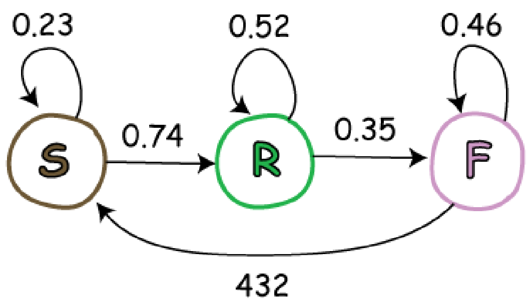
1. Spotted knapweed is an invasive species of noxious weed. They are a big problem in Canada.

Their lifecycle consists of three stages: seed, rosette, flower and can be modeled as follows:

1. Seed does not germinate but remains viable 0.23
2. Seed germinates into a rosette 0.74
3. Rosette remains rosette 0.52
4. Rosettes do not produce seeds
5. Rosette blooms into flowers 0.35
6. Flower remains flower 0.46
7. Flowers produce on average 432 seeds



As with the hippos, this population can be modeled with a matrix of the form:

$$K = \begin{bmatrix} b_1 & b_2 & b_3 \\ p_{1,1} & p_{1,2} & 0 \\ 0 & p_{2,3} & p_{3,3} \end{bmatrix}$$

Where b_i is the average amount of new seeds produced in the ith life stage, and $p_{i,j}$ is the probability something in stage i transitions to stage j.

(a) Say the population after one generation is $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ for the general matrix K, write out the product $K\vec{s}$.

Solution: Say $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$. We can write out the multiplication $K\vec{s}$:

$$\begin{bmatrix} b_1 & b_2 & b_3 \\ p_{1,1} & p_{1,2} & 0 \\ 0 & p_{2,3} & p_{3,3} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} b_1 s_1 + b_2 s_2 + b_3 s_3 \\ p_{1,1} s_1 + p_{1,2} s_2 \\ p_{2,3} s_2 + p_{3,3} \end{bmatrix}$$

(b) Using your previous answer, explain why $K\vec{s}$ models the amount of each life stage type in the next generation.

Solution: Lets look at each entry of

$$\begin{bmatrix} b_1 s_1 + b_2 s_2 + b_3 s_3 \\ p_{1,1} s_1 + p_{1,2} s_2 \\ p_{2,3} s_2 + p_{3,3} \end{bmatrix}.$$

The first entry describes how many of each stage become seeds. The next entry describes the number of s_1 that advance to s_2, and the number of s_2 that remain in s_2. The last entry is the same idea: the number of s_2 that become s_3 and the number of s_3 that remain s_3.

(c) Fill in the matrix K to model the spotted knapweed.

Solution:

$$K = \begin{bmatrix} .23 & 0 & 432 \\ .74 & .52 & 0 \\ 0 & .35 & .46 \end{bmatrix}.$$

(d) Write the characteristic polynomial of K. You do not need to simplify.

Solution:

$$\begin{vmatrix} .23-\lambda & 0 & 432 \\ .74 & .52-\lambda & 0 \\ 0 & .35 & .46-\lambda \end{vmatrix} = (.23-\lambda)[(.52-\lambda)(.46-\lambda) - 0] + 432[(.74)(.35) - 0]$$
$$= (.23-\lambda)(.52-\lambda)(.46-\lambda) + 432(.74)(.35)$$

(e) This has only one eigenvalue. It is

$$\lambda = \frac{121}{300} + \frac{1}{4050000}((3716328850998750000000 - 22143375000000\sqrt{28167003253545333})^{1/3} + 13500(1510472530 + 9\sqrt{28167003253545333})^{1/3})$$
$$\approx 5.22.$$

Explain the meaning of 5.22 for the spotted knapweed population.

Solution: This says that the spotted knapweed population will grow at an exponential rate 5.22^k after k generations.

(f) Write out a matrix that has the same nullspace as the λ-eigenspace. (You can either use λ or 5.22 in your answer.)

Solution: $\begin{bmatrix} .23-\lambda & 0 & 432 \\ .74 & .52-\lambda & 0 \\ 0 & .35 & .46-\lambda \end{bmatrix}$

(g) A compute solver gives an eigenvector, $\vec{u} = \begin{bmatrix} .86 \\ .13 \\ .01 \end{bmatrix}$. Explain the meaning of this vector in terms of the spotted knapweed population.

Solution: The population grows like $5.22^k\vec{u}$ over time and most $(86+13)\%$ of the population is in seed or rosette form.

(h) How would you recommend combatting the spotted knapweed?

Solution: You could pretty much say whatever for this. My opinion is that relatively few flowers are needed to maintain the population, because the flowers produce so many viable seeds each year. So, adding or removing a flower has the most significant effect on the long-term population change.

I would start a campaign about picking knapweed being the thing to do.

