressed as equalities.

2. Slack Variable : It is a non-negative variable added to the left-hand side of less than or equal [to] (≤) type constraint in order to convert the constraint into an equality. This variable represents imagina[ry] unused resources

product with zero profit per unit. Since the slack variable represents an idle resource therefore it has to be positive (or non negative). Further the contribution per unit of slack variable is always taken as **zero** in the objective function of LPP, because profits are not made out of unused resources. Suppose in a particular problem, the constraint appears as

Pare not made out of unused resources

$$2X_1 + 3X_2 \le 8$$

By adding a suitable non-negative variable say S_1 to the left hand side, the inequality constraint can be written as

$$2X_1 + 3X_2 + S_1 = 8$$

3. Surplus Variable : It is a non-negative variable subtracted from the left-hand side of a greater-than or equal to ($\ge$) type constraint in order to convert the constraint into an equality. The value of this variable can be interpreted as the excess amount of the resources utilized over and above the given level. The contribution per unit of Surplus variable is also taken as zero in the objective function. *e.g*

$$2X_1 + 3X_2 \ge 8$$

By subtracting a suitable variable say S_1 from the LHS, the inequality constraint can be written as

fictitious

$$2X_1 + 3X_2 - S_1 = 8$$

identity

4. Artificial Variable : It is a fictitious variable added to the left-hand side of greater-than or equal to ($\ge$) type constraint and equal to (=) type constraint in order to form the identity matrix. It is introduced into the model to obtain an initial solution. It is to note that the initial solution obtained using artificial variable is not a feasible solution to the given problem. It only provides a starting point and the artificial variable is driven out in the normal course of applying the simplex algorithm. Consider the inequality

$$2X_1 + 3X_2 \ge 8$$

In this inequality we introduce artificial variable A_1, along with surplus variable S_1 and it can be restated as

$$2X_1 + 3X_2 - S_1 + A_1 = 8$$

If the constraint is of equal to (=) type, for example

$$2X_1 + 3X_2 = 8$$

other eq = 0
all eq = 1

then only artificial variable is introduced, as given below

$$2X_1 + 3X_2 + A_1 = 8.$$

5. Basic Variable : The variables which form identity matrix in a particular simplex table are known as basic variables. In other words a variable is said to be a basic variable in an equation if it appears with a unit coefficient in that equation and with zero coefficient in all other equations.

6. Basic Solution : Given a linear programme with 'm' linear equations and 'n' unknowns. A solution is said to be basic if it is obtained by setting $(n - m)$ variables to zero and solving the remaining 'm' variables. 'm'-variables are the basic variables whereas $(n - m)$ variables are the non-basic variables. The coefficient of basic variables form an identity matrix.

7. Basic Feasible Solution : A feasible solution to the problem which is also a basic solution to the given linear prgramming problem is known as a basic feasible solution.

8. Optimal Basic Feasible Solution : A basic feasible solution is optimal if it maximizes or minimizes the value of the objective function of a linear programming problem.

$(n-m) = 0$

STEPS FOLLOWED IN SOLVING MAXIMIZATION PROBLEM WITH ALL '≤' CONSTRAINTS

also added to obj. function with zero co-efficients

Step 1. Convert the inequalities into equations by adding Non-negative slack variables (say S_1, S_2 etc). The contribution per unit of slack variables is assumed to be zero (or negative, if given). This slack variables are called **Basic variables** and other variables are called Non-basic variables.

Step 2. Calculate the value of each Basic Variable in the objective function by assuming the value of each Non-basic variable equal to zero.

Step 3. Find the value of 'Z' by placing the value of each basic and non-basic variables in objective function.

Step 4. Draw the Initial Simplex Table as follows :

1	$C_j \rightarrow$ 2	3	C_1	C_2 4	0	0	5
Profit per Unit	B. V. (Program)	Qty.	X_1	X_2	S_1	S_2	R. Ratio
C_1	S_1	....	....	......	1	0	
C_2	S_2	......	....	......	0	1	
Total Profits Z_j		0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			C_1	C_2	0	0	

Column 1. **Profit per unit** : In this columns 'Profits' or contributions are listed the objective coefficients of the BV that are included in the specific program. Hence the profits coefficients of S_1 and S_2 are all zero.

Column 2. **Program (Basic Variables)** : It constrains B.V. that are included in the solution. Initially, they are Slack variables.

Column 3. **Quantity (Solution Value) :** In this column, quantity are listed the values of the BV included in the solution. This is also called solution values. Any variable that are not listed under BV are the Non BV and the value of each non-BV is zero.

Note : *The total profit contribution from a specific program (i.e. Z_j) can be calculated by using the formula $C_1x_1 + C_2x_2$. The values in the Z_j row in the column for other variables are calculated by using the same formula i.e. $C_1x_1 + C_2x_2$.*

Column 4. **Variables :** The column involves the number of real Slack variables it depicts each of the variable with profit/unit as read from the objective function at its top. The body of sub table carries **coefficients** of the variables as read from the constraints.

Note : *The coefficients of the BV consists a UNIT MATRIX in this initial sub table.*

Column 5. **Replacement ratio :** This ratio is ascertained by dividing value in the Quantity Column of each row by corresponding Key Column Value. This ratio represents how much quantity of variable can be produced based on that row taking the Key Column Value.

Step 5. Mark the **Key Column** *i.e.* the column having **maximum positive value** in $C_j - Z_j$ row shown by ↑ sign depicting the opportunity cost or loss of **not** introducing one unit of variable of that column. This column represents the **selected incoming variable** in the following Simplex Table.

Step 6. Find out replacement ratio.

Step 7. Mark the **Key Row** *i.e.* the row having **minimum non-negative key** ratio shown by → sign. This row depicts that no basic variable will ever be negative. In this row **zero** may be considered and it represent the **selected outgoing variable** from the current Simplex Table.

replacement ratio

Step 8. Select the Key element and encircle 'O' it. It is the element at the intersection of key row and key column. This element is also called as Pivot element.

Step 9. Replace the outgoing variable (→) by the incoming variable (↑) with its contribution or profit per unit.

Step 10. Calculate the new values of the key row by using the following formula :

$$\text{New Values of Key Raw} = \frac{\text{Old value of the key row}}{\text{Key element}}$$

Step 11. Calculate the new values of other rows by using the following formula :

New Values of the row = Old value of the row – [New value of key Row × Key column element]

Step 12. Draw another modified simplex table with new values of each row.

Step 13. Repeat all the steps till the values of $C_j - Z_j$ row become either zero or negative which depicts that introduction of one unit of variable having negative values will add negative contribution towards objective function. Find the optimal solution from the final Simplex Table.

Important :

1. *In maximisation problem, the solution is optimum if $C_j - Z_j$ row is either zero or negative.*
2. *While solving the problem, if any tie exists in choosing key column or key row, the following rules may be followed.*
   - *(a) The column farthest to the left may be selected if these is a tie between two elements in the index row.*
   - *(b) The nearest ratio to the top may be selected when ever those is a tie between two ratio in the ratio column.*

Example 1. Use simplex method to solve the following LPP :

Max. $Z = 6X_1 + 8X_2$

Subject to constraints :

$$2X_1 + 3X_2 \le 16$$
$$4X_1 + 2X_2 \le 16$$
$$X_1 \ge 0,\ X_2 \ge 0$$

Solution :

Convert the inequalities into equations by introducing slack variables.

Max. $Z = 6X_1 + 8X_2 + 0S_1 + 0S_2$

Subject to constraints

$$2X_1 + 3X_2 + S_1 + 0S_2 = 16$$
$$4X_1 + 2X_2 + 0S_1 + S_2 = 16$$

where $X_1 \ge 0,\ X_2 \ge 0,\ S_1 \ge 0,\ S_2 \ge 0$

Prepare Initial Simplex Table :

Simplex Table I

Profit per Unit	Contribution per Unit		$C_j \to 6$	8	0	0	R. Ratio
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	
0	S_1	16	2	3	1	0	160/30 → Key Row
0	S_2	16	4	2	0	1	160/20
Total Contribution $Z_j = 0$			0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			6	8	0	0	

minimum but greater than zero

↑ Key Column

Hence outgoing variable is S_1 and incoming variable is X_2 *i.e.* X_2 replaces S_1. Key element is 3.

Calculation of New values of :

Key Ratio

$16 \div 3 = 16/3$
$2 \div 3 = 2/3$
$3 \div 3 = 1$
$1 \div 3 = 1/3$
$0 \div 3 = 0$

Non-Key Row (S_2)

$16 - (16/3 \times 2) = 16/3$
$4 - (2/3 \times 2) = 8/3$
$2 - (1 \times 2) = 0$
$0 - (1/3 \times 2) = -\frac{2}{3}$
$1 - (0 \times 2) = 1$

$0 - [\frac{1}{3} \times 2] = -2/3$

$2 - [1 \times 2]$
$2 - 2 = 0$

Simplex Table II

Profit per Unit	Contribution per Unit		$C_j \to 6$	8	0	0	R. Ratio
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	
8	X_2	$\frac{16}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{16/3}{2/3} = 8$
0	S_2	$\frac{16}{3}$	$\frac{8}{3}$	0	$\frac{-2}{3}$	1	$\frac{16/3}{8/3} = 2$ → Key Row
Total Contribution $Z_j = \frac{128}{3}$			$\frac{16}{3}$	8	$\frac{8}{3}$	0	
Opportunity Cost $(C_j - Z_j)$			$\frac{2}{3}$	0	$\frac{-8}{3}$	0	

↑ Key Column

Entering variable is X_1 and outgoing variable is S_2. Key element is 8/3.

$\frac{16}{3} - \frac{4}{3} = \frac{12}{3} = 4$ $\quad 4 - [\frac{2}{3} \times \frac{2}{1}] \Rightarrow 4 - [\frac{4}{3}]$ $\frac{12-4}{3} = \frac{8}{3}$

Calculation of new values of :

Key row

$$\frac{16/3}{8/3}, \quad \frac{8/3}{8/3}, \quad \frac{0}{8/3}, \quad \frac{-2/3}{8/3}, \quad \frac{1}{8/3}$$

i.e. $2, \quad 1 \quad 0, \quad \frac{-1}{4}, \quad \frac{3}{8}$

Non-Key Row (X_2)

$\frac{16}{3} - (2/3 \times 2) = 4, \quad \frac{2}{3} - (2/3 \times 1) = 0, \quad 1 - (2/3 \times 0) = 1$

$1/3 - (2/3 \times -1/4) = 1/2, \quad 0 - (2/3 \times 3/8) = -1/4$

Simplex Table III

Profit per Unit	Contribution per Unit	$C_j \rightarrow$	6	8	0	0	R. Ratio
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	
8	X_2	4	0	1	1/2	−1/4	
6	X_1	2	1	0	−1/4	3/8	
Total Contribution $Z_j = 44$			6	8	$\frac{25}{10}$	$\frac{25}{100}$	
Opportunity Cost $(C_j - Z_j)$			0	0	$\frac{-25}{10}$	$\frac{-25}{100}$	

Since all the values of $C_j - Z_j \le 0$, the solution is optimal, where $X_1 = 2$, $X_2 = 4$ and the Maximum Profit = $(2 \times 6) + (4 \times 8)$ = Rs. 44.

Example 2. A company manufactures 3 types of parts which use precious metal platinum and gold. Due to shortage of these precious metals, the govt. regulates the amount that may be used per day. The relevant data with respect to supply requirements and profit are summarised in the table shown below.

Product	Platinum Required per unit (gm.)	Gold Required per unit (gm.)	Profit per unit (Rs.)
A	2	3	500
B	4	2	600
C	6	4	1200

Daily allotments of platinum and gold are 160 gm and 120 gm respectively. How should the company divide the supply of scarce precious metals ? What is the optimum profit ? (I.C.W.A. June 1987)

Solution :

Formulation of LPP

The three types of product A, B and C are expressed in number of units as X_1, X_2 and X_3 respectively. So, Maximize $Z = 500\,X_1 + 600\,X_2 + 1200\,X_3$.

Subject to $\quad 2\,X_1 + 4\,X_2 + 6\,X_3 \le 160$

$\quad 3\,X_1 + 2\,X_2 + 4\,X_3 \le 120$

Where $\quad X_1, X_2, X_3 \ge 0$

Adding slack variables S_1, S_2 to convert the inequalities into equalities, where S_1 and $S_2 \ge 0$. The problem can be restated as.

Maximize $\quad Z = 500\,X_1 + 600\,X_2 + 1200\,X_3 + 0S_1 + 0S_2$

Subject to $\quad 2\,X_1 + 4\,X_2 + 6\,X_3 + S_1 + 0S_2 = 160$

$\quad 3\,X_1 + 2\,X_2 + 4\,X_3 + 0S_1 + S_2 = 120$

Where $\quad X_1, X_2, X_3, S_1, S_2 \ge 0$

The above changed problem can be put into the following Simplex Table I.

Simplex Table I

Profit per Unit	Contribution per Unit $C_j \rightarrow$		500	600	1200	0	0	R. Ratio
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	
0	S_1	160	2	4	6	1	0	160/6 = 26.67→ Key Row
0	S_2	120	3	2	4	0	1	30
Total Contribution $Z_j = 0$			0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			500	600	1200	0	0	

↑
Key Column

Entering variable is X_3 and Outgoing variable is S_1. Key element is 6.

Calculation of new values of :

Key Row

160/6 2/6 4/6 6/6 1/6 0/6

i.e. 160/6 1/3 2/3 1 1/6 0

Non-key Row (S_2)

$120 - (4 \times 160/6) = 80/6, \quad 3 - (4 \times 1/3) = 5/3$

$2 - (4 \times 2/3) = -2/3, \quad 4 - (4 \times 1) = 0$

$0 - (4 \times 1/6) = -2/3 \quad 1 - (4 \times 0) = 1$

Following the same procedure we develop next table II and III, as below :

Simplex Table II

Profit per Unit	Contribution per Unit $C_j \rightarrow$		500	600	1200	0	0	R. Ratio
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	
1200	X_3	160/6	1/3	2/3	1	1/6	0	80
0	S_2	80/6	5/3	–4/6	0	–2/3	1	8 → Key Row
Total Contribution $Z_j = 32000$			400	800	1200	200	0	
Opportunity Cost $(C_j - Z_j)$			100	–200	0	–200	0	
			↑ Key Column					

Entering variable is X_1 and Outgoing variable is S_2. Key element is 5/3.

Simplex Table III

$C_j \rightarrow$ ↓	Basic Variables	Qty.	X_1 (500)	X_2 (600)	X_3 (1200)	S_1 (0)	S_2 (0)	R. Ratio
1200	X_3	24	0	4/5	1	27/90	–1/5	
500	X_1	8	1	–2/5	0	–2/5	3/5	
Total Contribution $Z_j = 32800$			500	760	1200	160	60	
Opportunity Cost $(C_j - Z_j)$			0	–160	0	–160	–60	

Since all $C_j - Z_j$ are negative or zero, so we get the optimum solution.

$X_1 = 8, X_2 = 0$

$X_3 = 24.$

and Max. Profit $Z = 500 \times 8 + 600 \times 0 + 1200 \times 24$

$= 4000 + 0 + 28800$

$Z = \text{Rs. } 32800.$

Example 3.

Maximize $Z = 2X_1 + 5X_2$

Subject to constraints

$$X_1 + 4X_2 \leq 24$$
$$3X_1 + X_2 \leq 21$$
$$X_1 + X_2 \leq 9$$
$$X_1 \geq 0, \quad X_2 \geq 0$$

Solution :

Convert the inequalities into equations by introducing Slack Variables.

