Math 308: Midterm 1 Review

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1 Linear equations

1.1 Systems of Equations

1.1.1 Modeling

1. A scientist realize a curious relationship between the temperature and mass of 3 samples in an experiment. She finds that the mass of all three (in grams) sums to twice the sum of all three's temperatures (celsius). The mass of each sample minus the temperature is exactly 5. The mass of the first two samples are 3 and 5 grams, respectively. Represent this as a matrix.

   It row reduces to:

$$\left(\begin{array}{cccccc|c} m_1 & m_2 & m_3 & t_1 & t_2 & t_3 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 22 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 17 \end{array}\right)$$

   The scientist is measuring the quantity $\vec{m} \cdot \vec{t}$. Is this value nonnegative?

2. Ling, an engineer, was just hired to orchestrate the upkeep of 7 bridges in the San Francisco area. She has a budget of $700,000. Older bridges cost more per day to repair. In fact, the nth oldest bridge costs n thousand dollars per day. They need more work too. She plans to spend 2 days more on each bridge than the one that was built just after. For example, the oldest bridge will take 2 days longer than the second oldest bridge. How many days does she allocate to each bridge?

1.2 Linear systems and matrices

1.2.1 Echelon form

3. (a) Is the following system in echelon form? If not, how could it be changed to be in echelon form?

$$\begin{aligned} 2x_1 - 4x_2 + 2x_3 + x_4 &= 11 \\ 3x_4 &= 9. \\ x_2 - x_3 + 2x_4 &= 5 \end{aligned}$$

   (b) Explain in a sentence or two why a system in echelon form always has a solution.

4. Every system of equations can be put into echelon form.

   True False Not enough information

5. (a) Put the following matrix into a system of equations:

$$\left(\begin{array}{ccc|c} \frac{1}{a} & 1 & 2 & \frac{1}{a} \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{array}\right).$$

   In echelon form this matrix is:

$$\left(\begin{array}{ccc|c} 1 & a & 2\,a & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2\,a+2 & a-2 \end{array}\right)$$

   (b) Are there any values of a for which this system doesn't have any solutions?

6. Every linear system of equations can be represented by a matrix.

   True False Not enough information

7. Echelon form is unique.

   True False Not enough information

1.2.2 Row-reducing

8. Row reduce the following matrices and decide if they have 0, 1 or infinitely many solutions.

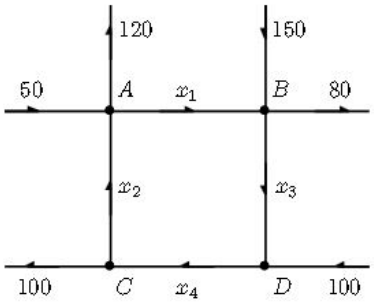
   (a)
   $$\left[\begin{array}{ccc|c} 2 & -2 & -3 & -2 \\ 1 & 0 & -2 & 1 \\ -1 & 1 & 0 & 2 \end{array}\right].$$

   (b)
   $$\left(\begin{array}{ccc|c} -1 & 3 & 9 & 7 \\ -1 & 2 & 6 & 5 \\ -2 & 7 & 21 & 16 \end{array}\right)$$

   (c)
   $$\left(\begin{array}{cc|c} 10 & -3 & -5 \\ -3 & 1 & -3 \\ 5 & -2 & 4 \end{array}\right)$$

1.3 Applications

9. Represent this traffic diagram as a system of equations:



10. Write the solution set for coefficients $ax + by + cz + d = 0$ of the planes which contain the points $(1, 0, 1)$ and $(0, 1, 1)$.

11. Write the matrix representing the solution set for the coefficients of a cubic $y = ax^3+bx^2+cx+d$ that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$.

2 Euclidean space

2.1 Vectors

12. Write the solution set to the system represented by this matrix in vector form:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -\frac{1}{2} \end{array}\right)$$

The vectors in the solution set belong to $\mathbb{R}^n$ for what n?

13. Every linear combination of nonzero vectors is never the zero vector.

True False Not enough information

2.2 Span

14. To check the span of three vectors you write the following matrix down:

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & a \\ 1 & 0 & 0 & b \\ 0 & h & 1 & c \end{array}\right)$$

A computer solver outputs:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & \frac{1}{2} & \frac{1}{2}\,a \\ 0 & 0 & -\frac{1}{2}\,h+1 & -\frac{1}{2}\,ah+c \end{array}\right)$$

(a) What type of objects belong to the span of these three vectors?

(b) For what value(s) of h is the span of these three vectors not equal to $\mathbb{R}^3$?

(c) For a value of h from part (b), describe the span as a plane equation.

(d) Describe a vector not in the span from the previous part.

15. If $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is a set of vectors that span $\mathbb{R}^n$, then we can find a unique solution to the vector equation

$$x_1\vec{u}_1 + \cdots + x_m\vec{u}_m = \vec{v}$$

for any vector $\vec{v}$ in $\mathbb{R}^n$.

True False Not enough information

2.3 Linear independence

16. In trying to determine if some set of vectors is linearly independent we consider the following augmented matrix:

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 4 & 0 \\ 2 & 1 & 0 & 8 & 0 \\ 4 & 2 & 1 & 21 & 0 \\ 1 & 0 & 1 & 9 & 0 \end{array}\right].$$

Chris did the legwork and put it in reduced echelon form:

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(a) Why is this set linearly dependent? Use the original definition provided in class.

(b) Which of the vectors can we write as a linear combination of the others? Which cannot be written as a linear combination of the others?

(c) Can we remove a vector from the set to make it linearly independent?

(d) Can we add another vector to the set to make them span $\mathbb{R}^4$?

17. Give an example of a linearly independent set of vectors that does not span $\mathbb{R}^3$.

18. If $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then any subset of these vectors is also linearly independent.

True False Not enough information