2. We consider the matrix $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

Solution:

$$\begin{vmatrix} 7-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = (7-\lambda)(1-\lambda)+8 = 15-8\lambda+\lambda^2 = (5-\lambda)(3-\lambda).$$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$.

(b) Do we have enough information to determine that A is diagonalizable? Why or why not?

Solution: A has two distinct, real eigenvalues, so it must be diagonalizable.

(c) Determine bases for the eigenspace(s) of A.

Solution:

$$A - 3I = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}.$$

So the eigenspace of $\lambda_1 = 3$ has basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

$$A - 5I = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

So the eigenspace of $\lambda_2 = 5$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(d) Find matrices P, P^{-1}, and D such that $A = PDP^{-1}$. Verify that your decomposition is correct.

Solution: Our matrices are

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},\ D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

It is not hard to then verify that

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}.$$

(e) Use your diagonalization to calculate e^A. Present your answer as a single 2×2 matrix.

Solution: Our diagonalization gives us a manageable way to calculate the matrix exponential.

$$\begin{aligned} e^A &= P \begin{bmatrix} e^3 & 0 \\ 0 & e^5 \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^3 & 0 \\ 0 & e^5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -e^3 + 2e^5 & e^3 - e^5 \\ -2e^3 + 2e^5 & 2e^3 - e^5 \end{bmatrix}. \end{aligned}$$

3. Chris and Matt are big pizza eaters, and also big on knowing their macro nutrients. Both agree on the same kind of pizza dough, but prefer different toppings. Matt likes feta cheese and sunflower seeds. Chris prefers egg and kale. These have macro profiles $\begin{bmatrix} \text{Fat} \\ \text{Carbs} \\ \text{Protein} \end{bmatrix} \in \mathbb{R}^3$ as follows:

$$\vec{d} = \begin{bmatrix} 3 \\ 42 \\ 7 \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} 21 \\ 4 \\ 14 \end{bmatrix}, \quad \vec{s} = \begin{bmatrix} 50 \\ 24 \\ 19 \end{bmatrix}, \quad \vec{e} = \begin{bmatrix} 10 \\ 1 \\ 13 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}.$$

Let $\mathcal{M} = \{\vec{d}, \vec{f}, \vec{s}\}$ and $\mathcal{C} = \{\vec{d}, \vec{e}, \vec{k}\}$. Also, let $M = [\vec{d}\,\vec{f}\,\vec{s}]$, and $C = [\vec{d}\,\vec{e}\,\vec{k}]$. These have inverses:

$$M^{-1} = \frac{1}{13990} \begin{pmatrix} -260 & 301 & 304 \\ -630 & -293 & 2028 \\ 560 & 105 & -870 \end{pmatrix}, \qquad C = \frac{1}{306} \begin{pmatrix} 179 & 16 & -129 \\ 35 & -2 & -3 \\ -581 & -28 & 417 \end{pmatrix}$$

(a) A friend offers Matt some carrot cake with profile $\vec{c} = \begin{bmatrix} 20 \\ 45 \\ 4 \end{bmatrix}$. Matt passes and says he would prefer to make a pizza. What quantities of each of his preferred ingredients should he use? It is okay to express your answer just in terms of the symbols for the matrices.

Solution:

$$\vec{c}_{\mathcal{M}} = M^{-1}\vec{c} = \frac{1}{13390} \begin{pmatrix} 9561 \\ -17673 \\ 12445 \end{pmatrix} \approx \begin{bmatrix} .68 \\ -1.26 \\ .89 \end{bmatrix}.$$

(b) Chris opts to have 100g of carrot cake, but wants to log the macros in terms of his fav pizza ingredients. Write an expression that describes this.

Solution:

$$\vec{c}_{\mathcal{C}} = C^{-1}\vec{c} = \frac{1}{306} \begin{pmatrix} 3784 \\ 598 \\ -11212 \end{pmatrix} \approx \begin{bmatrix} 12.36 \\ 1.95 \\ -36.640522876 \end{bmatrix}.$$

(c) Write a matrix and vector expression relating $\vec{c}_{\mathcal{C}}$ and $\vec{c}_{\mathcal{M}}$.

Solution: $\vec{c}_{\mathcal{C}} = C^{-1}M\vec{c}_{\mathcal{M}}$.

(d) What are the entries f_1, f_2, f_3 of $\vec{f}_{\mathcal{M}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}_{\mathcal{M}}$?

Solution: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{M}}$.

(e) Simplify the expression $M\vec{f}_{\mathcal{M}} - C^{-1}\vec{f} - C\vec{f}_{\mathcal{C}} - M^{-1}C\vec{f}_{\mathcal{C}}$.

Solution: We can write it as $\vec{f} - \vec{f}_{\mathcal{C}} - \vec{f} - \vec{f}_{\mathcal{M}} = \vec{f}_{\mathcal{C}} - \vec{f}_{\mathcal{M}}$.