Maximize $Z = 2X_1 + 5X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$X_1 + 4X_2 + S_1 + 0S_2 + 0S_3 = 24$$
$$3X_1 + X_2 + 0S_1 + S_2 + 0S_3 = 21$$

$$X_1 + X_2 + 0S_1 + 0S_2 + S_3 = 9$$
$$X_1 \geq 0, X_2 \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0$$

Simplex Table I

$C_j \rightarrow$ ↓	Basic Variables	Qty.	X_1 (2)	X_2 (5)	S_1 (0)	S_2 (0)	S_3 (0)	R. Ratio
0	S_1	24	1	4	1	0	0	24/4 = 6 → Key Row
0	S_2	21	3	1	0	1	0	21/1 = 21
0	S_3	9	1	1	0	0	1	9/1 = 9
Total Contribution $Z_j = 0$			0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			2	5	0	0	0	
				↑ Key Column				

Entering variable is X_2 and Outgoing variable is S_1. Key element is 4.

Simplex Table II

$C_j \rightarrow$ ↓	Basic Variables	Qty.	X_1 (2)	X_2 (5)	S_1 (0)	S_2 (0)	S_3 (0)	R. Ratio
5	X_2	6	1/4	1	1/4	0	0	24
0	S_2	15	11/4	0	–1/4	1	0	60/11
0	S_3	3	3/4	0	–1/4	0	1	4 → Key Row
Total Contribution $Z_j = 30$			5/4	5	5/4	0	0	
Opportunity Cost $(C_j - Z_j)$			3/4	0	–5/4	0	0	
			↑ Key Column					

Entering variable is X_1 and Outgoing variable is S_3. Key element is 3/4.

Simplex Table III

$C_j \rightarrow$ ↓	Basic Variables	Qty.	X_1 (2)	X_2 (5)	S_1 (0)	S_2 (0)	S_3 (0)	R. Ratio
5	X_2	5	0	1	1/3	0	–1/3	
0	S_2	4	0	0	2/3	1	–11/3	
2	X_1	4	1	0	–1/3	0	4/3	
Total Contribution $Z_j = 33$			2	5	1	0	1	
Opportunity Cost $(C_j - Z_j)$			0	0	–1	0	–1	

Since all the values at $C_j - Z_j \leq 0$, hence optimum solution is $X_1 = 4, X_2 = 5$

Maximum Profit = $(4 \times 2) + (5 \times 5) = \text{Rs. } 33$

A B C

x_1 x_2 x_3

Example 4. A manufacturer of leather belts makes three types of belts A, B and C which are processed on three machines M_1 and M_2 are M_3. Belt A requires 2 hours on machine M_1 and 3 hours on machine M_3. Belt B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 and Belt C requires 5 hours on machine M_2 and 4 hours on machines M_3. There are 8 hours of time per day available on machine M_1, 10 hours of time per day available on machine M_2 and 15 hours of time per day available on machine M_3. The profit gained from belt A is Rs. 3.00 per unit, from Belt B is Rs. 5.00 per unit, from belt C is Rs. 4.00 per unit. What should be the daily production of each types of belts so that the profit is maximum.

$2x_1 + 3x_2 + 0x_3 \le 8$

$0x_1 + 2x_2 + 5x_3 \le 10$

$3x_1 + 2x_2 + 4x_3 \le 15$

Solution :

The mathematical formulation of the given LPP is :

$2X_1 + 3X_2 \quad 0X_3 \le 8$

$0X_1 + 2X_2 + 5X_3 \le 10$

$3X_1 + 2X_2 + 4X_3 \le 15$

Maximize $\quad Z = 3X_1 + 5X_2 + 4X_3$

Subject to $\quad 2X_1 + 3X_2 \le 8$

$2X_2 + 5X_3 \le 10$

$3X_1 + 2X_2 + 4X_3 \le 15$

Where $\quad X_1, X_2, X_3 \ge 0$

After introduction of slack variables we convert inequalities into equations as given below :

Maximize $\quad Z = 3X_1 + 5X_2 + 4X_3 + 0S_1 + 0S_2 + 0S_3$

Subject to $\quad 2X_1 + 3X_2 + 0X_3 + S_1 + 0S_2 + 0S_3 = 8$

$0X_1 + 2X_2 + 5X_3 + 0S_1 + S_2 + 0S_3 = 10$

$3X_1 + 2X_2 + 4X_3 + 0S_1 + 0S_2 + S_3 = 15$

$X_1, X_2, X_3, S_1, S_2, S_3 \ge 0$

Simplex Table I

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	5 X_2	4 X_3	0 S_1	0 S_2	0 S_3	R. Ratio
5	S_1	8	2	3	0	1	0	0	8/3→ Key Row
0	S_2	10	0	2	5	0	1	0	5
2	S_3	15	3	2	4	0	0	1	
Contribution $Z_j = 0$			0	0	0	0	0	0	
rtunity Cost $(C_j - Z_j)$			3	5	4	0	0	0	

↑ Key Column

ering variable is X_2 and Outgoing variable is S_1. Key element is 3.

Simplex Table II

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	5 X_2	4 X_3	0 S_1	0 S_2	0 S_3	R. Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	∞
0	S_2	14/3	–4/3	0	5	–2/3	1	0	14/15→ Key Row
0	S_3	29/3	5/3	0	4	–2/3	0	1	
Total Contribution $Z_j = 40/3$			10/3	5	0	5/3	0	0	
Opportunity Cost $(C_j - Z_j)$			–1/3	0	+4	–5/3	0	0	

↑ Key Column

Entering variable is X_3 and Outgoing variable is S_2. Key element is 5.

Simplex Table III

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	5 X_2	4 X_3	0 S_1	0 S_2	0 S_3	R. Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	4
4	X_3	14/15	–4/15	0	1	–2/15	1/5	0	–ve
0	S_3	89/15	41/15	0	4	–2/15	–4/5	1	89/41→ Key Row
Total Contribution $Z_j = 256/15$			+34/15	5	4	17/15	4/5	0	
Opportunity Cost $(C_j - Z_j)$			+11/15	0	0	–17/15	–4/5	0	

$15 \times \frac{3}{15} - \frac{34}{15}$

↑ Key Column

Entering variable is X_1 and Outgoing variable is S_3. Key element is 41/15.

Simplex Table IV

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	5 X_2	4 X_3	0 S_1	0 S_2	0 S_3
5	X_2	50/41	0	1	0	15/41	8/41	–10/41
4	X_3	62/41	0	0	1	–6/41	5/41	4/41
3	X_1	89/41	1	0	0	–2/41	–12/41	15/41
Total Contribution $Z_j = 765/41$			3	5	4	45/41	24/41	–11/41
Opportunity Cost $(C_j - Z_j)$			0	0	0	–45/41	–24/41	–11/41

Since all $(C_j - Z_j)$ are ≤ 0, we have optimum solution

$$X_1 = \frac{89}{41},\ X_2 = \frac{50}{41},\ X_3 = \frac{62}{41} \quad \text{Max. Profit} = \frac{765}{41}$$

Example 5. Maximize $Z = 10X + 6Y + 4Z$

Subject to constraints :

$$X + Y + Z \le 100$$
$$10X + 4Y + 5Z \le 600$$
$$2X + 2Y + 6Z \le 300$$
$$X \ge 0,\ Y \ge 0,\ Z \ge 0$$

Solution :

Convert the inequalities into equations by introducing Slack Variables.

Max. $Z = 10X + 6Y + 4Z + 0S_1 + 0S_2 + 0S_3$

Subject to

$$X + Y + Z + S_1 + 0S_2 + 0S_3 = 100$$
$$10X + 4Y + 5Z + 0S_1 + S_2 + 0S_3 = 600$$
$$2X + 2Y + 6Z + 0S_1 + 0S_2 + S_3 = 300$$
$$X \ge 0,\ Y \ge 0,\ Z \ge 0,\ S_1 \ge 0,\ S_2 \ge 0,\ S_3 \ge 0$$

Simplex Table I

$C_j \rightarrow$ / ↓	Basic Variables	Qty.	X (10)	Y (6)	Z (4)	S_1 (0)	S_2 (0)	S_3 (0)	R. Ratio
0	S_1	100	1	1	1	1	0	0	100
0	S_2	600	10	4	5	0	1	0	60 → Key Row
0	S_3	300	2	2	6	0	0	1	150
Total Contribution $Z_j = 0$			0	0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			10	6	4	0	0	0	
			↑ Key Column						

Entering variable is X and Outgoing variable is S_2. Key element is 10.

Simplex Table II

$C_j \rightarrow$ / ↓	Basic Variables	Qty.	X (10)	Y (6)	Z (4)	S_1 (0)	S_2 (0)	S_3 (0)	R. Ratio
0	S_1	40	0	6/10	5/10	1	–1/10	0	400/6 → Key Row
10	X	60	i	4/10	5/10	0	1/10	0	150
0	S_3	180	0	12/10	5	0	–2/10	1	150
Total Contribution $Z_j = 600$			10	4	5	0	1	0	
Opportunity Cost $(C_j - Z_j)$			0	2	–1	0	–1	0	
				↑ Key Column					

Entering variable is Y and Outgoing variable is S_1. Key element is 6/10.

Simplex Table III

$C_j \rightarrow$ / ↓	Basic Variables	Qty.	X (10)	Y (6)	Z (4)	S_1 (0)	S_2 (0)	S_3 (0)	R. Ratio
6	Y	400/6	0	1	5/6	10/6	–1/6	0	
10	X	100/3	1	0	1/6	–2/3	1/6	0	
0	S_3	100	0	0	4	– 2	0	1	
Total Contribution $Z_j = \frac{2200}{3}$			10	6	40/6	10/3	4/6	0	
Opportunity Cost $(C_j - Z_j)$			0	0	–40/6	–10/3	–4/6	0	

Since all values in the $C_j - Z_j$ row are ≤ 0, Hence the optimum solution is

$$X = 100/3,\ Y = 400/6,\ Z = 2200/3$$

Maximum profit = $733\frac{1}{3}$

Example 6. Max. $Z = 4X + 3Y + 5Z - 150$

Subject to Constraints

$$2X + 3Y + 2Z \le 400$$
$$3X + 2Y + 2Z \le 350$$
$$X + 4Y + 2Z \le 300$$

where as, $X, Y, Z \ge 0$.

Solution :

After adding slack variables and assigning '0' co-efficient in the objective function

Max. $Z = 4X + 3Y + 5Z - 150 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$2X + 3Y + 2Z + S_1 = 400$$
$$3X + 2Y + 2Z + S_1 = 350$$
$$X + 4Y + 2Z + S_3 = 300$$

where as $X, Y, Z\ S_1, S_2, S_3 \ge 0$

Simplex Table I

$C_j \rightarrow$ / ↓	Basic Variables	Qty.	X (4)	Y (3)	Z (5)	S_1 (0)	S_2 (0)	S_3 (0)	R. Ratio
0	S_1	400	2	3	2	1	0	0	400/2 = 200
0	S_2	350	3	2	2	0	1	0	350/2 = 175
0	S_3	300	1	4	2	0	0	1	300/2 = 150 → Key Row
Total Contribution $Z_j = 0$			0	0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			4	3	5	0	0	0	
					↑ Key Column				

Entering variable is Z and Outgoing variable is S_3. Key element is 2.

Simplex Table II

$C_j \rightarrow$			4	3	5	0	0	0	R. Ratio
↓	Basic Variables	Qty.	X	Y	Z	S_1	S_2	S_3	
5	S_1	100	1	–1	0	1	0	–1	100/1 = 100
0	S_2	50	2	–2	0	0	1	–1	50/2 = 25 → Key Row
0	$\cancel{Z}$	150	1/2	2	1	0	0	1/2	150 × 2/1
Total Contribution Z_j = 750			5/2	10	5	0	0	5/2	
Opportunity Cost $(C_j - Z_j)$			3/2	–7	0	0	0	–5/2	

↑

Key Column

Entering variable is X and Outgoing variable is S_2. Key element is 2.

Simplex Table III

$C_j \rightarrow$			4	3	5	0	0	0
↓	Basic Variables	Qty.	X	Y	Z	S_1	S_2	S_3
0	S_1	75	0	0	0	1	–1/2	–1/2
4	X	25	1	–1	0	0	1/2	–1/2
5	Z	275/2	0	5/2	1	0	–1/4	3/4
Total Contribution Z_j = 1575/2			4	17/2	5	0	3/4	–7
Opportunity Cost $(C_j - Z_j)$			0	–11/2	0	0	–3/4	–7/4

Since all $C_j - Z_j$ are ≤ 0, we have optimum solution.

$X = 25$

$Y = 0$

$Z = 137.5$

Max. Profit $Z = 25 \times 4 + 0 \times 3 + 137.5 \times 5 - 150$

$= 637.5$ per day **Ans.**

Example 7. Using Simplex Method to solve the LPP .

Maximize $\quad Z = 3X_1 + 4X_2 + X_3$

Subject to constraints

$$X_1 + 2X_2 + 3X_3 \leq 90$$

$$2X_1 + X_2 + X_3 \leq 60$$

$$3X_1 + X_2 + 2X_3 \leq 80$$

Where $\quad X_1 \geq 0, X_2 \geq 0, X_3 \geq 0$

Solution :

Convert the inequalities into equations by introducing slack variables.

Max. $\quad Z = 3X_1 + 4X_2 + X_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$X_1 + 2X_2 + 3X_3 + S_1 + 0S_2 + 0S_3 = 90$$

$$2X_1 + X_2 + X_3 + 0S_1 + S_2 + 0S_3 = 60$$

$$3X_1 + X_2 + 2X_3 + 0S_1 + 0S_2 + S_3 = 80$$

Where $X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$

Simplex Table I

$C_j \rightarrow$			3	4	1	0	0	0	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	90	1	2	3	1	0	0	45 → Key Row
0	S_2	60	2	1	1	0	1	0	60
0	S_3	80	3	1	2	0	0	1	80
Total Contribution Z_j = 0			0	0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			3	4	1	0	0	0	

↑

Key Column

$1 - (0 \times 1) = 1 - 0$
$= 1$

Entering variable is X_2 and Outgoing variable is S_1. Key element is 2.

Simplex Table II

$C_j \rightarrow$			3	4	1	0	0	0	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	S_3	
4	X_2	45	1/2	1	3/2	1/2	0	0	90
0	S_2	15	3/2	0	–1/2	–1/2	1	0	10 → Key Row
0	S_3	35	5/2	0	1/2	–1/2	0	1	14
Total Contribution Z_j = 180			2	4	6	2	0	0	
Opportunity Cost $(C_j - Z_j)$			1	0	–5	–2	0	0	

↑

Key Column

Entering variable is X_1 and Outgoing variable is S_2. Key element is 3/2.

