#6: Derivatives of Polynomials, the Number e, Product and Quotient Rule, Trig Derivatives Chapter 3.1,3.2

Contents

#6: Derivatives of Polynomials, the Number e, Product and Quotient Rule, Trig Derivatives
Chapter 3.1,3.2

1 Derivatives of Polynomials and the Number e

1.1 Derivatives of Polynomials

Goal: Derive Shortcuts to Computing Derivatives.

1.1.1 Intro

You may have noticed that it is somewhat time consuming to compute derivatives using limits. Fortunately, there is a better way. Today we will begin to hint at a way to sidestep the formal definition of the derivative. Our grand scheme is to find a general derivative formula for certain functions (today we will talk about polynomials), and with this we will be able to apply it to specific cases. First a warmup.

Example: "The Derivative of a Constant" Let $f(x) = c$ with c a constant. We see that $f'(c) = 0$

Diagram

Computation

Example: Let $f(x) = x$, $g(x) = x^2$ and $h(x) = x^3$. We can grind out the computations to get

$$f'(x) = 1, \qquad g'(x) = 2x, \qquad h'(x) = 3x^2$$

1.1.2 Follow the Rules!

It seems that above a pattern appears... It turns out that for the function $f(x) = x^n$ with n an integer (i.e. whole number) the derivative is nx^{n-1} this is called the power rule.

THE POWER RULE	If $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Proof. It is a fact that we can write

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}.$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \cdots = a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} = na^{n-1}$$

□

This is great. Now we just need a couple of rules that follow from our limit laws.

DERIVATIVE RULES: Let f and g be differentiable functions and let c be a constant.

1. CONSTANT MULTIPLE RULE:

$$(c(f(x))' = cf'(x)$$

2. SUM RULE:

$$(f(x) + g(x))' = f'(x) + g'(x)$$

3. DIFFERENCE RULE:

$$(f(x) - g(x))' = f'(x) - g'(x)$$

We aren't quite yet ready for a product or quotient rule, but with the above we can now easily compute some derivatives that one page ago would have been rather time consuming for example,

Example: $\frac{d}{dx}\left(x^{10} + 3x^9 + 3x^{-2}\right) = 10x^9 + 27x^2 - 6x^{-3}$

It turns out that the power rule works when n is any real number. So, we have the general version of the power rule is

THE POWER RULE (General Version)

For any real number r, if $f(x) = x^r$ then $f'(x) = rx^{r-1}$.

This is super powerful. An example would be

Example: $\frac{d}{dx}\left(\pi x^\pi + 3x^{7/3}\right) = \pi^2 x^{\pi-1} + 7x^{5/3}$

! Note that the power rule *does not* apply to exponential functions. For example,

$$\frac{d}{dx}(3^x) \neq x(3^{x-1})$$

We will deal with these functions in a week or so.

However, there is one exponential function that is a piece of cake to differentiate. We discuss that next.

1.2 The Number e.

There are several different ways to define the number e, for our purposes we will use derivatives to define e.

Definition: e is the number so that $\frac{d}{dx}(e^x) = e^x$.

Fact: e is unique and irrational (not expressible as a fraction), we can approximate it with

$$e \approx 2.7182$$

Extraneous, but Interesting Fact

e is a super important number and literally comes up everywhere. You **don't need to know the following fact for this course** but if you take Math 307 you will be able to prove the following equality (known as Euler's Identity where $i = \sqrt{-1}$)

$$e^{\pi i} = -1$$

Example: $f(x) = x^3 + e^x$, then $f'''(x) = 6 + e^x$.

2 Product and Quotient Rule

In the last section we had the sum and difference rule for derivatives. But what about products and quotients. For example what should the derivative of $f(x)g(x)$ be? A natural guess would be

$$\text{Natural (but wrong) Guess:} \qquad \frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$$

Counterexample: $f(x) = x$ $g(x) = x^2$. Since we know $f(x)g(x) = x^3$ we know $(f(x)g(x))' = 3x^2$. However, the wrong guess way we obtain $f'(x) \cdot g'(x) = 1 \cdot 2x = 2x$ and we have the false statement $3x^2 = 2x$.

It turns out the right guess is called the product rule.

2.1 The Product Rule

PRODUCT RULE:

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)\right) &= f'(x)g(x) + g'(x)f(x)\\ &= \text{`Derivative of the first times the second plus the Derivative of the second times the first'}\end{aligned}$$

Proof. For the proof see page 184 of the book. □

Example: $f(x) = x$, $g(x) = x^2$

Example: $f(x) = x^3e^x$

2.2 The Quotient Rule

Like with the product rule, the natural guess for the quotient rule turns out to be wrong. Instead we get the following formula:

QUOTIENT RULE:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

$$= \frac{f'g - g'f}{g^2}$$

Proof. See page 185 of text. Uses $\delta x, \delta y$ definition of derivative. □

In words the quotient rule is *the derivative of the top times the bottom minus the derivative of the bottom times the top all divided by the bottom squared.*

Example: $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$

Computation

All of the rules we have derived thus far are summarized in the table on page 187. Let's do an example that uses everything all at once.

