Real Analysis Lectures©

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1 Metric Spaces

Proposition 1 (Lindelof's Property)**.** *A metric space is separable iff there exists a countable family of open sets, $\{O_i\}$. Such that for any open set $O = \bigcup_{j \in J} O_j$.*

Proof. $D = \{x_n\}$. $\overline{D} = X$.

$$\{B(x_i, r_j)\}_{x_i \in D} r_j \in \mathbb{Q}^+$$

□

Definition 1. *Given (X, ρ), $E \subseteq X$ is sequentially compact in E if any sequence in X has a convergent subsequence in E.*

Definition 2. *$E \subseteq X$ is **totally bounded** if for all $\epsilon > 0$ there exist $x_1, \ldots, x_m \in X$ such that $E \subseteq \bigcup B(x_i, \epsilon)$.*

Theorem 1. *Let $E \subseteq X$. The following are equivalent.*

1. *E is complete and totally bounded.*
2. *Bolzano-Weirstrauss Property: E is sequentially compact.*
3. *Heine-Borel: E is compact.*

(1) $\iff$ (2). □

Example: There exists a space in which closed and bounded does not imply compact.

$$l^2 = \{\{x_i\} : \sqrt{\sum (x_i)^2} < \infty\}$$

$$\rho(x, y) = \sqrt{\sum (x_i - y_i)^2}$$

The closed ball is not compact. Will show that $\overline{B}$ is not compact. Consider the sequence

$$e_i = (0, \ldots, 1, 0, \ldots)$$

The sequence $\{e_i\}$ has no convergent subsequence.

Theorem 2 (Cantor Intersection Theorem)**.** *Let C_i be a decreasing family of nonempty compact sets. This implies $\bigcap C_i \neq \emptyset$.*

Definition 3. *A **closed rectangle** is $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$.*
*A **open rectangle** is $R = \prod^d (a_k, b_k)$.*

$$|R| = \prod (b_i - a_i) = \text{ Volume}$$

Definition 4. *A family of rectangles is **almost disjoint** if their interiors are disjoint.*

Lemma 1 (Finite Additivity)**.** *If R is an almost disjoint union. $R = \bigcup^n R_j$ then $|R| = \sum^n |R_j|$*

Lemma 2. *$R \subseteq \bigcup R_j$ implies $|R| \leq \sum |R_j|$*

Theorem 3. *Every open set, $O \subseteq \mathbb{R}$ can be written as a disjoint union of countably many open intervals.*

Proof. Fix $x \in O$ and take the component containing x. Call this I_x. We claim that

$$O = \bigcup_{x \in O} I_x$$

Note that if $I_x \cap I_y \neq \emptyset$ then $I_x = I_y$ by maximality. So we can refine to (by using say a well ordering of $\mathbb{Q}$)

$$O = \bigcup^{\infty} I_{x_n}$$

With the I_{x_n} disjoint. □

Theorem 4. *Every open set $O \in \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.*

Proof. Start with a grid by cubes of side 1, then subdivide them. So at stage n you have a girde with cubes of side 2^{-n} and select those that are in O and tentatively select those which have nonempty intersection with ∂O. Get rid of those that are outside. Continue as $n \to \infty$. Let $\mathcal{Q}$ be the set of cubes which were selected at some stage.

$$\bigcup_{Q \subseteq O} Q \subseteq O$$

It remains to show that $O \subseteq \bigcup_{Q \in O}$. We use the fact that the grid of cubes of side 2^{-k} have diameter $2^{-k}\sqrt{d}$. Hence there exists k such that $2^{-k}\sqrt{d} < r$.

□

HOMEWORK NOTE: (*) the Whitney Decomposition can be made into cubes also.

Definition 5.

$$|O| = \sum_{Q \in \mathcal{Q}} |Q_i|$$

2 Lebesgue Measure

2.1 Exterior Measure

Definition 6. *Define a map* $m_* : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^+ \cup \{\infty\}$

$$m_*(E) = \inf\left\{\sum |Q_i| : E \subseteq \bigcup^{\infty} Q_j, Q_j \text{ are closed cubes}\right\}$$

2.2 Examples

- $m_*(\{x\}) = 0$
- $m_*(Q) = |Q|$ with Q a closed cube.