Simplex Table III

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	4 X_2	1 X_3	0 S_1	0 S_2	0 S_3	R. Ratio
4	X_2	40	0	1	5/3	2/3	–1/3	0	
3	X_1	10	1	0	–1/3	–1/3	2/3	0	
0	S_3	10	0	0	4/3	1/3	–5/3	1	
Total Contribution Z_j = 190			3	4	17/3	5/3	2/3	0	
Opportunity Cost $(C_j - Z_j)$			0	0	– 14/3	–5/3	–2/3	0	

Since all $C_j - Z_j \leq 0$, the solution is optimum at $X_1 = 10$, $X_2 = 40$, $X_3 = 0$

Thus, value of $Z = (10 \times 3) + (40 \times 4) = 190$

EXERCISE 4.1

1. Solve the LP Problem using simplex method.

Maximize $Z = 5X_1 + 3X_2$

Subject to constraints

$$3X_1 + 5X_2 \leq 15$$
$$5X_1 + 2X_2 \leq 10$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 20/19$, $X_2 = 45/19$, Max. $Z = 235/19$]

2. Solve the LP Problem using simplex method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$2X_1 + X_2 \leq 5$$
$$X_1 + X_2 \leq 3$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 2$, $X_2 = 1$, Max. $Z = 8$]

3. Solve the LP Problem using simplex method.

Maximize $Z = 5X_1 + 7X_2$

Subject to constraints

$$X_1 + X_2 \leq 4$$
$$3X_1 - 8X_2 \leq 24$$
$$10X_1 + 7X_2 \leq 35$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 0$, $X_2 = 4$, Max. $Z = 28$]

4. Solve the LP Problem using simplex method.

Maximize $Z = 8X + 16Y$

Subject to constraints

$$X + Y \leq 200$$
$$Y \leq 125$$
$$3X + 6Y \leq 900$$
$$X, Y \geq 0$$

[GNDU (April) 2002]

[**Ans.** $X = 50$, $Y = 125$, Max. $Z = 2400$]

5. Solve the LP Problem using simplex method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$2X_1 + X_2 \leq 40$$
$$X_1 + X_2 \leq 24$$
$$2X_1 + 3X_2 \leq 60$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 16$, $X_2 = 8$, Max. $Z = 64$]

6. Solve the LP Problem using simplex method.

Maximize $Z = 3X_1 + 5X_2$

Subject to constraints

$$3X_1 + 2X_2 \leq 18$$
$$X_1 \leq 4$$
$$X_2 \leq 6$$
$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 2$, $X_2 = 6$, Max. $Z = 36$]

7. Solve the LP Problem using simplex method.

Maximize $Z = 2X_1 + X_2$

Subject to constraints

$$X_1 + 2X_2 \leq 10$$
$$X_1 + X_2 \leq 6$$
$$X_1 - X_2 \leq 2$$
$$X_1 - 2X_2 \leq 1$$
$$X_1, X_2 \geq 0 \text{ (Non-negativity constraint)}$$

[Meerut (Maths.) 77, Punjab Univ. April 2002]

[**Ans.** $X_1 = 4$, $X_2 = 2$, Max. $Z = 10$]

8. Solve the LP Problem using simplex method.

Maximize $Z = 4X_1 + 5X_2 + 9X_3 + 11X_4$

Subject to constraints

$$X_1 + X_2 + X_3 + X_4 \leq 15$$

$$7X_1 + 5X_2 + 3X_3 + 2X_4 \leq 120$$

$$3X_1 + 5X_2 + 10X_3 + 15X_4 \leq 100$$

$X_1, X_2, X_3, X_4 \geq 0$ (Non-negativity constraint)

[**Ans.** $X_1 = 50/7, X_2 = 0, X_3 = 55/7, X_4 = 0$, Max. $Z = 695/7$]

9. The consumer product corporation wishes to plan its advertising strategy. There are two media under consideration magazine I and II. Magazine I has a reach of 2000 potential customers and magazine 2 has a reach of 3000 potential customers. The cost per page of advertising is Rs. 600 and Rs. 900 in magazine I and II respectively. The firm has a monthly budget of Rs. 9000. There is an important requirement that the total reach for the income group under Rs. 25,000 per annum per should not exceed 3000 potential customers. The reach in magazine I and II for this income group is 300 and 150 potential customer. How many pages should be bought in the magazine to maximize the customer reach.

[**Ans.** $X_1 = 0, X_2 = 10$, Max. $Z = 30000$]

10. A small furniture manufacturer produces wooden chairs and tables. The manufacturer wants to determine the product mix that maximizes profits. Each chair generates Rs. 15 of profits, whereas each table generates Rs. 30 in profits. There are two bottleneck operations in the production process ; finishing and assembly. Each chair needs 4 hours of finishing and 2 hours of assembly while each table requires 5 hours of finishing and 4 hours of assembly. There is a capacity of 200 hours in finishing and 240 hours in assembly. Finally, forecast indicate a demand potential of 40 chairs and 28 tables. Management considers these forecasts as upper bounds on production. (*i*) Determine the optimum product mix. (*ii*) What is shadow price of finishing capacity, and Assembly capacity ?

[**Ans.** $X_1 = 15, X_2 = 28$, Max. $Z = 1065$]

11. Solve the LP Problem using simplex method.

Maximize $Z = X_1 + X_2 + 3X_3$

Subject to constraints

$$3X_1 + 2X_2 + X_3 \leq 3$$

$$2X_1 + X_2 + 2X_3 \leq 2$$

$$X_1, X_2, X_3 \geq 0$$

[*G.N.D.U. B.Com. III 2005*]

[**Ans.** $X_1 = 0, X_2 = 0, X_3 = 1$, Max. $Z = 3$]

12. Solve the LP Problem using simplex method.

Maximize $Z = 3X_1 + 2X_2$

Subject to constraints

$$2X_1 + X_2 \leq 10$$

$$X_1 + 3X_2 \leq 6$$

$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 24/5, X_2 = 2/5$, Max. $Z = 76/5$]

13. Solve the LP Problem using simplex method.

Maximize $Z = 5X_1 + 3X_2$

Subject to constraints

$$X_1 + X_2 \leq 2$$

$$5X_1 + 2X_2 \leq 10$$

$$3X_1 + 8X_2 \leq 12$$

$$X_1, X_2 \geq 0$$

[**Ans.** $X_1 = 2, X_2 = 0$, Max. $Z = 10$]

14. A company is engaged in the production of three products A, B and C. Product A and B are processed in three operations I, II and III, whereas product C is processed in operation I and II only. The maximum capacities of operations per week of I, II and III are 100 hours, 120 hours and 80 hours respectively. The times required to produce one item in each of the operations are given below :

Operations	Time (hrs.) required to produce one unit of A	B	C
I	3	1	2
III	1	4	1
III	2	3	0

Profit (Rs.) per unit of A, B and C is Rs. 10, Rs. 12 and Rs. 8 respectively. Find the optimum production of A, B and C so as to maximize profit.

[*I.C.W.A. June 1988*]

[**Ans.** $X_1 = 0, X_2 = 20, X_3 = 40$, Max. $Z = 560$]

15. Solve the LP Problem using simplex method.

Maximize $Z = 8X_1 + 19X_2 + 7X_3$

Subject to constraints

$$3X_1 + 4X_2 + X_3 \leq 25$$

$$X_1 + 3X_2 + 3X_3 \leq 50$$

$$X_1, X_2, X_3 \geq 0$$

[**Ans.** $X_1 = 0, X_2 = 25/9, X_3 = 125/9$, Max. $Z =$

16. Solve the LP Problem using simplex method.

Maximize $Z = 2X_1 + 5X_2$

Subject to constraints

$$X_1 + 3X_2 \le 3$$
$$3X_1 + 2X_2 \le 6$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 12/7, X_2 = 3/7$, Max. $Z = 39/7$]

17. Solve the LP Problems using simplex method.

Maximize $Z = 2X + 5Y$

Subject to restrictions

$$X + Y \le 600$$
$$X \le 400$$
$$Y \le 300$$
$$X, Y \ge 0$$

[**Ans.** $X = 300$, $Y = 300$, Max. $Z = 2100$]

18. Solve the LP Problem using simplex method.

Maximize $Z = 10X_1 + 6X_2$

Subject to constraints

$$X_1 + X_2 \le 2$$
$$2X_1 + X_2 \le 4$$
$$3X_1 + 8X_2 \le 12$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 2, X_2 = 0$, Max. $Z = 20$]

19. A company manufactures 3 products A, B and C which are manufactured in two departments (I) and (II). The company has enough demand for these products. However, the production capacity is limited. The relevant data are as follows :

Product	Department (I) (hours)	Department (II) (hours)	Expected Profit/unit
A	60	3	2,100
B	50	2	1,600
C	40	1	1,300
Availability	4,000	144	

Find out the optimum product mix and maximum profit. If the company wants to expand its production capacity, which production department should be given priority and why ?

[**Ans.** $X_1 = 88/3, X_2 = 0, X_3 = 56$, Max. $Z = 134400$]

20. Solve the LP Problem using Simplex Method

Maximize $Z = 40X_1 + 35X_2 + 30X_3$

Subject to constraints

$$2X_1 + 3X_2 + 2X_3 \le 120$$
$$4X_1 + 3X_2 + X_3 \le 160$$

$$3X_1 + 2X_2 + 4X_3 \le 100$$
$$X_1 + X_2 + X_3 \le 40$$
$$X_1, X_2, X_3 \ge 0 \text{ (Non-negativity constraint)}$$

[**Ans.** $X_1 = 20, X_2 = 20, X_3 = 0$, Max. $Z = 1500$]

21. A company produces two types of products say type A and B. Product B is of superior quality and product A is of a lower quality. Profits on the two types of products are Rs. 30 and 40 respectively. The data on resources required, availability of resources are given below :

	Requirements Product A	Product B	Capacity available per month
Raw materials (kg.)	60	120	12,000
Machining (hours per piece)	8	5	600
Assembly (man hour)	3	4	500

How should the company manufacture the two types of products in order to have a maximun overall profit ?

[*I.C.W.A. June 1985*

[**Ans.** $X_1 = 200/11, X_2 = 1000/11, X_3 = 0$, Max. $Z = 46000/1$

22. Solve the LP Problem using simplex method.

Maximize $R = 2X - 3Y + Z$

Subject to restrictions

$$3X + 6Y + Z \le 6$$
$$4X + 2Y + Z \le 4$$
$$X - Y + Z \le 3$$
$$X, Y, Z \ge 0 \text{ (Non-negativity constraint)}$$

[**Ans.** $X = 1/3, Z = 8/3$, Max. $Z = 1$

optimum solution – All values in index row ≥ 0

SIMPLEX METHOD FOR MINIMIZATION LPP

The solution procedure for the minimization linear programming problem is similar to the one maximization problem, except few differences. In case of a minimization problem, the entering var will be the one having the highest negative value in the index $(C_j - Z_j)$ row and the optimal solution w obtained when all the values in the index row become greater than or equal to ($\ge$) zero. Minimiz problem can also be solved by converting the given problem into an equivalent maximization problem then applying the usual simplex method.

Example 8. Use Simplex method to solve the LPP

Minimize $\quad Z = 3X_1 + 2.5X_2$

Subject to constraints

$$2X_1 + 4X_2 \ge 40$$
$$5X_1 + 2X_2 \ge 50$$
$$X_1 \ge 0, \; X_2 \ge 0$$

Solution :

Introducing surplus variables S_1 and S_2 in first and second constraints respectively, the problem can be restated as

Minimize $Z = 3X_1 + 2.5X_2 + 0S_1 + 0S_2$

Subject to

$$2X_1 + 4X_2 - S_1 + 0S_2 = 40$$

$$5X_1 + 2X_2 + 0S_1 - S_2 = 50$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

In order to provide an initial feasible solution, introduce artificial variables A_1 and A_2 in the problem. Being fictitious variables, they must be driven out of the system. In order to drive out the artificial variables, a value 'M' (very high value) is assigned to each one of them. In case of minimization LPP, we assign +M coefficient to each of the artificial variables in the objective function, whereas, in case of maximization type problem, each of the artificial variables introduced has a coefficient $-M$. The method adopted is Big M method*.

The problem can now be restated as :

Minimize $Z = 3X_1 + 2.5X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to

$$2X_1 + 4X_2 - S_1 + 0S_2 + A_1 + 0A_2 = 40$$

$$5X_1 + 2X_2 + 0S_1 - S_2 + 0A_1 + A_2 = 50$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0.$$

And not S_1 & S_2 as we have to remove Artificial variables

Simplex Table I

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	2.5 X_2	0 S_1	0 S_2	M A_1	M A_2	R. Ratio
M	A_1	40	2	4	−1	0	1	0	20
M	A_2	50	5	2	0	−1	0	1	10 → Key Row
Total Contribution $Z_j = 90M$			7M	6M	−M	−M	M	M	
Opportunity Cost $(C_j - Z_j)$			3−7M	2.5−6M	M	M	0	0	

↑ Key Column

Only see the coefficient of M i.e which one is most negative

Entering variable is X_1 and Outgoing variable is A_2. Key element is 5.

Imp

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*The big M Method : This method is applied when we have some constraints of '≥' and /or '=' type in the linear programming problem. A variant of the simplex method, this method is also known as Charne's Penalty method. In this method while applying simplex method as and when an artificial variable leaves the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Charne Penalty Method

Simplex Table II

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	2.5 X_2	0 S_1	0 S_2	M A_1	R. Ratio
M	A_1	20	0	16/5	−1	2/5	1	6.25 → Key Row
3	X_1	10	1	2/5	0	−1/5	0	25
Total Contribution $Z_j = 20M + 30$			3	16M+6/5	−M	2M−3/5	M	
Opportunity Cost $(C_j - Z_j)$			0	2.5−16M+6/5	M	−2M+3/5	0	

↑ Key Column

Entering variable is X_2 and Outgoing variable is A_1. Key element is 16/5.

Simplex Table III

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	2.5 X_2	0 S_1	0 S_2	R. Ratio
2.5	X_2	6.25	0	1	−5/16	1/8	
3	X_1	7.5	1	0	2/16	−2/8	
Total Contribution $Z_j = 38.125$			3	2.5	−.406	−.438	
Opportunity Cost $(C_j - Z_j)$			0	0	+.406	+.438	

Since all $C_j - Z_j$ are ≥ 0, the optimal solution has been reached where $X_1 = 7.5$, $X_2 = 6.25$, $Z = 38.125$

Example 9. Using Surplus and Artificial Variables, solve the following :

Minimize $Z = 5X_1 + 6X_2$

Subject to $2X_1 + 5X_2 \geq 1500$

$3X_1 + X_2 \geq 1200$

where $X_1, X_2 \geq 0$

[C.A. (Final) May 198

Solution :

In order to convert inequalities into equations we will introduce surplus and artificial variables in the above problem assigning zero coefficient to surplus and M to artificial variables in the above constraints.

Min $Z = 5X_1 + 6X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to $2X_1 + 5X_2 - S_1 + 0S_2 + A_1 + 0A_2 = 1500$

$3X_1 + X_2 + 0S_1 - S_2 + 0A_1 + A_2 = 1200$

Where $X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

Simplex Table I

$C_j \rightarrow$			5	6	0	0	M	M	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	A_2	
M	A_1	1500	2	5	–1	0	1	0	300 → Key Row
M	A_2	1200	3	1	0	–1	0	1	1200
Total Contribution Z_j = 2700 M			5 M	6M	–M	–M	M	M	
Opportunity Cost $(C_j - Z_j)$			5–5M	6–6M	M	M	0	0	

↑ Key Column

Entering variable is X_2 and leaving variable is A_1. Key element is 5.

Simplex Table II

$C_j \rightarrow$			5	6	0	0	M	M	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	A_2	
6	X_2	300	2/5	1	–1/5	0	–	0	750
M	A_2	900	13/5	0	1/5	–1	–	1	4500/13 → Key Row
Total Contribution Z_j = 1800+900M			$\frac{12+13M}{5}$	6	$\frac{-6+M}{5}$	–M	–	M	
Opportunity Cost $(C_j - Z_j)$			$\frac{13-13M}{5}$	0	$\frac{6-M}{5}$	M	–	0	

↑ Key Column

Entering variable is X_1 and Outgoing variable is A_2. Key element is 13/5.

Simplex Table III

$C_j \rightarrow$			5	6	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	
6	X_2	2100/13	0	1	–3/13	2/13	
5	X_1	4500/13	1	0	1/13	–5/13	
Total Contribution Z_j = 2700			5	6	–1	– 1	
Opportunity Cost $(C_j - Z_j)$			0	0	1	1	

Since all the elements in $C_j - Z_j$ row are +ve or zero so this is the optimal solution. Where X_1 = *4500/13*, $X_2 = 2100/13$ and $Z_j = 2700$

Example 10. *Food A contains 20 units of vitamin X and 40 units of vitamin Y per gram. Food B contains 30 unit each of vitamin X and Y. The daily minimum human requirement of vitamin X and Y are 900 unit and 1200 units respectively. How many grams of each type of good should be consumed so as to minimise the cost, if food A cost Rs. 0.60 per gram and food B costs Rs. 0.80 per gram.*

just to remove decimals ×100 ×100 unit per gram

[I.C.W.A. Dec., 1986]

Solution :

The mathematical formulation of the given LPP is given below : In units

Min. $Z = 60\,X_1 + 80\,X_2$

Subject to $20\,X_1 + 30\,X_2 \geq 900$

$40\,X_1 + 30\,X_2 \geq 1200$

$X_1\,,\,X_2 \geq 0$

We introduce surplus and artificial variable and assign '0' co-efficient to surplus variable and 'M' to artificial variables in the objective function.