Example: Let $f(x) = \frac{3x^3e^x+x}{x-1}$, find $f'(x)$.

Computation

If you do drift back to the early days remember that the elementary functions are:

1. Polynomials
2. Rational Functions

3. Root Functions
4. Exponential Functions
5. Trig Functions
6. Logarithmic Functions

At this point we can handle derivatives of 1,2,3 and the special case of 4 with e^x. Next up, trig functions.

3 Derivatives of Trig Functions

3.1 $\frac{d}{dx}(\sin x) = \cos x$

Goal: Differentiate $\sin x$.

WARMUP: Let's use our derivative sketching prowess to get a feel for the derivative.

Diagram

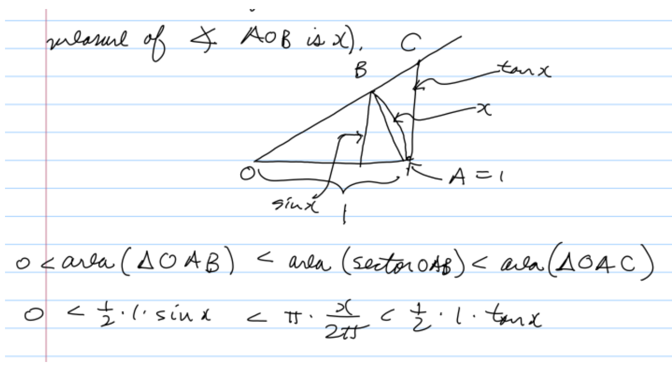
Let $f(x) = \sin x$. We will dive right in and see where the limit definition takes us:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\ &= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \end{aligned}$$

We know that $\lim_{h\to 0} \sin x = \sin x$ and $\lim_{h\to 0} \cos x = \cos x$ since x is a constant with respect to h. So we need to figure out how to compute the two limits

1. $$\lim_{h\to 0} \frac{\sin h}{h}$$
2. $$\lim_{h\to 0} \frac{\cos h - 1}{h}$$

1. We will give a geometric argument that relies on the squeeze theorem.



Computation

And we finally get

$$\boxed{\lim_{h\to 0}\frac{\sin h}{h} = 1}$$

2. This is a little more straightforward. We use a pseudo conjugate and also the limit we just proved in 1.

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)}\\
&= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h + 1)}\\
&= -\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{\sin h}{\cos h + 1}\\
&= -(1)\cdot(0) = 0
\end{aligned}$$

So we have

$$\boxed{\lim_{h\to 0}\frac{\cos h - 1}{h} = 0}$$

Recall we reached the point

$$(\sin x)' = \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

We can now shut the door and write

$$(\sin x)' = \sin x(0) + \cos x(1) = \cos x$$

Hence,

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

3.2 Trig Derivatives

We can prove that $(\cos x)' = -\sin x$ using a similar approach. (If you are feeling very ambitious try to work through it on your own.)

From this all of the other trig derivatives fall out the quotient and product rule. We have the following table of the major trig derivatives that you should know inside and out.

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin xx$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
6. $\frac{d}{dx}(\cot x) = -\csc^2 x$

Example: Let's compute $(\tan x)'$ for practice.

Computation

Example: Let $f(x) = \sin x + \cos x$, find $f^{(45)}(x)$.

Computation

4 Intro to Maxima and Minima

4.1 Intro

Now that we can take the derivatives of a variety of elementary functions let's talk about one of the most powerful applications of derivatives: finding local maxima and minima.

Definition 1. *We say that a* ***local maximum*** *of $f(x)$ occurs at $x = a$ if for all x near a it holds that $f(x) \leq f(a)$.*
We say that a ***local minimum*** *of $f(x)$ occurs at $x = a$ if for all x near a it holds that $f(x) \geq f(a)$.*

Diagram

Notice that the slope of the tangent lines at these points is 0. This is an important fact.

Fact: If f has a local max or min at $x = a$ then $f'(x) = 0$.

! Notice that $f'(x) = 0$ doesn't imply that there is a local max or min. For example $f(x) = x^3$ at $x = 0$.

4.2 Examples

We will get more technical about this when we cover optimization. But for now let's look at some examples of how we can use this.

Example: Below is a graph of $f'(x)$. Where are the local maxima of $f(x)$?

Diagram

Example: Coffee costs $C(x) = 1 + x - \sqrt{x}$ dollars per ounce. What amount of coffee is the cheapest per ounce to purchase?

Computation

#6: Derivatives of Polynomials, the Number e, Product and Quotient Rule, Trig Derivatives
Chapter 3.1,3.2

Example: Using calculus, for $a > 0$, find the minimum of $ax^2 + bx + c$.

Computation

To summarize at this point we can find minima and maxima by

1. Taking the derivative and setting equal to zero.
2. At each point $x = a$ where $f'(a) = 0$ decide if the derivative is positive or negative for $x < a$ and $x > a$.
3. Plot on a numberline.
4. If (reading left to right) the sign switches from - to + then it is a local min and if it switches from + to - then it is a local max. Otherwise it is neither.
5. Plug back into your original function to obtain the value of the max or min.

! If you are asked to do this on the midterm it will be simple. So don't overthink it.