  Proof. It is clear that $m_*(Q) \leq |Q|$. Consider any cover $Q \subseteq \bigcup Q_j$. Want to show that $|Q| \leq \sum |Q_j|$. Use compactness of Q. For each j there exists S_j an open cube such that $Q_j \subseteq S_j$ and $|S_j| < |Q_j| + \epsilon/2^j$. So we have $Q \subseteq \bigcup S_j$, and we can refine to a finite cover $Q \subseteq \bigcup_{j=1}^N S_j$. Now that it is finite we have from the lemma 2 from last lecture

  $$|Q| \leq \sum^N |S_j| \leq \sum^N (|Q_j| + \epsilon/2^j) < \sum |Q_j| + \epsilon$$

  □

- Q is an open cube implies $m_*(Q) = |Q|$.

  Proof.
  $$m_*(Q) \leq m_*(\overline{Q}) = |\overline{Q}| = |Q|$$
  To show the other way just put a closed cube inside Q that is arbitrarily close to the volume of Q. □

- $m_*(\mathbb{R}^d) = +\infty$
- $\mathcal{C}$ the ternary Cantor set.
- R a rectangle implies that $m_*(R) = (R)$

  Proof. Can prove $|R| \leq m_*(R)$ with a compactness argument like with the cube. To show that $|R| \geq m_*(R)$

  □

Observation 1.

$$\forall E \subseteq \mathbb{R}^d, \forall \epsilon > 0, \quad \exists \{Q_j\}^\infty : \quad m_*(E) \geq \sum^\infty |Q_j| - \epsilon$$

Observation 2 (Monotonicity)**.**

$$E \subseteq E' \Rightarrow m_*(E) \leq m_*(E')$$

Observation 3 (Countable Sub-Additivity)**.**

$$E \subseteq \bigcup E_j \Rightarrow m_*(E) \leq \sum m_*(E_j)$$

Observation 4 (Regularity)**.** *Let O denote an open set.*

$$m_*(E) = \inf\{m_*(O) : O \supset E\}$$

Proof. The fact that $m_*(E) \leq \inf\{m_*(O) : O \supset E\}$ is clear from monotonicity. To prove the other inequality, fix $\epsilon > 0$ and chose a covering of E by Q_j in such a way that

$$M_*(E) + \epsilon/2 \geq \sum |Q_j| = \sum m_*(Q_j)$$

For each cube let S_j be a slightly larger open cube. We know from observation 2 that

$$m_*(O) \leq \sum m_*(S_j) = \sum |S_j| \leq \sum (|Q)j| + \epsilon/2^{j+1}) = \sum |Q_j| + \epsilon/2 \leq m_*(E) + \epsilon/2 + \epsilon/2$$

□

Observation 5 (Metric Measure)**.** *If you have $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$ then*

$$m_*(E) = m_*(E_1) + m_*(E_2)$$

Proof. Obvious that $m_*(E) \leq m_8(E_1) + m_*(E_2)$. Choose $\delta < d(E_1, E_2)$ and choose a covering $E \subseteq \bigcup Q_j$ such that

$$\sum^{\infty} |Q_j| \leq m_*(E) + \epsilon$$

Can subdivide Q_j so that $diam(Q_j) < \delta$. Thus Q_j can intersect only one of E_1 and E_2. Then you can define $J_1 = \{j : Q_j \cap E_1 \neq \emptyset\}$ and $J_2 = \{j : Q_j \cap E_2 \neq \emptyset\}$.

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \leq \sum_{j \in \mathbb{N}} |Q_j| \leq m_*(E) + \epsilon$$

□

Observation 6. *If $E = \bigcup Q_j$, almost disjoint closed cubes. Implies $m_*(E) = \sum m_*(Q_j) = \sum |Q_j|$.*

Proof. ‘ $\leq'$ follows from countable subadditivity. For ‘ $\geq'$ enough to show that for all $N \in \mathbb{N}$ we have

$$m_*(E) \geq \sum^{N} |Q_j|$$

then $N \to \infty$. Fix N and $\epsilon > 0$. Approximate Q_j from inside and use Observation 5. Choose cubes with thickness $\epsilon/2^j$ thickness.

□

2.3 Measurable Sets and Lebesque Measure

Definition 7. *We say that $E \subseteq \mathbb{R}^d$ is measurable if for all $\epsilon > 0$ there exists $O \supset E$ such that $m_*(O \setminus E) < \epsilon$. We call the Lebesgue measure of measurable set E by*

$$m(E) = m_*(E)$$

2.3.1 Properties

1. Open Sets are measurable.

2. $M_*(E) = 0 \Rightarrow E$ is measurable.