Objective Function Min. $= 60\,X_1 + 80\,X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to $20\,X_1 + 30\,X_2 - S_1 + 0S_2 + A_1 + 0A_2 = 900$

$40\,X_1 + 30\,X_2 + 0S_1 - S_2 + 0A_1 + A_2 = 1200$

where $X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

Simplex Table I

$C_j \rightarrow$			60	80	0	0	M	M	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	A_2	
M	A_1	900	20	30	–1	0	1	0	45
M	A_2	1200	40	30	0	–1	0	1	30 → Key Row
Total Contribution Z_j = 2100M			60	60M	–M	–M	M	M	
Opportunity Cost $(C_j - Z_j)$			60–60M	80–60M	M	M	0	0	

↑ Key Column

Entering variable is X_1 and Outgoing variable is A_2. Key element is 40.

Simplex Table II

$C_j \rightarrow$			60	80	0	0	M	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	
M	A_1	300	0	15	–1	1/2	1	20 → Key Row
60	X_1	30	1	3/4	0	–1/40	0	40
Total Contribution Z_j = 1800 + 300 M			60	45 + 15 M	–M	–3 +M/2	M	
Opportunity Cost $(C_j - Z_j)$			0	35-15M	M	3–M/2	0	

↑ Key Column

Entering variable is X_2 and Outgoing variable is A_1. Key element is 15.

Simplex Table III

$C_j \rightarrow$			60	80	0	0
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2
80	X_2	20	0	1	–1/15	1/30
60	X_1	15	1	0	1/20	–1/20
Total Contribution Z_j = 2500			60	80	–7/3	– 1/3
Opportunity Cost $(C_j - Z_j)$			0	0	7/3	1/3

Since all $C_j - Z_j \geq 0$, we are having optimum solution.

$X_1 = 15,\ X_2 = 20\ Z =$ Rs. 25 **Ans.**

Example 11. Min. $Z = 4X_1 + 2X_2$

Subject to $3X_1 + X_2 \geq 27$

$X_1 + X_2 \geq 21$

$X_1 + 2X_2 \geq 30$

Where X_1 and $X_2 \geq 0$

Solution :

Introducing the surplus variables S_1, S_2, S_3 with zero coefficients in order to get equalities and artificial variables A_1, A_2, A_3 with '+ M' coefficients in order to get unit matrix.

Min. $Z = 4X_1 + 2X_2 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2 + MA_3$

Subject to $3X_1 + X_2 - S_1 + 0S_2 + 0S_3 + A_1 + 0A_2 + 0A_3 = 27$

$X_1 + X_2 + 0S_1 - S_2 + 0S_3 + 0A_1 + A_2 + 0A_3 = 21$

$X_1 + 2X_2 + 0S_1 + 0S_2 - S_3 + 0A_1 + 0A_2 + A_3 = 30$

Where as : $X_1, X_2, S_1, S_2, S_3, A_1, A_2, A_3 \geq 0$

Simplex Table I

$C_j \rightarrow$			4	2	0	0	0	M	M	M	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	A_1	A_2	A_3	R. Ratio
M	A_1	27	3	1	–1	0	0	1	0	0	9→ Key Row
M	A_2	21	1	1	0	–1	0	0	1	0	21
M	A_3	30	1	2	0	0	–1	0	0	+1	30
Total Contribution Z_j = 78 M			5M	4M	–M	–M	–M	M	M	M	
Opportunity Cost $(C_j - Z_j)$			4-5M	2-4M	M	M	M	0	0	0	
			↑ Key Column								

Entering variable is X_1 and Outgoing variable is A_1. Key element is 3.

Simplex Table II

$C_j \rightarrow$			4	2	0	0	0	M	M	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	A_2	A_3	R. Ratio
4	X_1	9	1	1/3	–1/3	0	0	0	0	27
M	A_2	12	0	2/3	1/3	–1	0	1	0	18
M	A_3	21	0	5/3	1/3	0	–1	0	1	63/5→Key Row
Total Contribution Z_j = 36+33M			4	4+7M/3	–4+2M/3	–M	–M	M	M	
Opportunity Cost $(C_j - Z_j)$			0	2-7M/3	4-2M/3	M	M	0	0	
				↑ Key Column						

Entering variable is X_2 and Outgoing variable is A_3. Key element is 5/3.

Simplex Table III

$C_j \rightarrow$			4	2	0	0	0	M	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	A_2	R. Ratio
4	X_1	24/5	1	0	–6/15	0	1/5	0	24
M	A_2	18/5	0	0	1/5	–1	2/5	1	9→Key Row
2	X_2	63/5	0	1	1/5	0	–3/5	0	21
T. Contribution Z_j = 222 + 18M/5			4	2	–6+M/5	–M	2+2M/5	M	
Opportunity Cost $(C_j - Z_j)$			0	0	6-M/5	M	2-2M/5	0	
							↑ Key Column		

Entering variable is S_3 and Outgoing variable is A_2. Key element is 2/5.

Simplex Table IV

$C_j \rightarrow$			4	2	0	0	0	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	R. Ratio
4	X_1	3	1	0	–1/2	1/2	0	
0	S_3	9	0	0	1/2	–5/2	1	
2	X_2	18	0	1	1/2	–3/2	0	
Total Contribution Z_j = 48			4	2	–1	–1	0	
Opportunity Cost $(C_j - Z_j)$			0	0	1	1	0	

In the fourth simplex table all the elements of row $C_j - Z_j$ are either zero or positive, so optimal solution is : $X_1 = 3,\ X_2 = 18,\ Z = 48$ **Ans.**

Example 12. Min $Z = 4X_1 + X_2$

Subject to $3X_1 + 4X_2 \geq 20$

$-X_1 - 5X_2 \leq -15$

where $X_1, X_2 \geq 0$

[M.A. Part I, G.N.D.U. April 2001]

Solution :

The given problem can be presented as below after multiplying constraint 2 by –1 :

Min. $Z = 4X_1 + X_2$

Subject to $3X_1 + 4X_2 \geq 20$

$X_1 + 5X_2 \geq 15$

$X_1, X_2 \geq 0$

Subtracting Surplus variable with zero coefficient and adding artificial variable with + M coefficient the given problem becomes

Min. $Z = 4X_1 + X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to $3X_1 + 4X_2 - S_1 + 0S_2 + A_1 + 0A_2 = 20$

$X_1 + 5X_2 + 0S_1 - S_2 + 0A_1 + A_2 = 15$

where $X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

Simplex Table I

$C_j \rightarrow$			4	1	0	0	M	M	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	A_2	R. Ratio
M	A_1	20	3	4	–1	0	1	0	5
M	A_2	15	1	5	0	–1	0	1	3→Key Row
T. Contribution Z_j = 35M			4M	9M	–M	–M	M	M	
Opportunity Cost $(C_j - Z_j)$			4-4M	1–9M	M	M	0	0	

↑ Key Column

Entering variable is X_2 and Outgoing variable is A_2. Key element is 5.

Simplex Table II

$C_j \rightarrow$			4	1	0	0	M	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	R. Ratio
M	A_1	8	11/5	0	–1	4/5	1	40/11→Key Row
1	X_2	3	1/5	1	0	–1/5	0	15
T. Contribution Z_j = 3 + 8M			1+11M/5	1	–M	–1+4M/5	M	
Opportunity Cost $(C_j - Z_j)$			19–11M/5	0	M	1–4M/5	0	

↑ Key Column

Entering variable is X_1 and Outgoing variable is A_1. Key element is 11/5.

Simplex Table III

$C_j \rightarrow$			4	1	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	R. Ratio
4	X_1	40/11	1	0	–5/11	4/11	10→ Key Row
1	X_2	25/11	0	1	1/11	–3/11	–ve
Total Contribution Z_j = 185/11			4	1	–19/11	13/11	
Opportunity Cost $(C_j - Z_j)$			0	0	19/11	–13/11	

↑ Key Column

Entering variable is S_2 and Outgoing variable is X_1. Key element is –4/11.

Simplex Table IV

$C_j \rightarrow$			4	1	0	0
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2
0	S_2	10	11/4	0	–5/4	1
1	X_2	5	3/ 4	1	–1/4	0
Total Contribution Z_j = 5			3/ 4	1	–1/4	0
Opportunity Cost $(C_j - Z_j)$			13/4	0	1/ 4	0

Since all the elements in $C_j - Z_j$ row are +ve or zero. Thus the solution is optimal

$X_1 = 0, \; X_2 = 5, \; Z = 1 \times 5 = 5$ **Ans.**

Example 13. Min $Z = 4X_1 + 3X_2 + X_3$

Subject to $X_1 + 2X_2 + 4X_3 \geq 12$

$3X_1 + 2X_2 + X_3 \geq 8$

Where $X_1, X_2, X_3 \geq 0$

Solution :

Introducing surplus variables S_1 and S_2 with zero coefficient in order to get equalities and Artificial variables A_1, A_2 with '+ M' coefficient in order to get unit matrix.

Min. $Z = 4X_1 + 3X_2 + X_3 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to $X_1 + 2X_2 + 4X_3 - S_1 + 0S_2 + A_1 + 0A_2 = 12$

$3X_1 + 2X_2 + X_3 + 0S_1 - S_2 + 0A_1 + A_2 = 8$

Where $X_1, X_2, X_3, S_1, S_2, A_1, A_2 \geq 0$

Now reformulated problem can be placed in the following table.

Simplex Table I

$C_j \rightarrow$			4	3	1	0	0	M	M	
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	A_2	A_3	R. Ratio
M	A_1	12	1	2	4	–1	0	1	0	3→Key Row
M	A_2	8	3	2	1	0	–1	0	1	8
Total Contribution Z_j = 20M			4M	4M	5M	–M	–M	M	M	
Opportunity Cost $(C_j - Z_j)$			4-4M	3-4M	1-5M	M	M	0	0	

↑ Key Column

Entering variable is X_3 and Outgoing variable is A_1. Key element is 4.

Simplex Table II

C_j →			4	3	1	0	0	M	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	A_2	
1	X_3	3	1/4	1/2	1	–1/4	0	0	12
M	A_2	5	11/4	3/2	0	1/4	–1	1	20/11→Key row
Total Contribution Z_j = 3 + 5M			1+11M/4	1+3M/2	1	–1+1M/4	–M	M	
Opportunity Cost $(C_j – Z_j)$			15–11M/4	5–3M/2	0	1–M/4	M	0	

↑
Key Column

Entering variable is X_1 and Outgoing variable is A_2. Key element is 11/4.

Simplex Table III

C_j →			4	3	1	0	0	R. Ratio
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	
1	X_3	28/11	0	4/11	1	–3/11	1/11	–ve
4	X_1	20/11	1	6/11	0	1/11	–4/11	20→key row
T. Contribution Z_j = 108/11			4	28/11	1	1/11	–15/11	
Opportunity Cost $(C_j – Z_j)$			0	5/11	0	–1/11	15/11	

↑
Key column

Entering variable is S_1 and Outgoing variable is X_1. Key element is 1/11.

Simplex Table IV

C_j →			4	3	1	0	0	
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	
1	X_3	8	3	2	1	0	–1	
0	S_1	20	11	6	0	0	–1	
Total Contribution Z_j = 8			3	2	1	0	–1	
Opportunity Cost $(C_j – Z_j)$			1	1	0	0	1	

As all elements of $C_j – Z_j$, row are either zero or positive, so no element will enter, hence optimal solution is arrived

$X_1 = 0$; $X_2 = 0$; $X_3 = 8$; $Z_{min} = 8$ **Ans.**

EXERCISE 4.2

1. Solve the LP Problem using simplex method.
   Minimize $Z = 2X_1 + 9X_2 + X_3$
   Subject to constraints
   $X_1 + 4X_2 + 2X_3 \geq 5$
   $3X_1 + X_2 + 2X_3 \geq 4$
   $X_1, X_2, X_3 \geq 0$

   [**Ans.** $X_1 = 0, X_2 = 0, X_3 = 5/2$, Minimum $Z = 5/2$]

2. Solve the LP Problem using simplex method.
   Minimize $Z = 0.6X_1 + X_2$
   Subject to restrictions
   $10X_1 + 4X_2 \geq 20$
   $5X_1 + 5X_2 \geq 20$
   $X_1, X_2 \geq 0$

   [**Ans.** $X_1 = 4, X_2 = 0$, Minimum $Z = 2.4$]

3. Solve the LP Problem using simplex method.
   Minimum $Z = 4X_1 + 8X_2 + 3X_3$
   Subject to restrictions
   $X_1 + X_2 \geq 2$
   $2X_1 + X_3 \geq 5$
   $X_1, X_2, X_3 \geq 0$

   [**Ans.** $X_1 = 5/2, X_2 = 0, X_3 = 0$, Minimum $Z = 10$]

4. Solve the LP Problem using simplex method.
   Minimize $Z = X_1 + X_2$
   Subject to restrictions
   $2X_1 + X_2 \geq 4$
   $X_1 + 7X_2 \geq 7$
   $X_1, X_2 \geq 0$

   [**Ans.** $X_1 = 21/13, X_2 = 10/13$, Minimum $Z = 31/13$]

5. Solve the LP Problem using big-M method :
   Minimize $Z = 400X_1 + 700X_2 + 800X_3 + 1000X_4$
   Subject to restrictions
   $X_1 + X_3 + 2X_4 \geq 40$
   $X_2 + X_3 + X_4 \geq 30$
   $X_1, X_2, X_3, X_4 \geq 0$

   [**Ans.** $X_1 = 0, X_2 = 0, X_3 = 20, X_4 = 10$, Minimize $Z = 26000$]

6. A dietician in a hospital is to arrange a special diet using three foods P, Q and R. Each gm of food P contain 20 units of calcium, 10 units of iron, 10 unit of vitamin A, 20 units of cholesterol. Each gm of food Q contain 10 units of calcium, 10 units of iron, 20 units of vitamin A, and 24 units of cholesterol. Each gm of R contain 10 units of calcium 10 units of iron, 10 units of vitamin A and 18 units of cholesterol. If the minimum daily requirement are 300 units calcium. 200 units of iron, 240 units of vitamin A. How many gm of each food should be used to meet the min requirements at the same time minimise the cholesterol intake. What is the minimum cholesterol intake ?

[**Ans.** $X_1 = 10, X_2 = 4, X_3 = 6$, Minimize Z = 404]

7. A product is manufactured by blending three different raw materials. The finished product should meet certain quality requirements. Given the following data what is your recommendation with regard to quantity for raw materials to be blended, which will meet the quality requirements with minimum cost.

Quality characteristics	Contribution to quality			Minimum quality requirements
	A	B	C	
1	3	0	1	10
2	5	1	2	15
3	1	2	0	8
Cost of raw materials per unit in Rs.	2	5	3	

[*I.C.W.A. June, 1990*]

[**Ans.** $X_1 = 8, X_2 = 0, X_3 = 0$, Minimize Z = 16]

8. Formulate linear programming model for the following problem and solve the problem using simplex method.

A company sells two types of fertilizers, one is liquid and the other is dry. The liquid fertilizer contains 2 units of chemical A and 4 units of chemical B per jar and the dry fertilizer contains 3 units of each of the chemicals A and B per carton. The liquid fertilizers sells for Rs. 3 per jar and the dry fertilizer sells for Rs. 4 per carton. A farmer requires at least 90 units of the chemical A and at least 120 units of the chemical B for his farm. How many of each type of fertilizer should the farmer purchase to minimize the cost while meeting his requirements ? [*I.C.W.A. Dec. 1985*]

[**Ans.** $X_1 = 15, X_2 = 20$, Minimum Z = 125]

9. Solve the LP Problem using simplex method.

Minimize $Z = 4X_1 + X_2$

Subject to restrictions

$$3X_1 + 4X_2 \geq 20$$
$$-X_1 - 5X_2 \leq -15$$
$$X_1, X_2 \geq 0$$

[*C.A., May 1986, Nagpur Univ. M.B.A. 1990*]

[**Ans.** $X_1 = 0, X_2 = 5$, Minimum Z = 5]

MIXED CONSTRAINTS PROBLEMS

Example 14. Use simplex method to solve the LPP.

