(I) Compute the following integrals using integration by parts.

(a) $\int \ln(x)dx$

$$u = \ln x, dv = dx$$

This gives

$$= x\ln x - x + C$$

(b) $\int (\sin x)e^x dx$

$$u = \sin x, dv = e^x$$

This yields

$$= \cos x e^x - \int \cos x e^x dx$$

Appy integration by parts again with $u = \cos x, dv = e^x$ to get

$$\int \sin x e^x dx = \cos x e^x - [\sin x e^x - \int -\sin x e^x dx].$$

Add the $-\int \sin x e^x$ term to both sides and divide by 2 to obtain

$$\int \sin x e^x dx = \frac{e^x \cos x - e^x \sin x}{2} + C.$$

(c) $\int (\sin\theta\cos\theta)\, e^{\left(\sin^2 \frac{\theta}{2}\right)}\, d\theta$

Make the substitution $\sin^2(\theta/2) = \frac{1}{2}(1-\cos 2\theta)$ to rewrite as

$$= \int \cos\theta \sin\theta e^{\frac{1}{2}(1-\cos 2\theta)} d\theta = e^{1/2}\int \cos\theta\sin\theta e^{-\cos(2\theta)} d\theta$$

Make the substitution $w = \cos(2\theta)$ to rewrite as

$$= e^{\frac{1}{2}} \int w e^{-\frac{1}{2}w} dw.$$

Now we do integration by parts with $u = w$ and $dv = e^{-\frac{1}{2}w}$. This gives

$$= e^{1/2}\left[-2we^{\frac{1}{2}w} + 2\int e^{-\frac{1}{2}w} dw\right] = e^{1/2}\left[-2we^{\frac{1}{2}w} - 4e^{\frac{1}{2}w}\right] + C.$$

(II) Compute the following integrals by using a trig identity and an appropriate u-substitution.

(a) $\int (\sin x)^3 (\cos x)^2\, dx$

We can use the fact that $\sin^2 x = 1 - \cos^2 x$ to rewrite as

$$= \int \sin x(1-\cos^2 x)\cos^2(x) dx.$$

Now let $u = \cos x$ and this becomes

$$\int (1-u^2)u^2 du = \int u^2 - u^4 du = \frac{u^3}{3} - \frac{u^5}{5} + C.$$

(b) $\int (\sin x)^4 (\cos x)^3 dx.$

This is similar to the previous problem, instead we use the fact that $\cos^2 x = 1 - \sin^2 x$. This lets us rewrite as

$$\int \sin^4(x)(1-\sin^2 x)\cos x dx.$$

Letting $u = \sin x$ this becomes

$$= \int u^4(1-u^2)du.$$

Which you can integrate with comfort and ease.

(c) $\int (\sin x)^2 (\cos x)^4 dx.$ *Hint:* $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$; $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$.

Using the hint this integral becomes

$$= \int \frac{1}{2}(1-\cos(2\theta))(\frac{1}{2}(1+\cos(2\theta)))^2 d\theta$$

Simplifying we can write as

$$\frac{1}{8}\int \underbrace{-\cos^3(2\theta)}_{(i)} - \underbrace{\cos^2(2\theta)}_{(ii)} + \underbrace{\cos(2\theta)+1}_{(iii)} d\theta$$

We integrate (i), (ii) and (iii) separately then add them up.

(i) We use the fact that $\cos^2\theta = 1 - \sin^2\theta$ then make the u-substitution $u = \sin 2\theta$ to write

$$\int \cos^3(2\theta)d\theta = \int \cos 2\theta(1-\sin^2 2\theta)d\theta = \frac{1}{2}\int 1+u^2 du = \frac{\sin\theta}{2} + \frac{\sin^3\theta}{6}.$$

(ii) We use the double angle identity $\cos^2(2\theta) = \frac{1}{2}(1+\cos(4\theta))$ to write

$$\int \cos^2(2\theta)d\theta = \frac{1}{2}\int \frac{1}{2}(1+\cos(4\theta)d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right].$$

(iii) This integral is straightforward

$$\int \cos(2\theta)+1 d\theta = \frac{1}{2}\sin 2\theta + \theta.$$

The answer is then

$$-(i)-(ii)+(iii)+C = -\frac{\sin\theta}{2} - \frac{\sin^3\theta}{6} - \frac{1}{2}\int \frac{1}{2}(1+\cos(4\theta)d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right] + \frac{1}{2}\sin\theta + \theta + C.$$

(d) $\int t\cos^2(t)\, dt$

We use integration by parts with $u = t$ and $du = \cos^2(t)$ and the fact that $\cos^2(t) = \frac{1}{2}(1+\cos(2t))$ to write

$$= t\left[\frac{1}{2}[t + \frac{1}{2}\sin(2t)]\right] - \int t + \frac{1}{2}\sin(2t)dt = t\left[\frac{1}{2}[t+\frac{1}{2}\sin(2t)]\right] - \frac{t^2}{2} + \frac{1}{4}\cos(2t) + C.$$

(III) Once Matt explains how, use trig substitution to compute the following integrals.

(a) $\int \frac{u^3}{1-u^2}du.$

Let $u = \sin\theta$. This means $du = \cos\theta d\theta$. Making this substitution gives

$$= \int \frac{\sin^3\theta}{1-\sin^2\theta}d\theta = \int \frac{\sin^3\theta}{\cos^2\theta}d\theta.$$

Integrating $\sin^3\theta/\cos^2\theta$ requires writing $\sin^\theta = 1-\cos^2\theta$. More explicitly,

$$\int \frac{\sin^3\theta}{\cos^2\theta}d\theta = \int \sin\theta\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)d\theta$$

Let $w = \cos\theta$ and we have

$$= \int \frac{1-w^2}{w^2}dw = \int w^{-2} - 1dw = -\frac{1}{\cos\theta} - \cos\theta.$$

Lastly we have to go back to the variable u. If you draw a triangle and think about it you will get $\cos\theta = \frac{u}{\sqrt{u^2-1}}$ so the answer is

$$-\frac{\sqrt{u^2-1}}{u} - \frac{u}{\sqrt{u^2-1}} + C.$$

(b) $\int \sqrt{\frac{x^2+9}{x^4}}\,dx$

I'm too exhausted to write out all of the steps. Basically start with the substitution $\tan\theta = x$ and if you are careful and persistent you will eventually get

$$\ln\left|\frac{\sqrt{x^2+4}}{3} + \frac{x}{3}\right| - \frac{\sqrt{x^2+4}}{4} + C$$