   Proof. By Obs 3, for all $\epsilon > 0$ there exists $O \supset E$ such that $M_*(O) < \epsilon$. So we have

   $$m_*(O \setminus E) \leq m_*(O) < \epsilon$$

   □

3. The class of all measurable sets is closed under countable unions, so E_j measurable implies that $\bigcup E_j$ is also measurable. Use the $\epsilon/2^j$ trick.

4. Closed sets are measurable.

   Proof. $\boxed{\text{Step 1}}$ Enough to do this for F closed (compact), $m_*(F) < \infty$, because we can us concentric balls intersected with F and property 3. $\boxed{\text{Step 2}}$ We need a lemma:

   Lemma 3. *If K is compact, F is closed and $K \cap F = \emptyset$, this implies $d(K, F) > 0$.*

   Proof. One proof is in the book, uses open cover of K. Alternate proof is by contradiction. Suppose there exists $x_n \in K : d(x_n, F) \to 0$. Since K is compact there exists a convergent subsequence $\{x_{n_k}\} \to x \in K$. Then by a Δ-inequality argument we get $d(x, F) = 0$ and since F is closed this implies that $x \in F$, a contradiction.

   □

   Assume that $m_*(F) < \infty$, then find o open containing F. with $m_*(O) < M_*(F) + \epsilon$ by observation 3. So we have

   $$O \setminus F, \text{ is open}$$

   This implies there exists Q_j closed, almost disjoint cubes such that

   $$O \setminus F = \bigcup Q_j$$

   By countable subadditivity we have

   $$m_*(O \setminus F) \leq \sum m_*(Q_j)$$

   We will show that

   $$\sum_{j=1}^{n} m_*(Q_j) \leq \epsilon, \quad \forall N$$

   Then we will take $N \to \infty$. Fix N, let $K_i = \bigcup_{j=1}^{N} Q_j$, is compact. We have $K \bigcup F = \emptyset$ and so by the lemma $d(K, F) > 0$, hence by Observation 5 we have $m_*(F \cup K) = m_*(F) + m_*(k)$. Next note that $K \cup F \subseteq O$.

   $$m_*(O) \geq m_*(K \cup F) = m_*(F) + m_*(K) = m_*(F) + \sum^{N} m_*(Q_j)$$

By Observation 6 we have

$$\sum^{N} m_*(Q_j) \leq m_*(O) - m_*(F) < \epsilon$$

Thus for all N this holds and we have

$$\sum^{\infty} m_*(Q_j) \leq \epsilon$$

□

5. If E is measurable then E^C is measurable.

   Proof. For every n there exists $O_n \supset E$ such that $m_*(O \setminus E) < 1/n$. Since O_n^C is closed, it is measurable. Hence

   $$S = \bigcup^{\infty} O_n^C$$

   is also measurable. So we know that $S \subseteq E^C$. Note that

   $$S^C = \bigcap^{\infty} O_n$$

   We then have

   $$E^C \setminus S \subseteq O_n \setminus E$$
   $$E^C = S \cup (E^C \setminus S)$$

   We just showed that S is measurable, and we know that $E^C \setminus S$ has measure 0. □

6. A countable intersection of measurable sets is measurable.

   Proof.

   $$\left(\bigcap E_j\right)^C = \bigcup E_j^C$$

□

Theorem 5 (Countable Additivity of Lebesgue Measure)**.** *Let $\{E_j\}$ be disjoint measurable. This implies that*

$$m\left(\bigcup E_j\right) = \sum m(E_j)$$

Proof. We know ' $\leq$' by subadditivity. To show the other inequality first assume that all E_j are bounded. First claim for every j thre exists closed $F_j \subseteq E_j$ such that $m_*(E_j \setminus F_j) \leq \epsilon/2$. We can do this by using complements. Since F_j is compact and we know that $F_1, \ldots, F_N$ are disjoint. This implies $m\left(\bigcup^N F_j\right) = \sum^N m(F_j)$ by the Lemma and Observation 5.

$$\bigcup^{N} F_j \subseteq E$$

$$\begin{aligned} m(E) &\geq \sum^{N} m(F_j) \\ &\geq \sum^{N} m(E_j) - \epsilon \end{aligned}$$

□

Definition 8. *A* ***σ-algebra*** *of subsets of X is a collection of subsets, containing $\emptyset$, X, which is closed under countable unions and complements. It is endowed with a measure*

$$\mu : \mathcal{M} \subseteq 2^X \to [0, \infty]$$

is a measure if $\mu(\emptyset) = 0$ and is countably additive on disjoint sets.