Maximize $Z = 4X_1 + 5X_2 - 3X_3$

Subject to Constraints :

$$X_1 + X_2 + X_3 = 10$$
$$X_1 - X_2 \geq 1$$
$$2X_1 + 3X_2 + X_3 \leq 30$$

Solution :

By introducing slack, surplus and artificial variables in the set of constraints and assigning appropriate costs to these variables in the objective function, the problem can be stated in standard form as follows :

Maximize $Z = 4X_1 + 5X_2 - 3X_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to the constraints

$$X_1 + X_2 + X_3 + 0S_1 + 0S_2 + A_1 + 0A_2 = 10$$
$$X_1 - X_2 + 0X_3 - S_1 + 0S_2 + 0A_1 + A_2 = 1$$
$$2X_1 + 3X_2 + X_3 + 0S_1 + S_2 + 0A_1 + 0A_2 = 30$$
$$X_1, X_2, X_3, S_1, S_2, A_1, A_2 \geq 0$$

Simplex Table I

$C_j \rightarrow$			4	5	–3	0	0	–M	–M	
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	A_1	A_2	R. Ratio
–M	A_1	10	1	1	1	0	0	1	0	10
–M	A_2	1	1	–1	0	–1	0	0	1	1→ Key row
0	S_2	30	2	3	1	0	1	0	0	15
Total Contribution Z_j = –11M			–2M	0	–M	M	0	–M	–M	
Opportunity Cost $(C_j - Z_j)$			4+2M	5	–3+M	–M	0	0	0	

↑

Key Column

Entering variable is X_1 and Outgoing variable is A_2. Key element is 1.

Simplex Table II

$C_j \rightarrow$			4	5	–3	0	0	–M	
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	A_2	R. Ratio
–M	A_1	9	0	2	1	1	0	–1	9/2→Key row
4	X_1	1	1	–1	0	–1	0	1	–ve
0	S_2	28	0	5	1	2	1	–2	28/5
Total Contribution Z_j = –9M + 4			4	–2M–4	–M	–4–M	0	M+4	
Opportunity Cost $(C_j - Z_j)$			0	1+2M	–3+M	4 + M	0	–4	

↑

Key Column

Entering variable is X_2 and Outgoing variable is A_1. Key element is 2.

Simplex Table III

$C_j \rightarrow$			4	5	–3	0	0
↓	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2
5	X_2	9/2	0	1	1/2	1/2	0
4	X_1	11/2	1	0	1/2	–1/2	0
0	S_2	11/2	0	0	–3/2	–1/2	1
Total Contribution Z_j			4	5	9/2	1/2	0
Opportunity Cost $(C_j - Z_j)$			0	0	–15/2	–1/2	0

Since all entries in the net evaluation row are either negative or zero, further improvement in the objective function is not possible. Hence, the optimal solution is being attained and is given by

$$X_1 = \frac{11}{2},\ X_2 = \frac{9}{2},\ X_3 = 0,\ S_1 = \frac{11}{2} \text{ and maximum } Z = \frac{89}{2}.$$

Example 15. Max. $Z = 20\,X_1 + 10\,X_2$

Subject to
$$X_1 + X_2 = 150$$
$$X_1 \leq 40$$
$$X_2 \geq 20$$
$$X_1, X_2 \geq 0$$

Solution :

We introduce Slack, Surplus and Artificial variables in the constraints and convert them into equations and assign '0' co-efficient to slack and surplus variables and –M to artificial variables.

Max. $Z = 20\,X_1 + 10\,X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

$$X_1 + X_2 + 0S_1 + 0S_2 + A_1 + 0A_2 = 150$$
$$X_1 + 0X_2 + S_1 + 0S_2 + 0A_1 + 0A_2 = 40$$
$$0X_1 + X_2 + 0S_1 - S_2 + 0A_1 + A_2 = 20$$

where $X_1, X_2, S_1, S_1, A_1, A_2 \geq 0$

The initial basic feasible solution can be displayed in the following simplex table :

Simplex Table I

$C_j \rightarrow$			20	10	0	0	–M	–M	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	A_2	R. Ratio
–M	A_1	150	1	1	0	0	1	0	150
0	S_1	40	1	0	1	0	0	0	∞
–M	A_2	20	0	1	0	–1	0	1	20→Key row
Total Contribution Z_j = –170 M			–M	–2M	0	M	–M	–M	
Opportunity Cost $(C_j - Z_j)$			20 + M	10 + 2M	0	–M	0	0	

↑ Key Column

Entering variable is X_2 and Outgoing variable is A_2. Key element is 1.

Simplex Table II

$C_j \rightarrow$			20	10	0	0	–M	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	R. Ratio
–M	A_1	130	1	0	0	1	1	130
0	S_1	40	1	0	1	0	0	40→Key row
10	X_2	20	0	1	0	–1	0	∞
Total Contribution Z_j = 200-130 M			–M	10	0	–10–M	–M	
Opportunity Cost $(C_j - Z_j)$			20 + M	0	0	10 + M	0	

↑ Key Column

Entering variable is X_1 and leaving variable is S_1. Key element is 1.

Simplex Table III

$C_j \rightarrow$			20	10	0	0	–M	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	R. Ratio
–M	A_1	90	0	0	–1	1	1	90→Key row
20	X_1	40	1	0	1	0	0	∞
10	X_2	20	0	1	0	–1	0	–ve
Total Contribution Z_j = 1000–90 M			20	10	20 + M	–10–M	–M	
Opportunity Cost $(C_j - Z_j)$			0	0	–20–M	10 + M	0	

↑ Key Column

Entering variable is S_2 and leaving variable is A_1. Key element is 1.

Simplex Table IV

$C_j \rightarrow$			20	10	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	
0	S_2	90	0	0	–1	1	
20	X_1	40	1	0	1	0	
10	X_2	110	0	1	–1	0	
Total Contribution Z_j = 1900			20	10	10	0	
Opportunity Cost $(C_j - Z_j)$			0	0	–10	0	

$X_1 = 40$, $X_2 = 110$, $S_1 = 0$, $S_2 = 90$, $Z_j = 1900$ **Ans.**

Example 16. Max. $Z = 2\,X_1 + 3\,X_3$

Subject to $X_1 + X_2 + 2\,X_3 \leq 5$

$0 - 2 = -2$

$$2X_1 + 3X_2 + 4X_3 = 12$$

whereas $X_1, X_2, X_3 \geq 0$

Max. problem $-M$

Solution : In order to get equality and unit matrix introduce slack variable S_1 with zero co-efficient and artificial variable A_1 with '$-M$' co-efficient respectively. The given problem can be formulated as follows:

Max. $Z = 2X_1 + 3X_3 + 0S_1 - MA_1$

Subject to $X_1 + X_2 + 2X_3 + S_1 + 0A_1 = 5$

$2X_1 + 3X_2 + 4X_3 + 0S_1 + A_1 = 12$

whereas $X_1, X_2, X_3, S_1, A_1 \geq 0$

$12 - (4 \times \frac{5}{2}) = 2$
$2 - (4 \times \frac{1}{2}) = 0$
$3 - (4 \times \frac{1}{2}) = 1$
$4 - (4 \times 1) = 0$
$0 - (4 \times \frac{1}{2}) = -2$
$1 - (4 \times 0) = 1$

Simplex Table I

$C_j \rightarrow$ ↓			2	0	3	0	$-M$	
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	A_1	R. Ratio
0	S_1	5	1	1	2	1	0	5/2→Key row
$-M$	A_1	12	2	3	4	0	-1	3
Total Contribution $Z_j = -12M$			$-2M$	$-3M$	$-4M$	0	$-M$	
Opportunity Cost $(C_j - Z_j)$			$2 + 2M$	$3M$	$3 + 4M$	0	0	

Key Column

see the minimum +ve

$-M - (-M)$
$-M + M = 0$

Entering variable is X_3 and leaving variable is S_1. Key element is 2.

Simplex Table II

$C_j \rightarrow$ ↓			2	0	3	0	$-M$	
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	A_1	R. Ratio
0	X_3	5/2	1/2	1/2	1	1/2	0	5
$-M$	A_1	2	0	1	0	-2	1	2→Key Row
Total Contribution $Z_j = 15/2 - 2M$			3/2	$3/2 - M$	3	$3/2 + 2M$	$-M$	
Opportunity Cost $(C_j - Z_j)$			1/2	$-3/2 + M$	0	$-3/2 - 2M$	0	

Key Column

Entering variable is X_2 and leaving variable is A_1, where key element is 1.

Simplex Table III

$C_j \rightarrow$ ↓			2	0	3	0	
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	R. Ratio
3	X_3	3/2	1/2	0	1	3/2	3→Key Row
0	X_2	2	0	1	0	-2	∞
Total Contribution $Z_j = 9/2$			3/2	0	3	9/2	
Opportunity Cost $(C_j - Z_j)$			1/2	0	0	$-9/2$	

Key column

Entering variable is X_1 and Outgoing variable is X_3. Key element is 1/2.

Simplex Table IV

$C_j \rightarrow$ ↓			2	0	3	0	
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	R. Ratio
2	X_1	3	1	0	2	3	
0	X_2	2	0	1	0	-2	
Total Contribution $Z_j = 6$			2	0	4	6	
Opportunity Cost $(C_j - Z_j)$			0	0	-1	-6	

All element of row $C_j - Z_j$ are either zero or negative, so we are having optimal solution :

$X_1 = 3$, $X_2 = 2$, $X_3 = 0$, $Z = 6$ **Ans.**

Example 17. Solve the following LPP.

Max. $Z = X_1 + 2X_2 + 3X_3 - X_4$

Subject to $X_1 + 2X_2 + 3X_3 = 15$

$2X_1 + X_2 + 5X_3 = 20$

$X_1 + 2X_2 + X_3 + X_4 = 10$

where $X_1, X_2, X_3, X_4 \geq 0$

In case of minimization $-M$

Solution : Since the constraints of the given system are equations introduce the Artificial Variables A_1, A_2. The problem thus becomes

Max. $Z = X_1 + 2X_2 + 3X_3 - X_4 - MA_1 - MA_2$

Subject to restriction,

$X_1 + 2X_2 + 3X_3 + 0X_4 + A_1 + 0A_2 = 15$ A3 ?

$2X_1 + X_2 + 5X_3 + 0X_4 + 0A_1 + A_2 = 20$

$X_1 + 2X_2 + X_3 + X_4 + 0A_1 + 0A_2 = 10$

$X_1, X_2, X_3, X_4, A_1, A_2 \geq 0$

Simplex Table I

$C_j \rightarrow$ ↓			1	2	3	-1	$-M$	$-M$	
	Basic Variables	Qty.	X_1	X_2	X_3	X_4	A_1	A_2	R. Ratio
$-M$	A_1	15	1	2	3	0	1	0	5
$-M$	A_2	20	2	1	5	0	0	1	4→Key row
-1	X_4	10	1	2	1	1	0	0	10
Total Contribution $Z_j = -35M - 10$			$-2M - 1$	$-3M - 2$	$-8M - 1$	-1	M	M	
Opportunity Cost $(C_j - Z_j)$			$+3M + 2$	$+3M + 4$	$8M + 4$	0	0	0	

Key Column

Entering variable is X_3 and Outgoing variable is A_2. Key element is 5.

Simplex Table II

$C_j \rightarrow$			1	2	3	–1	–M	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	X_3	X_4	A_1	R. Ratio
–M	A_1	3	–1/5	7/5	0	0	1	15/7→Key row
3	X_3	4	2/5	1/5	1	0	0	20
–1	X_4	6	3/5	9/5	0	1	0	30/9
Total Contribution Z_j = –3 M + 6			3+M/5	–6–7M/5	3	–1	M	
Opportunity Cost $(C_j - Z_j)$			2–M/5	16+7M/5	0	0	0	

↑
Key Column

Entering variable is X_2 and Outgoing variable is A_1. Key element is 7/5.

Simplex Table III

$C_j \rightarrow$			1	2	3	–1	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	X_3	X_4	R.Ratio
2	X_2	15/7	–1/7	1	0	0	–Ve
3	X_3	25/7	3/7	0	1	0	25/3
–1	X_4	15/7	6/7	0	0	1	15/6→ Key row
al Contribution Z_j = 90/7			1/7	2	3	–1	
ortunity Cost $(C_j - Z_j)$			6/7	0	0	0	

↑
Key Column

Entering variable is X_1 and Outgoing variable is X_4. Key element is 6/7.

Simplex Table IV

→			1	2	3	–1	
	Basic Variables	Qty.	X_1	X_2	X_3	X_4	
	X_2	15/6	0	1	0	1/6	
	X_3	15/6	0	0	1	3/6	
	X_1	15/6	1	0	0	7/6	
ntribution Z_j = 15			1	2	3	3	
ity Cost $(C_j - Z_j)$			0	0	0	–4	

e all element in $C_j - Z_j$ are either zero or –ve, hence, optimal solution has been obtained X_1 = 15/6, X_3 = 15/6, X_4 = 0, Z_j = 15.

Example 18. Solve the LP Problem using simplex method. whenever co-efficient is zero, use sign (+)

Minimize $Z = X_1 + 4X_2$

Subject to
$$X_1 + 3X_2 \geq 4000$$
$$X_1 + 2X_2 \leq 3500$$
$$X_1 + X_2 \geq 2000$$

where $X_1, X_2 \geq 0$

as minimisation problem

We introduce surplus, slack and artificial variable and convert the inequalities into equations and assign '0' coefficient to the slack and surplus variables and '+M' to the artificial variables.

Min. $Z = X_1 + 4X_2 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2$

Subject to
$$X_1 + 3X_2 - S_1 + 0S_2 + 0S_3 + A_1 + 0A_2 = 4000$$
$$X_1 + 2X_2 + 0S_1 + S_2 + 0S_3 + 0A_1 + 0A_2 = 3500$$
$$X_1 + X_2 + 0S_1 + 0S_2 - S_3 + 0A_1 + A_2 = 2000$$

Where $X_1, X_2, S_1, S_2, S_3, A_1, A_2 \geq 0$

Simplex Table I

$C_j \rightarrow$			1	4	0	0	0	M	M	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	A_1	A_2	R. Ratio
M	A_1	4000	1	3	–1	0	0	1	0	4000/3→Key ro = 1333
0	S_2	3500	1	2	0	1	0	0	0	3500/2 = 175
M	A_2	2000	1	1	0	0	–1	0	1	2000/1 = 20
Total Contribution Z_j = 6000 M			2M	4M	M	0	–M	M	M	
Opportunity Cost $(C_j - Z_j)$			1–2M	4–4M	M	0	M	0	0	

↑
Key Column

Entering variable is X_2 and Outgoing variable is A_1. Key element is 3.

Simplex Table II

$C_j \rightarrow$			1	4	0	0	0	M	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	A_2	R. Ra
4	X_2	4000/3	1/3	1	–1/3	0	0	0	4000
0	S_2	2500/3	1/3	0	2/3	1	0	0	2500
M	A_2	2000/3	2/3	0	1/3	0	–1	1	1000→K
Total Contribution Z_j = 2000M + 16000/3			4/3+2M/3	4	–4/3 + M/3	0	–M	M	
Opportunity Cost $(C_j - Z_j)$			–1/3 – 2M/3	0	–M/3 + 4/3	0	M	0	

↑
Key Column

Entering variable is X_1 and Outgoing variable is A_2. Key element is 2/3.

Simplex Table III

$C_j \rightarrow$ ↓	Basic Variables	Qty.	1 X_1	4 X_2	0 S_1	0 S_2	0 S_3	R. Ratio
4	X_2	1000	0	1	–1/2	0	1/2	2000
0	S_2	500	0	0	1/2	1	1/2	1000→Key row
1	X_1	1000	1	0	1/2	0	–3/2	–ve
Total Contribution Z_j = 5000			1	4	–3/2	0	1/2	
Opportunity Cost $(C_j - Z_j)$			0	0	3/2	0	–1/2	

↑ Key Column

Entering variable is S_3 and Outgoing variable is S_2. Key element is 1/2.

Simplex Table IV

$C_j \rightarrow$ ↓	Basic Variables	Qty.	1 X_1	4 X_2	0 S_1	0 S_2	0 S_3
4	X_2	500	0	1	–1	–1	0
0	S_3	1000	0	0	1	2	1
1	X_1	2500	1	0	2	3	0
Total Contribution Z_j = 4500			1	4	–2	–1	0
Opportunity Cost $(C_j - Z_j)$			0	0	2	1	0

Since all element in the index row are positive or zero, we are having optimal solution.

$X_1 = 2500$; $X_2 = 500$; $Z_j = 4500$ **Ans.**

Example 19. Use Simplex Method to solve the following LPP.

Min. $Z = 3X_1 + 8X_2$

Subject to $X_1 + X_2 = 200$ → only Introduce artificial variable

$X_1 \geq 80$

$X_2 \leq 60$

Where $X_1, X_2 \geq 0$ *[BBA III Yr. G.N.D.U. April 2002]*

Solution :

Converting the inequalities into equations by adding slack, surplus and artificial variables in the set of constraints, the problem can be re-written as follows :

Minimize $Z = 3X_1 + 8X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

$X_1 + X_2 - 0S_1 + 0S_2 + A_1 + 0A_2 = 200$

$X_1 + 0X_2 - S_1 + 0S_2 + 0A_1 + A_2 = 80$

$0X_1 + X_2 + 0S_1 + S_2 + 0A_1 + 0A_2 = 60$

Where as $X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$

From the above system of equations, the initial basic feasible solution can be shown in the following implex table.

Simplex Table I

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	8 X_2	0 S_1	0 S_2	M A_1	M A_2	R. Ratio
M	A_1	200	1	1	0	0	1	0	200
M	A_2	80	1	0	–1	0	0	1	80→Key row
0	S_2	60	0	1	0	1	0	0	
Total Contribution Z_j = 280M			2M	M	–M	0	M	M	
Opportunity Cost $(C_j - Z_j)$			3–2 M	8–M	M	0	0	0	

↑ Key Column

Entering variable is X_1 and Outgoing variable is A_2. Key element is 1.

Simplex Table II

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	8 X_2	0 S_1	0 S_2	M A_1	R. Ratio
M	A_1	120	0	1	1	0	1	120→Key row
3	X_1	80	1	0	–1	0	0	–80
0	S_2	60	0	1	0	1	0	∞
Total Contribution Z_j = 240 + 120M			3	M	M–3	0	M	
Opportunity Cost $(C_j - Z_j)$			0	8–M	3–M	0	0	

60-(0x
0-(0x
1-(0x
0-(0x
1-(0x
0-(0x

↑ Key Column

Entering variable is S_1 and Outgoing variable is A_1. Key element is 1.

Simplex Table III

$C_j \rightarrow$ ↓	Basic Variables	Qty.	3 X_1	8 X_2	0 S_1	0 S_2
0	S_1	120	0	1	1	0
3	X_1	200	1	1	0	0
0	S_2	60	0	1	0	1
Total Contribution Z_j = 600			3	3	0	0
Opportunity Cost $(C_j - Z_j)$			0	5	0	0

All the values of $C_j - Z_j$ row are '0' or more than zero, so the optimum solution is arrived.

$X_1 = 200,\ X_2 = 0,\ Z$ = Rs. 600 **Ans.**

Example 20. Use Simplex Method to solve the following LPP.

Minimize $Z = 8X_1 + 4X_2$

Subject to constraints

$$3X_1 + X_2 \geq 27$$
$$X_1 + X_2 = 21$$

where $X_1 \geq 0, X_2 \geq 0$

Solution convert the inequalities into equations

Minimize $Z = 8X_1 + 4X_2 + 0S_1 + MA_1 + MA_2$

Subject to

$$3X_1 + X_2 - S_1 + A_1 + 0A_2 = 27$$
$$X_1 + X_2 + 0S_1 + 0A_2 + A_2 = 21$$

where as $X_1, X_2, S_1, A_1, A_2 \geq 0.$

Simplex Table I

$C_j \rightarrow$			8	4	0	M	M	
↓	Basic Variables	Qty.	X_1	X_2	S_1	A_1	A_2	R. Ratio
M	A_1	27	3	1	–1	1	0	9→Key row
M	A_2	21	1	1	0	0	1	21
l Contribution Z_j = 48 M			4M	2M	–M	M	M	
ortunity Cost $(C_j - Z_j)$			8–4M	4–2M	M	0	0	

↑ Key column

ntering variable is X_1 and Outgoing variable is A_1. Key element is 3.

Simplex Table II

6			8	4	0	M	
	Basic Variables	Qty.	X_1	X_2	S_1	A_2	R. Ratio
	X_1	9	1	1/3	–1/3	1/3	27
	A_2	12	0	2/3	1 /3	2/3	18→Key row
ntribution Z_j = 72 + 12 M			8	8+2/3M	–8+M/3	8+2M/3	
ity Cost $(C_j - Z_j)$			0	–4 –2M/3	8–M/3	–8– M/3	

↑ Key Column

ng variable is X_2 and Outgoing variable is A_2. Key element is 2/3.

Simplex Table III

$C_j \rightarrow$			8	4	0
↓	Basic Variables	Qty.	X_1	X_2	S_1
8	X_1	3	1	0	–1/2
4	X_2	18	0	1	1/2
Total Contribution Z_j = 96			8	4	–4
Opportunity Cost $(C_j - Z_j)$			0	0	4

Since all elements in $C_j - Z_j$ row are zero or positive. This table gives optimal solution as below

$$X_1 = 3, X_2 = 18, Z_j = 96.$$

EXERCISE 4.3

1. Solve the LP Problem using simplex method.

   Maximum $Z = 60X_1 + 80X_2$

   Subject to restrictions

   $$X_1 \leq 400$$
   $$X_2 \geq 200$$
   $$X_1 + X_2 = 500$$
   $$X_1, X_2 \geq 0$$

   [**Ans.** $X_1 = 0, X_2 = 500$, Max. $Z = 40000$]

2. Solve the LP Problem using simplex method.

   Maximum $Z = 2X_1 + 3X_3$

   Subject to restrictions

   $$X_1 + X_2 + 2X_3 \leq 5$$
   $$2X_1 + 3X_2 + 4X_3 = 12$$
   $$X_1, X_2, X_3 \geq 0 \text{ (Non-negativity constraint)}$$

   [**Ans.** $X_1 = 3, X_2 = 2, X_3 = 0$, Max. $Z = 6$]

3. Solve the LP Problem using simplex method.

   Maximum $Z = 30X_1 + 20X_2$

   Subject to restrictions

   $$-X_1 - X_2 \geq -8$$
   $$-6X_1 - 4X_2 \leq -12$$
   $$5X_1 + 8X_2 = 20$$
   $$X_1, X_2 \geq 0 \text{ (Non-negativity constraint)}$$

   [**Ans.** $X_1 = 4, X_2 = 0$, Max. $Z = 12$

4. Solve the LP Problem using simplex method.

Maximum $Z = 3X_1 + 5X_2 + 4X_3$

Subject to restrictions

$$2X_1 + 3X_2 \le 8$$
$$2X_2 + 5X_3 \le 10$$
$$3X_1 + 2X_2 + 4X_3 = 15$$
$X_1, X_2, X_3 \ge 0$ (Non-negativity constraint)

[**Ans.** $X_1 = 2.165, X_2 = 1.24, X_3 = 1.504$, Max. $Z = 18.611$]

5. Solve the LP Problem using simplex method.

Maximum $Z = 60{,}000X_1 + 1{,}20{,}000X_2$

Subject to restrictions

$$9000X_1 + 12000X_2 \le 720000 \text{ or}$$
$$3X_1 + 4X_2 \le 240$$
$$X_1 \ge 2$$
$$X_2 \ge 3$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 2, X_2 = 58.50$, Max. $Z = 7140000$]

6. Solve the LP Problem using simplex method.

Minimize $Z = 10X_1 + 30X_2$

Subject to restrictions

$$6X_1 + 10X_2 \le 720$$
$$500X_1 + 500X_2 \le 40000$$
$$X_1 \ge 30,$$
$$X_2 \ge 20,$$
$$X_1, X_2 \ge 0.$$

[**Ans.** $X_1 = 30, X_2 = 50$, Max. $Z = 1800$]

7. Solve the following LPP

Minimize $Z = 5X_1 + 8X_2$

Subject to restrictions

$$X_1 + X_2 = 5$$
$$X_1 \le 4$$
$$X_2 \ge 2$$
$$X_1, X_2 \ge 0$$

[**Ans.** $X_1 = 3, X_2 = 2$, Minimum $Z = 31$]

8. Solve the LP Problem using simplex method.

Minimize $Z = 2X_1 + X_2$

Subject to restrictions

$$5X_1 + 10X_2 - X_3 = 8$$
$$X_1 + X_2 + X_4 = 1$$
$$X_1, X_2, X_3, X_4 \ge 0$$

[**Ans.** $X_1 = 0, X_2 = 4/5, X_3 = 0, X_4 = 1/5$, Minimum $Z = 4/5$]

9. A small jewellry manufacturing company employs a person who is highly skilled gem cutter, and it wishes to use this person at least 6 hours per day for this purpose. On the other hand, the polishing facilities can be used in any amount upto 8 hours per day. The company specializes in three kinds of semiprecious stones P, Q and R. Relevant cutting, polishing and cost requirements are listed in the following table. How many gemstones of each type should be processed each day to minimize the cost of the finished stones ? What is the minimum cost ?

	P	Q	R
Cutting	2 hr.	1 hr.	1 hr.
Polishing	1 hr.	1 hr.	2 hr.
Cost per stone	Rs. 30	Rs. 30	Rs. 10

[**Ans.** $X_1 = 4/3, X_2 = 0, X_3 = 10/3$, Minimum $Z = 220/$

10. Solve the LP Problem using simplex method.

Minimum $Z = 8X_1 + 4X_2 + 2X_3$

Subject to restrictions

$$4X_1 + 2X_2 + X_3 \le 8$$
$$3X_1 + 2X_2 \le 10$$
$$X_1 + X_2 + X_3 = 4$$
$X_1, X_2, X_3 \ge 0$ (Non-negativity constraint) [*G.N.D.U. (April) 19*

[**Ans.** $X_1 = 0, X_2 = 0, X_3 = 4$, Minimum Z

11. A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintal metal and no more than 35 quintals of Y-metal. The firm can purchase the scrap from suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of w of the scraps supplied by A and B is given below :

Metal	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs. 200 per quintal and that of B's Rs. 400 per quintal. The firm w determine the quantities that it should buy from the two suppliers so that total cost is min Using the linear programming technique, solve it by simplex method.

[*Gujarat Univ. M.B.*

[**Ans.** $X_1 = 100, X_2 = 100$, Minimum Z

when not able to decide the outgoing variable
eg Max = − 9 Min = −

EXCEPTIONAL CASES

In this section, important cases are discussed which commonly occur during simplex problems.

CASE I : UNBOUNDED SOLUTION

Unbounded solution exists when the feasible region is unbounded so that one or more of the variables can be increased indefinitely. The simplex method provides a clear indication that the solution is unbounded, when the ratios in the ratio column of a simplex table are negative or they equal infinity.

Consider the following example.

Example 21. Max. $Z = 2X_1 + X_2$

Subject to constraints

$$X_1 - X_2 \le 10$$
$$2X_1 - X_2 \le 40$$
$$X_1, X_2 \ge 0$$

Solution :

Convert the inequalities into equation by introducing slack variables

Max. $Z = 2X_1 + X_2 + 0S_1 + 0S_2$

Subject to constraints

$$X_1 - X_2 + S_1 + 0S_2 = 10$$
$$2X_1 - X_2 + 0S_1 + S_2 = 40$$

where $X_1, X_2 \ge 0$

Simplex Table I

$C_j \rightarrow$ ↓			2	1	0	0	
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	R. Ratio
0	S_1	10	1	−1	1	0	10 → Key row
0	S_2	40	2	−1	0	1	20
Total Contribution $Z_j = 0$			0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			2	1	0	0	

↑ Key column

Entering variable is X_1 and Outgoing variable is S_1. Key element is 1.

Simplex Table II

$C_j \rightarrow$ ↓			2	1	0	0	
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	R. Ratio
2	X_1	10	1	−1	1	0	−ve
0	S_2	20	0	1	−2	1	20 → Key row
Total Contribution $Z_j = 20$			2	−2	2	0	
Opportunity Cost $(C_j - Z_j)$			0	3	2	0	

↑ Key column

Entering variable is X_2 and outgoing variable is S_2. Key elements is 1.

Simplex Table III

$C_j \rightarrow$ ↓			2	1	0	0	
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	R. Ratio
2	X_1	30	1	0	−1	1	−30 as there is
1	X_2	20	0	1	−2	1	−20 no minimum
Total Contribution Z_j		80	2	1	−4	3	the value
Opportunity Cost $(C_j - Z_j)$			0	0	4	−3	

↑ Key column

outgoing

On the basis of $C_j - Z_j$ row, the entering variable is S_1, but it not possible to decide the departing variable as replacement ratio column contains negative elements. Hence the given LPP has **unbounded solution.**

Example 22. Maximize $Z = 5X_1 + 6X_2 + X_3$

Subject to restriction

$$9X_1 + 3X_2 - 2X_3 \le 5$$
$$4X_1 + 2X_2 - X_3 \le 2$$
$$X_1 - 4X_2 + X_3 \le 3$$

whereas $X_1, X_2, X_3 \ge 0$

Solution :

Convert the inequalities into equations by introducing slack variables.

Maximize $Z = 5X_1 + 6X_2 + X_3 + 0S_1 + 0S_2 + 0S_3$

Subject to restriction

$$9X_1 + 3X_2 - 2X_3 + S_1 + 0S_2 + 0S_3 = 5$$
$$4X_1 + 2X_2 - X_3 + 0S_1 + S_2 + 0S_3 = 2$$
$$X_1 - 4X_2 + X_3 + 0S_1 + 0S_2 + S_3 = 3$$
$$X_1, X_2, X_3, S_1, S_2, S_3 \ge 0$$

Initial Simplex Table I

$C_j \rightarrow$ ↓			5	6	1	0	0	0	
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	S_3	R. Ratio
0	S_1	5	6	3	−2	1	0	0	5/3
0	S_2	2	4	2	−1	0	1	0	1→Key row
0	S_3	3	1	−4	1	0	0	1	−ve
Total Contribution $Z_j = 0$			0	0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			5	6	1	0	0	0	

↑ Key Column

R. Ratio = Replacement ratio

Entering variable is X_2 and Outgoing variable is S_2. Key element is 2.