We proved that the class of all measurable sets in $\mathbb{R}^d$ is a σ-algebra, with m the measure.

Definition 9. *:*

- *$E_k \nearrow E$ means that $E_k \subseteq E_{k+1}$ and $E = \bigcup_k = 1^\infty E_k$.*

- *$E_k \searrow E$ means that $E_k \supset E_{k+1}$ for all k and $E = \bigcap E_k$.*

Corollary 1. *:*

1. *If $E_k \nearrow E$ and the E_k are measurable then E is measurable and $m(E) = \lim m(E_k)$*
2. *Same for $E_k \searrow E$. Extra hypothesis that the $m(E_k) < \infty$.*

Theorem 6. *Let E be measurable and $\epsilon > 0$.*

1. *There exists $O \subseteq E$: $m(O \setminus E) < \epsilon$.*
2. *There exists $F \subseteq E$ with F closed such that $m(E \setminus F) < \epsilon$.*
3. *Assuming $m(E) < \infty$, there exists compact $K \subseteq E$ with $m(E \setminus K) < \epsilon$.*
4. *Suppose that $m(E) < \infty$, then there exists a finite union of closed cubes $F = \bigcup^N Q_j$ such that $m(E \triangle F) < \epsilon$. (note that the symmetric difference $E \triangle F = (E \setminus F) \cup (F \setminus E)$)*

Proof. :

1. By definition.
2. Since E^C is measurable take $O \subseteq E^C$ such that $m(O \setminus E^C) < \epsilon$. Let $F = O^C \subseteq E$. We have $m(E \setminus F) < \epsilon$.
3. If E is bounded use (2). Write $E = \bigcup^\infty (E \cap \mathbb{B}_k)$. This implies that $m(E) = \lim m(E \cap \mathbb{B}_k) < \infty$. So there exists k so that $m(E \cap \mathbb{B}_k) > m(E) - \epsilon/2$. By (2) there exists closed and bounded $F \subseteq E \cap \mathbb{B}_k \subseteq E$ with
$$m(F) > m(E \cap \mathbb{B}_k) = \epsilon/2 > m(E) - \epsilon$$
Hence $F = K$ is compact.
4. By (i) there exist) $\subseteq E$ such that $m(O \setminus E) < \epsilon/2$. Let $O = \bigcup^\infty Q_j$ of almost disjoint closed cubes. Let $F_k = \bigcup^k Q_j$ we have $G_k \nearrow O$. So $m(G_k) \to m(0)$. Thus, there exists N such that $m(G_N) > m(O) - \epsilon/2$. We then have
$$m(E \triangle F) = m(E \setminus F) + m(F \setminus E)$$
with $F = G_N$, $E \setminus G_N \subseteq) \setminus G_N$.
$$m(E \setminus G_N) \leq m(O \setminus G_N) = m(O) - m(G_N) < \epsilon/2$$
Also,
$$m(G_N \setminus E) \leq m(O \setminus E) < \epsilon/2$$

□

3 General Measures

3.1 σ-algebras

There are the two trivial σ-algebras. Also we showed that measurable sets form a σ-algebra.

Definition 10. *The **Borel σ-algebra** denoted by $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ algebra containing all open sets.*

$$F_\sigma = \{ \text{ countable unions of closed sets } \}$$
$$G_\delta = \{ \text{ countable intersection of open sets} \}$$
$$F_{\sigma\delta} = \{ \text{countable intersections of } F_\sigma \text{ sets} \}$$

3.1.1 Remarks

- Every closed set is G_δ.
- For every measurable set there exists E' such that there is G_δ such that $E' \supset E$, $m(E' \backslash E) = 0$. And, there exists $E'' \subseteq E$, E'' is F_δ such that $m9E \setminus E'') = 0$.

Definition 11. *X is any set $2^X \supset \mathcal{M}$ is a σ-algebra if closed under countable unions, complements, $\emptyset \in \mathcal{M}$.*

Definition 12. *A measure on $(X, \mathcal{M})$ is $\mu : \mathcal{M} \to [0, +\infty]$ such that $\mu(\emptyset) =)$ and countably additive on disjoint sets in $\mathcal{M}$.*

3.1.2 Examples

- Can take $\mathcal{M} =$ the Lebesgue Measurable Sets.
- Can take $\mathcal{M}$ to be the Borel sets.
- A discrete measure on a set $\{z_n\}^\infty \subseteq X$, $\mathcal{M} = 2^X$ would be

$$\{\mu_n\}^\infty \subseteq [0, \infty]$$
$$\mu(E) = \sum \mu_n$$

- Special case is $\mu_n = 1$ for all n. Called the **counting measure**.

3.2 General Exterior Measure

Definition 13. *The function $\mu_* : 2^X \to [0, +\infty]$ is an exterior measure if it satisfies*

1. $\mu_*(\emptyset) = 0$
2. *Monotonicity*
3. *Countable Sub Additivity*

Definition 14. *Given μ_*, a set E is called **measurable** if for every $A \subseteq X$,*

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^C)$$

Theorem 7. *$\mathcal{M} = \{E$ that are measurable$\}$ is a σ-algebra and $\mu_*\big|_{\mathcal{M}}$ is a measure.*

Proof. Clearly $\emptyset \in \mathcal{M}$. And if $E \in \mathcal{M}$ then $E^C \in \mathcal{M}$ by symmetry. Next we show that $\mathcal{M}$ is closed under finite unions. So, let $E_1, E_2 \in \mathcal{M}$, we want to show that $E_1 \cup E_2 \in \mathcal{M}$. We know that

$$\mu_*(A) = \mu_*(A \cap E_1) + \mu_*(A \cap E^C)$$

$$\mu_*(A \cap E_1) = \mu_*(A \cap E_1 \cap E_2) + \mu_*(A \cap E_1^{\cap} E_2^C)$$
$$\mu_*(A \cap E_1^C) = \mu_*(A \cap E_1 C \cap E_2) + \mu_*(A \cap E_1^C \cap E_2^C)$$

Putting these together we obtain

$$\mu_*(A) = \mu_*(A \cap E_1 \cap E_2) + \mu_*(A \cap E_1^{\cap} E_2^C) + \mu_*(A \cap E_1 C \cap E_2) + \mu_*(A \cap E_1^C \cap E_2^C)$$

Applying subadditivity we obtain

$$\mu_*(A) \geq \mu_*(A \cap (E_1 \cup E_2)) + \mu_*(A \cap (E_1 \cup E_2)^C)$$

Note that

$$E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$$

We now need to show that it is closed under countable unions. It suffices to show that $\mathcal{M}$ is closed under unions of disjoint sets (exercise), and μ_* is countably additive on those. Suppose we have a sequence

$$\{E_j\} \subseteq \mathcal{M}, \qquad E_j \cap E_k = \emptyset$$

Let $G_n = \bigcup^n E_j$ and let $G = \bigcup^\infty E_j$. We know that $\mu_*(G_n) = \sum^n \mu_*(E_j)$. For all $A \subseteq X, E_n \in \mathcal{M}$. Observe that

$$\mu_*(G_n \cap A) = \mu_*(E_n \cap (G_n \cap A)) + \mu_*(E_n^C \cap (G_n \cap A))$$

By induction we obtain

$$\mu_*(G_n \cap A) = \sum^n \mu_*(E_j \cap A0$$

We wrap it up by observing

$$\mu_*(A) \geq \sum^\infty \mu_*(A \cap E_j) + \mu_*(A \cap G^C)$$

□

3.2.1 Remarks

- Only '$\geq$' needed to be proved in (1).
- If $\mu_*(E) = 0$, then E is measurable.

Definition 15. *$(X, \mathcal{M}, \mu)$ is **complete** if for all $E \in \mathcal{M}$, $\mu(E) = 0$ implies that for all $F \subseteq E$, $F \in M$.*

3.3 Metric Measures Preview

Let (X, d) be a metric spa e. μ_* on X is a metric exterior mesure if

$$\forall A, B \subseteq X, \quad d(A, B) > 0 \Rightarrow \mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$$

Theorem 8. *If this holds, then $\mathcal{M}$ contains Borel Sets.*

3.4 Metric Outer Measures

Definition 16. *(X, d) is a metric space. We have μ_* is a **metric** exterio measure if*

$$d(A, B) > 0 \Rightarrow \mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$$

Theorem 9. *If μ_* is a metric exterior measure then all Borel sets are measurable.*

Proof. By definition we need to show that for every $A \subseteq X$ and Borel set E we have $\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \setminus E)$. It is enough to show that $\mu_*(A) \geq \mu_*(A \cap F) + \mu_*(A \setminus F)$ for closed sets F. Can assume that $\mu_*(A) < \infty$.