Simplex Table II

$C_j \rightarrow$ ↓			5	6	1	0	0	0	R. Ratio
	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	2	3	0	−1/2	1	−3/2	0	−ve
6	X_2	1	2	1	−1/2	0	1/2	0	−ve
0	S_3	7	9	0	−1	0	2	1	−ve
Total Contribution $Z_j = 6$			12	6	−3	0	3	0	
Opportunity Cost $(C_j - Z_j)$			−7	0	4	0	−3	0	

↑ Key Column

On the basis of $C_j - Z_j$ row, the entering variable is X_3 but it not possible to decide departing variable from R. ratio as it contains all negative elements. Hence the given LPP has **unbounded solution.**

CASE II : INFEASIBLE SOLUTION

(artificial variable in BV column.

Solution is feasible, if it satisfies all the constraints and non negative conditions. But, sometimes conditions are not fulfilled so that there is no feasible solution to the problem. Such situation is called infeasibility and such solution is called an infeasible solution. In Simplex problem infeasibility arises when the solution obtained is optimal but at least one of the artificial variables, which should have been driven to zero, is present as a positive basic variable in the final solution.

Example 23. Max. $Z = 3X_1 + 2X_2$

Subject to
$$2X_1 + X_2 \leq 2$$
$$3X_1 + 4X_2 \geq 12$$
$$X_1 \geq 0, X_2 \geq 0$$

[I.C.W.A. (Final) Dec. 1994]

Solution :

Introduce surplus, slack and artificial variables to convert inequalities into equations, and assign '0' coefficient to slack and surplus variables and '−M' coefficient respectively to artificial variable in the objective function, resultant objective function and constraints are :

Max. $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 - MA_1$

Subject to $2X_1 + X_2 + S_1 + 0S_2 + 0A_1 = 2$

$3X_1 + 4X_2 + 0S_2 - S_2 + A_1 = 12$

where $X_1, X_2, S_1, S_2, A_1 \geq 0$

both −S2 & A1 occur but only A1 shall be taken in simplex table

A Max = −MA
Min = +MA

Simplex Table I

$C_j \rightarrow$ ↓			3	2	0	0	−M	R. Ratio
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	
0	S_1	2	2	1	1	0	0	2→Key row
−M	A_1	12	3	4	0	−1	1	3
Total Contribution $Z_j = -12M$			−3M	−4M	0	M	−M	
Opportunity Cost $(C_j - Z_j)$			3 + 3M	2 + 4M	0	−M	0	

↑ Key column

Entering variable is X_2 and Outgoing variable is S_1. Key element is 1.

Simplex Table II

$C_j \rightarrow$ ↓			3	2	0	0	−M	
	Basic Variables	Qty.	X_1	X_2	S_1	S_2	A_1	
2	X_2	2	2	1	1	0	0	
−M	A_1	4 +ve	−5	0	−4	−1	1	
Total Contribution $Z_j = 4 - 4M$			5M + 4	2	2+4M	M	−M	
Opportunity Cost $(C_j - Z_j)$			−1−5M	0	−2−4M	−M	0	

Since all values in $C_j - Z_j$ row are 0 or −ve, solution is optimal but since artificial variable appears as basic variable, the problem has **infeasible solution.**

$x_1, x_2, x_3 \geq 0$, $C_j -$

CASE III : MULTIPLE OPTIMAL SOLUTION

We have already seen in the previous chapter that the solution to a LPP may or may not be unique. I the simplex method, the existence of multiple optimal solutions to a LPP problem is indicated if one of th non-basic variables has a zero coefficient in the final $C_j - Z_j$ row. If we enter this zero co-efficient no basic variable in the solution, there will be no change in the numerical value of the objective functi although alternate optimal solution will be obtained.

Example 24.

Maximize $Z = 3X_1 + 5X_2 + 5X_3$

Subject to restrictions

$$0 \cdot 1 X_1 + 0 \cdot 25 X_2 + 0X_3 \leq 120$$
$$0 \cdot 20 X_1 + 0 \cdot 30 X_2 + 0 \cdot 40 X_3 \leq 260$$
$$X_1, X_2, X_3 \geq 0$$

Solution :

Convert the inequalities into equations by introducing slack variables.

Max. $Z = 3X_1 + 5X_2 + 5X_3 + 0S_1 + 0S_2$

Subject to $0 \cdot 1\, X_1 + 0 \cdot 25\, X_2 + 0X_3 + S_1 + 0S_2 = 120$

$0 \cdot 2\, X_1 + 0 \cdot 3\, X_2 + 0 \cdot 4\, X_3 + 0S_1 + S_2 = 260$

$X_1, X_2, X_3, S_1, S_2 \geq 0$

Initial Simplex Table I

$C_j \rightarrow$			3	5	5	0	0	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	R. Ratio
0	S_1	120	10/100	25/100	0	1	0	∞
0	S_2	260	20/100	30/100	40/100	0	1	650→Key row
Total Contribution $Z_j = 0$			0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			3	5	5	0	0	

↑ Key column

Entering variable is X_3 and Outgoing variable is S_2. Key element is 40/100.

Simplex Table II

$C_j \rightarrow$			3	5	5	0	0	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	R. Ratio
0	S_1	120	10/100	25/100	0	1	0	480→Key row
5	X_3	650	50/100	75/100	1	0	25/100	867
Total Contribution $Z_j = 3250$			250/100	375/100	5	0	125/100	
Opportunity Cost $(C_j - Z_j)$			50/100	125/100	0	0	−125/100	

↑ Key column

Entering variable is X_2 and Outgoing variable is S_1. Key element is 25/100.

Simplex Table III

$C_j \rightarrow$			3	5	5	0	0	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2	R. Ratio
5	X_2	480	40/100	1	0	4	0	1200
5	X_3	290	20/100	0	1	−3	25/100	1450
Total Contribution $Z_j = 3850$			3	5	5	5	125/100	
Opportunity Cost $(C_j - Z_j)$			0	0	0	−5	−125/100	

$X_1 = 0,\ X_2 = 480,\ X_3 = 290,\ Z = 3850.$

Zero under a non-basic variable X_1 in the final net evaluation row indicates that the problem has **multiple solutions**. If X_1 were to be brought into the solution in place of X_2 the following table would result.

Simplex Table IV

$C_j \rightarrow$			3	5	5	0	0
$\downarrow$	Basic Variables	Qty.	X_1	X_2	X_3	S_1	S_2
3	X_1	1200	1	250/100	0	10	0
5	X_3	50	0	−50/100	1	−5	25/100
Total Contribution $Z_j = 3850$			3	5	5	5	125/100
Opportunity Cost $(C_j - Z_j)$			0	0	0	−5	−125/100

$X_1 = 1200,\ X_2 = 0,\ X_3 = 50,\ Z = 3850$ same

Example 25. Maximize $Z = 6X_1 + 4X_2$

Subject to restriction

$$2X_1 + 3X_2 \leq 30$$
$$3X_1 + 2X_2 \leq 24$$
$$X_1 + X_2 \geq 3$$

Where $X_1, X_2 \geq 0$

Solution :

Convert the inequalities into equation by introducing slack, surplus and artificial variables.

~~Max~~ $Z = 6X_1 + 4X_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$

Subject to

$$2X_1 + 3X_2 + S_1 + 0S_2 + 0S_3 + 0A_1 = 30$$
$$3X_1 + 2X_2 + 0S_1 + S_2 + 0S_3 + 0A_1 = 24$$
$$X_1 + X_2 + 0S_1 + 0S_2 - S_3 + A_1 = 3$$
$$X_1, X_2, S_1, S_2, S_3, A_1 \geq 0$$

Initial Simplex Table I

$C_j \rightarrow$			6	4	0	0	0	−M	
$\downarrow$	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	A_1	R. Ratio
0	S_1	30	2	3	1	0	0	0	15
0	S_2	24	3	2	0	1	0	0	8
−M	A_1	3	1	1	0	0	−1	1	3→Key row
Total Contribution $Z_j = -3$ M			−M	−M	0	0	M	−M	
Opportunity Cost $(C_j - Z_j)$			6 + M	4 + M	0	0	−M	0	

↑ Key Column

Entering variable is X_1 and Outgoing variable is A_1. Key element is 1.

Simplex Table II

$C_j \rightarrow$			6	4	0	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	R. Ratio
0	S_1	24	0	1	1	0	2	12
0	S_2	15	0	-1	0	1	3	5→Key row
6	X_1	3	1	1	0	0	-1	-ve
Total Contribution $Z_j = 18$			6	6	0	0	-6	
Opportunity Cost $(C_j - Z_j)$			0	-2	0	0	6	

↑ Key Column

Entering variable is S_3 and Outgoing variable is S_2. Key element is 3.

Simplex Table III

$C_j \rightarrow$			6	4	0	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	R. Ratio
0	S_1	14	0	5/3	1	-2/3	0	42/5→Key row
0	S_3	5	0	-1/3	0	1/3	1	-
6	X_1	8	1	2/3	0	1/3	0	12
Total Contribution $Z_j = 48$			6	4	0	2	0	
Opportunity Cost $(C_j - Z_j)$			0	0	0	-2	0	

↑ Key Column

Since $C_j - Z_j$ elements are zero or –ve. Hence we got optimal solution $X_1 = 8$, $X_2 = 0$, $Z_j = 48$. But $X_2 = 0$ in $C_j - Z_j$ row in final Simplex table means, we have alternative solution also. Therefore, the solution is not unique. Hence, we can bring X_2 into the basis in place of S_1. The new optimum table is obtained as follows

Simplex Table IV

$C_j \rightarrow$			6	4	0	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	S_3	R. Ratio
4	X_2	42/5	0	1	3/5	-2/5	0	
0	S_3	39/5	0	0	1/5	1/5	1	
6	X_1	12/5	1	0	-2/5	3/5	0	
Total Contribution $Z_j = 48$			6	4	0	2	0	
Opportunity Cost $(C_j - Z_j)$			0	0	0	-2	0	

The alternative solution is $X_1 = 12/5$, $X_2 = 42/5$; Max. $Z = 48$.

CASE IV : FREE OR UNRESTRICTED VARIABLES

A variable which can be positive, negative or zero is called free or unrestricted variable. Such a variable (s) is replaced by the difference of two non-negative variables. After solution of the problem, the value of the original unrestricted variable is obtained.

Example 26.

Min $Z = 4X_1 + 2X_2$

Subject to constraint.

$$3X_1 + X_2 \geq 27$$
$$-X_1 - X_2 \leq 21$$
$$X_1 + 2X_2 \geq 30$$

$X_1 = (X_3 - X_4)$
$X_2 = (X_5 - X_6)$

When X_1, X_2 are unrestricted in sign. [I.C.W.A. (Final) Mar. 1992]

Solution :

Since X_1, X_2 are unrestricted in sign

1) slack
2) surplus

Let $X_1 = X_3 - X_4$

$X_2 = X_5 - X_6$

Now, Min $Z = 4(X_3 - X_4) + 2(X_5 - X_6)$

S.t.

$$3(X_3 - X_4) + 1(X_5 - X_6) \geq 27$$
$$-1(X_3 - X_4) - 1(X_5 - X_6) \leq 21$$
$$1(X_3 - X_4) + 2(X_5 - X_6) \geq 30$$

OR

$$3X_3 - 3X_4 + X_5 - X_6 \geq 27$$
$$-X_3 + X_4 - X_5 + X_6 \leq 21$$
$$X_3 - X_4 + 2X_5 - 2X_6 \geq 30$$

Where $\boxed{X_3, X_4, X_5, X_6 \geq 0}$

Introduce Slack, Surplus and Artificial variables and convert constraint inequalities into equations and assign '0' co-efficient to the surplus and slack variables and '+ M' to the Artificial variables in the objective function.

Min $Z = 4X_3 - 4X_4 + 2X_5 - 2X_6 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2$

$$3X_3 - 3X_4 + X_5 - X_6 - S_1 + A_1 + 0S_2 + 0S_3 + 0A_2 = 27$$
$$-X_3 + X_4 - X_5 + X_6 + 0S_1 + 0A_1 + S_2 + 0S_3 + 0A_2 = 21$$
$$X_3 - X_4 + 2X_5 - 2X_6 + 0S_1 + 0A_1 + 0S_2 + S_3 + A_2 = 30$$

where as $X_3, X_4, X_5, X_6, S_1, S_2, S_3, A_1, A_2 \geq 0$

Simplex Table I

$C_j \rightarrow$			4	-4	2	-2	0	0	0	M	M	
↓	Basic Variables	Qty	X_3	X_4	X_5	X_6	S_1	S_2	S_3	A_1	A_2	R. Ratio
M	A_1	27	3	-3	1	-1	-1	0	0	1	0	9→ Key Row
0	S_2	21	-1	1	-1	1	0	1	0	0	0	-ve
M	A_2	30	1	-1	2	-2	0	0	-1	0	1	30
Total Contribution Z_j = 57 M			4M	-4M	3M	-3M	-M	0	-M	+M	M	
Opportunity Cost $(C_j - Z_j)$			4-4M	-4+4M	2-3M	-2+3M	M	0	M	0	0	

↑
Key Column

Entering variable is X_3 and Outgoing variable is A_1. Key element is 3.

Simplex Table II

$C_j \rightarrow$			4	-4	2	-2	0	0	0	M	M	
↓	Basic Variables	Qty	X_3	X_4	X_5	X_6	S_1	S_2	S_3	A_1	A_2	R. Ratio
4	X_3	9	1	-1	1/3	-1/3	-1/3	0	0	-	0	27
0	S_2	30	0	0	-2/3	2/3	-1/3	1	0	-	0	-ve
M	A_2	21	0	0	5/3	-5/3	1/3	0	-1	-	1	63/5→ Key row
Total Contribution Z_j = 36 + 21M			4	-4	5M+4/3	-4-5M/3	M-4/3	0	-M	-	M	
Opportunity Cost $(C_j - Z_j)$			0	0	2-5M/3	5M-2/3	4-M/3	0	M	-	0	

↑
Key Column

Entering variable is X_5 and Outgoing variable is A_2. Key element is 5/3. X_5 $X_6, X_4 \neq 0$

Simplex Table III

$C_j \rightarrow$			4	-4	2	-2	0	0	0
↓	Basic Variables	Qty.	X_3	X_4	X_5	X_6	S_1	S_2	S_3
4	X_3	24/5	1	-1	0	0	-6/15	0	+1/5
0	S_2	192/5	0	0	0	0	-1/5	1	-2/5
2	X_5	63/5	0	0	1	-1	1/5	0	-3/5
Total Contribution Z_j = 44.4			4	-4	2	-2	-6/5	0	-2/5
Opportunity Cost $(C_j - Z_j)$			0	0	0	0	6/5	0	2/5

Consider X_4 as incoming and solve for one time more.

From this table we get a different optimum solution $X_1 = 24/5$, $X_2 = 63/5$, $Z_j = 44.5$

As in opportunity cost column X_4 and X_6 depicts value equal to zero and these variable are not present in basic variable column, it mean alternative solution exists.

? multiple optimal solution

Example 27. Solve the following LPP

$\leq$ S

$\geq$ -S+a

= a

Max. $Z = 8X_1 - 4X_2$

Subject to $4X_1 + 5X_2 \leq 20$

$-X_1 + 3X_2 \geq -23$

When $X_1 \geq 0$, X_2 is unrestricted in sign.

Solution :

In order to have RHS value as +ve, we muliply the constraint second with –1. Replace the unrestricted variable X_2 with non-negative variables $X_3 - X_4$. The resultant constraints equation and the objective function are given below :

Max. $Z = 8X_1 - 4X_3 + 4X_4$

Subject to $4X_1 + 5X_3 - 5X_4 \leq 20$

$X_1 - 3X_3 + 3X_4 \leq 23$

$X_1, X_3, X_4 \geq 0$

Introduce slack variables S_1 and S_2 in constraint and assign 'o' Co-efficient to slack variables in the objective function.

Max. $Z = 8X_1 - 4X_3 + 4X_4 + 0S_1 + 0S_2$

$4X_1 + 5X_3 - 5X_4 + S_1 + 0S_2 = 20$

$X_1 - 3X_3 + 3X_4 + 0S_1 + S_2 = 23$

$X_1, X_3, X_4, S_1, S_2, \geq 0$

Simplex Table I

$C_j \rightarrow$			8	-4	4	0	0	
↓	Basic Variables	Qty.	X_1	X_3	X_4	S_1	S_2	R. Ratio
0	S_1	20	4	5	-5	1	0	5→Key row
0	S_2	23	1	-3	3	0	1	23
Total Contribution Z_j = 0			0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			8	-4	4	0	0	

↑
Key column

Entering variable is X_1 and Outgoing variable is S_1. Key element is 4.

Simplex Table II

$C_j \rightarrow$			8	-4	4	0	0	
↓	Basic Variables	Qty.	X_1	X_3	X_4	S_1	S_2	R. Ratio
8	X_1	5	1	5/4	-5/4	1/4	0	-ve
0	S_2	18	0	-17/4	17/4	-1/4	1	72/17→Key row
Total Contribution Z_j = 40			8	10	-10	2	0	
Opportunity Cost $(C_j - Z_j)$			0	-14	14	-2	0	

↑
Key column

obtained means all of

Entering variable is X_4 and Outgoing variable is S_2. Key element is 17/4.

Simplex Table III

$C_j \rightarrow$			8	–4	4	0	0
↓	Basic Variables	Qty.	X_1	X_3	X_4	S_1	S_2
8	X_1	175/17	1	0	0	3/17	5/17
4	X_4	72/17	0	–1	1	–1/17	4/17
Total Contribution Z_j = 1688/17			8	–4	4	20/17	56/17
Opportunity Cost $(C_j - Z_j)$			0	0	0	–20/17	–56/17

Since all $C_j - Z_j$ are ≤ 0. The optimal solution is obtained. meaning & Alternative solution exists

$X_1 = \frac{175}{17}$; $X_3 = (X_3 - X_4) = 0 - \frac{72}{17} = -\frac{72}{17}$; $Z = \frac{1112}{17}$; $X_1 = \frac{175}{17}$ and $X_4 = \frac{72}{17}$.

(i) when value of basic variable = 0 in bi column

(ii) when there is a tie in the replacement ratio

CASE V : DEGENERACY

Degeneracy in liner programming problem is said to arise when a BFS contains a smaller number of non-zero variables than the number of Independent constraints. In other words, a solution is said to be degenerated when value of some BV are zero or **when there is a tie between the replacement ratio of two or more variables.**

However, if the Replacement ratio is **zero** for two or more basic variables, degeneracy may result the simplex routine to cycle indefinitely *i.e.* the solution which we have obtained in one iteration may repeat again after few iterations and therefore no optimum solution may be obtained under such circumstances. A simple way to get over the situation is to go back to the iteration stage where the tie had occurred and choose the other variable as leaving variable and then proceed with the solution of the problem in the usual manner. Alternatively, the following procedure may be followed for deciding the key row and leaving variable.

Steps to resolve degeneracy

means finding R R

Following procedure can be used to avoid cycling due to degeneracy :

tep 1 Divide the co-efficient of slack variables by the corresponding positive number of the key column in the row, starting from Left to Right. Division is made to that slack variables whose degeneracy is detected.

ep 2 The row which contains **smallest** ratio becomes the key row.

In case any artificial variable is involved in the problem of degeneracy, select it as leaving variable cily without following the given procedure to resolve the degeneracy.

mple 28. Maximize $Z = 3X_1 + 9X_2$

Subject to constraints

$$X_1 + 4X_2 \le 8$$
$$X_1 + 2X_2 \le 4$$
$$X_1, X_2 \ge 0$$

Solution : Convert the inequalities into equations by introducing slack variables.

Maximize $Z = 3X_1 + 9X_2 + 0S_1 + 0S_2$

Subject to

$$X_1 + 4X_2 + S_1 + 0S_2 = 8$$
$$X_1 + 2X_2 + 0S_1 + S_2 = 4$$
$$X_1, X_2, S_1, S_2 \ge 0$$

Initial Simplex Table I

$C_j \rightarrow$			3	9	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	R. Ratio
0	S_1	8	1	4	1	0	2 → Key row
0	S_2	4	1	2	0	1	2
Total Contribution Z_j = 0			0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			3	9	0	0	

↑ Key Column

We have resolved the degeneracy as

	S_1	S_2	
S_1	1/4	0/4	→ Least Positive
S_2	0/2	1/2	

Simplex Table II

$C_j \rightarrow$			3	9	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	R. Ratio
9	X_2	2	1/4	1	1/4	0	8
0	S_2	0	1/2	0	–1/2	1	0 → Key row
Total Contribution Z_j = 18			9/4	9	9/4	0	
Opportunity Cost $(C_j - Z_j)$			3/4	0	–9/4	0	

↑ Key Column

Entering variable is X_1 and outgoing variable is S_2. Key element is 1/2.

Simplex Table III

$C_j \rightarrow$			3	9	0	0	
↓	Basic Variables	Qty.	X_1	X_2	S_1	S_2	
9	X_2	2	0	1	1/2	–1/2	
3	X_1	0	1	0	–1	2	
Total Contribution Z_j = 18			3	9	3/2	3/2	
Opportunity Cost $(C_j - Z_j)$			0	0	–3/2	–3/2	

Since all the elements in $C_j - Z_j$ row is either zero or –ve. Hence optimal solution has been obtained where $X_1 = 0$, $X_2 = 2$, $Z_j = 18$.

Example 29. Solve the Linear programming problem, using simplex method :

Maximize $Z = 3x_1 + 2x_2$

Subject to

$$4x_1 + 3x_2 \le 12$$
$$4x_1 + x_2 \le 8$$
$$4x_1 - x_2 \le 8$$
$$x_1, x_2 \ge 0$$

(G.N.D.U. B.A./BSc. II Q.T-A., April, 2004, Sept., 2005)

Sol. Introducing slack variables S_1, S_2 and S_3 the given LPP becomes

Maximize $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$4x_1 + 3x_2 + S_1 + 0S_2 + 0S_3 = 12$$
$$4x_1 + x_2 + 0S_1 + S_2 + 0S_3 = 8$$
$$4x_1 - x_2 + 0S_1 + 0S_2 + S_3 = 8$$
$$x_1, x_2, S_1, S_2, S_3 \ge 0$$

9/4 1/4 0/4

0/4 0/4 1/4

Simplex Table I

C_j ↓	→ Basic Variables	Qty.	3 x_1	2 x_2	0 S_1	0 S_2	0 S_3	R. Ratio
0	S_1	12	4	3	1	0	0	3
0	S_2	8	4	1	0	1	0	2 → Key Row
0	S_3	8	4	–1	0	0	1	2
Total Contribution $Z_j = 0$			0	0	0	0	0	
Opportunity Cost $(C_j - Z_j)$			3	2	0	0	0	

↑ Key column

nearest to the top

Simplex Table II

C_j ↓	→ Basic Variables	Qty.	3 x_1	2 x_2	0 S_1	0 S_2	0 S_3	R. Ratio
0	S_1	4	0	2	1	–1	0	2 → Key Row
3	x_1	2	1	1/4	0	1/4	0	8
0	S_3	0	0	–2	0	–1	1	
Total Contribution $Z_j = 6$			3	3/4	0	3/4	0	
Opportunity Cost $(C_j - Z_j)$			0	5/4	0	–3/4	0	

↑ Key column

degenerate

S_1 S_3 can also be

0/2

Simplex Table III

C_j ↓	→ Basic Variables	Qty.	3 x_1	2 x_2	0 S_1	0 S_2	0 S_3	R. Ratio
2	x_2	2	0	1	1/2	–1/2	0	
3	x_1	3/2	1	0	–1/8	3/8	0	
0	S_3	4	0	0	1	–2	1	
Total Contribution $Z_j = 17/2$			3	2	5/8	1/8	0	
Opportunity Cost $(C_j - Z_j)$			0	0	–5/8	–1/8	0	

As all values in the index $(C_j - Z_j)$ row are less than or equal to zero, so the solution is optimum, where

$$x_1 = 3/2,\ x_2 = 2,\ Z = 17/2.$$

QUESTIONS

1. Define the following terms :
   (*i*) Basic variable (*ii*) Basic solution (*iii*) Basic feasible solution and (*iv*) Degenerate basic solution.
2. Establish the difference between (*i*) Feasible solution (*ii*) Basic feasible solution (*iii*) Degenerate basic feasible solution and (*iv*) Optimum basic feasible solution.
3. Explain the terms (in the context of LPP) : (*a*) Non-negativity constraints (*b*) Feasible solutions (*c*) Objective function.
4. Explain the simplex procedure to solve a linear programming problem.
5. In solving a linear programming problem by Simplex method, explain how you will move from a given basic feasible solution to another basic feasible with an improved value of the objective function.
6. Show that if there is an optimal basic feasible solution to a linear programming problem and for some a_j not in the basis, $z_j - c_j = 0$, $y_{ij} \le 0$ for all i, then a non-basic alternative optima will exist.
7. (*a*) If a linear programming problem possesses (*i*) unbounded solution (*ii*) multiple solutions (*iii*) infeasible solution, how can you detect it in course of simplex computations ?
   (*b*) How do you identify multiple solution in a linear programming problem ?
8. Write the difference in the Simplex solution procedure for a maximization problem and a minimization problem of a linear programming.
9. Explain the use of artificial variables in L.P.

10. Briefly describe the graphic and simplex method of solving a linear programming problem. Why is simplex method considered superior to graphic method ? [G.N.D.U. B.Com. III 2003]

11. Explain the penalty method or 'Big-M' method for solving a linear programming problem.

12. Define the following and indicate their significance to decision making with LP and the Simplex method

(*i*) Key Row (*ii*) Key Column (*iii*) Key element

(*iv*) Degeneracy (*v*) Cycling (*vi*) Multiple optima.

13. How are Key column and Key row determined in a Simplex table containing a non-optimal solution ?

14. Explain the meaning of 'infeasibility' and 'unboundedness'. How can each of these be detected while applying Simplex technique ?

15. Explain the term 'Artificial Variable' and its use in L.P. Why do we need them ?

16. What is meant by degeneracy in LPP ? Explain the method of resolving the degeneracy.

17. What is a redundant constraint ? How such constraints influence analysis and final solution of a LPP ?

18. Why is the simplex method a better technique than the graphical approach for most real cases ? [GNDU 1995, 97] Discuss the advantages and limitations of LPP.

19. What is difference between slack, surplus and artificial variables ? How do they differ to their use for solving LPP using Simplex method ? [G. N.D.U. 2005]

EXERCISE 4.4

1. Solve the LP Problem using simplex method.

Maximize $Z = 2X_1 + X_2$

Subject to restrictions

$2X_1 - X_2 \leq 10$

$X_1 \leq 8$

$X_1, X_2 \geq 0$

[**Ans.** Unbounded solution. As S_1 is entering]

2. Solve the LP Problem using simplex method.

Maximize $Z = 3X_1 + 5X_2$

Subject to restrictions

$X_1 + 2X_2 \leq 6$

$X_1 \leq 10$

$X_2 \geq 1$

$X_1, X_2 \geq 0$

[**Ans.** $X_1 = 4, X_1 = 1, Z = 17$]

3. Solve the LP Problem using simplex method.

Maximize $Z = X_1 - 2X_2 - 3X_3$

Subject to restrictions

$-2X_1 + X_2 + 3X_3 = 2$

$2X_1 + 3X_2 + 4X_3 = 1$

$X_1, X_2, X_3 \geq 0$

[**Ans.** Solution is infeasible as A_1 is present in final table]

4. Solve the following

Maximize $Z = 6X_1 + 4X_2$

Subject to restrictions

$X_1 + X_2 \leq 5$

$X_2 \geq 8$

$X_1, X_2 \geq 0$

[**Ans.** Solution is infeasible as A_1 is present in final table]

5. Solve the LP Problem using simplex method.

Maximize $Z = 4X_1 + 3X_2$

Subject to restrictions

$X_1 + X_2 \leq 50$

$X_1 + 2X_2 \geq 80$

$3X_1 + 2X_2 \geq 140$

$X_1, X_2 \geq 0$ (Non-negativity constraint)

[**Ans.** Solution is infeasible as A_1 is present in fina table]

6. Solve the following

Minimize $Z = 2X_1 + 3X_2 + 5X_3$

Subject to restrictions

$3X_1 + 10X_2 + 5X_3 \leq 15$

$X_1 + 2X_2 + X_3 \geq 4$

$33X_1 - 10X_2 + 9X_3 \leq 60$

$X_1, X_2, X_3 \geq 0.$

[**Ans.** Solution is infeasible as A_1 is present in final table]

7. Solve the LP Problem using simplex method.

Maximize $Z = 4X_1 + 8X_2$

Subject to constraints

$X_1 + 2X_2 \leq 10$

$X_1 + X_2 \leq 8$

$X_1, X_2 \geq 0$

[**Ans.** $X_1 = 0, X_2 = 5$ OR $X_1 = 6, X_2 = 2$, Maximum $Z = 40$]

8. Solve the LP Problem using simplex method.

Maximize $Z = 2X_1 + 3X_2$

Subject to constraints

$$6X_1 + 9X_2 \le 100$$
$$2X_1 + X_2 \le 20$$
$$X_1, X_2 \ge 0.$$

[**Ans.** $X_1 = 0, X_2 = 100/9$ OR $X_1 = 20/3, X_2 = 20/3$, Maximum $Z = 100/3$]

9. Solve the LP Problem using simplex method.

Minimize $Z = 4X_1 + 2X_2$

Subject to restrictions

$$3X_1 + X_2 \ge 27$$
$$-X_1 - X_2 \le 21$$
$$X_1 + 2X_2 \ge 30$$

X_1, X_2 are unrestricted in sign.

[**Ans.** Min. $Z = 44.4$, $X_1 = 24/5$, $X_2 = 63/5$]

10. Solve the LP Problem using simplex method.

Maximize $Z = 2X_1 + X_2 + 4X_3$

Subject to restrictions

$$-2X_1 + 4X_2 \le 4$$
$$X_1 + 2X_2 + X_3 \ge 5$$
$$2X_1 + 3X_3 \le 2$$

$X_1, X_2 \ge 0$ and X_3 is unrestricted in sign.

[**Ans.** Max. $Z = 27/8$, $X_1 = 7/4$, $X_2 = 15/8$, $X_3 = -1/2$]

11. Solve the LP Problem using simplex method.

Maximize $Z = 3X_1 + 2X_2 + X_3$

Subject to restrictions

$$2X_1 + 5X_2 + X_3 = 12$$
$$3X_1 + 4X_2 = 11$$

$X_2, X_3 \ge 0$ X_1 is unrestricted in sign.

[**Ans.** Max. $Z = 47/3$, $X_1 = 11/3$, $X_2 = 0$, $X_3 = 14/3$]

12. Solve the LP Problem using simplex method.

Maximize $Z = 5X_1 + 2X_2 + 10X_3$

$$X_1 - X_3 \le 10$$
$$X_2 - X_3 \ge 10$$
$$X_1 + X_2 + X_3 \le 10$$
$$X_1, X_2, X_3 \ge 0$$

[**Ans.** $X_1 = 0, X_2 = 10, X_3 = 0$, Maximum $Z = 20$]

13. Solve the LP Problem using simplex method.

Maximize $Z = 1000X_1 + 4000X_2 + 5000X_3$

Subject to restrictions

$$3X_1 + 3X_3 \le 22$$
$$X_1 + 2X_2 + 3X_3 \le 14$$
$$3X_1 + 2X_2 \le 14$$
$$X_1, X_2, X_3 \ge 0$$

[**Ans.** $X_1 = 0, X_2 = 7, X_3 = 0$, Maximum $Z = 28$