Let $B_n = \{x \in A \setminus F : d(x, F) \geq \frac{1}{n}\}$. Notice that $B_n \nearrow$, ie $B_n \subseteq B_{n+1}$ and thus since F is closed

$$\bigcup B_n = A \setminus F$$

By definition we have $d(B_n, F) \geq \frac{1}{n}$, therefore using monotonicity and the fact we have a metric measure

$$\mu_*(A) \geq \mu_*((A \cap F) \cup B_n) = \mu_*(A \cap F) + \mu_*(B_n)$$

Thus it is enough to show that $\lim \mu_*(B_n) = \mu_*(A \setminus F)$. Let

$$C_n := B_{n+1} \setminus B_n$$

We will show that

$$d(C_{n+1}, B_n) \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

Take $x \in C_{n+1}$ and $y \in B_n$. We know that $x \notin B_{n+1}$ which implies that $d(x, F)$ is less than $\frac{1}{n+1}$. And since $y \in B_n$ we know that $d(y, F) \geq \frac{1}{n}$. Thus

$$d(y, F) \leq d(y, x) + d(x, F)$$

Therefore

$$d(y, x) \geq d(y, F) - f(x, F) \leq \frac{1}{n(n+1)}$$

Next, observe that

$$\mu_*(B_{2k+1} \geq \mu_*(C_{2k} \cup B_{2k-1}) = \mu_*(C_{2k}) + \mu_*(B_{2k-1}) \geq \cdots$$

Where the equality above follows from the last claim and the metric exterior measure property. So, by induction we have

$$\mu_*(B_{2k+1}) \geq \sum_{j=1}^{k} \mu_*(C_{2j})$$

And similarly,

$$\mu_*(B_{2k}) \geq \sum_{j=1}^{k} \mu_*(C_{2j-1})$$

Since all of these are contained in$A$ we have

$$\sum^{\infty} \mu_*(C_j) < \infty$$

We are almost done. Next we have from subadditivity that

$$\mu_*(A \setminus F) \leq \mu_*(B_n) + \sum_{j=n}^{\infty} \mu_*(C_j)$$

But since the sum converges we then have

$$\mu_*(A \setminus F) \leq \lim_{n\to\infty} \mu_*(B_n) \leq \mu_*(A \setminus F)$$

Hence

$$\lim \mu_*(B_n) = \mu_*(A \setminus F)$$

□

Proposition 2 (Regularity of Borel Measures that are Finite on Balls)**.** *Suppose μ is a measure on a metric space such that $\mu(\mathbb{B}_r(x)) < \infty$ for all $r < \infty$. Then for every Borel set E and for every $\epsilon > 0$ there exists an open $\mathcal{O}$ and closed F such that $F \subseteq E \subseteq \mathcal{O}$ and*

$$\mu(\mathcal{O} \setminus E) < \epsilon, \quad \mu(E \setminus F) < \epsilon$$

Observation 7. *Suppose $F^* = \bigcup F_k$ with the F_k closed. Then for every $\epsilon > 0$ there exist $F \subseteq F^*$ such that $\mu(F^* \setminus F) < \epsilon$.*

Proof. Since $F_k \nearrow F^*$ we have $\mu(F_k) \to \mu(F^*)$. If $\mu(F^*)$ is finite we are done. Otherwise, let $B_n = \mathbb{B}_n(x_0)$, consider

$$F_k \cap (\overline{B}_n \setminus B_{n+1}) \underset{k\to\infty}{\nearrow} F^* \cap (\overline{B}_n \cap B_{n+1})$$

Then do the $\epsilon/2^n$ trick.

□

Observation 8. *Suppose $F^* = \bigcup F_k$ with the F_k closed. Then for every $\epsilon > 0$ there exist $F \subseteq F^*$ such that $\mu(F^* \setminus F) < \epsilon$.*

Proof. Since $F_k \nearrow F^*$ we have $\mu(F_k) \to \mu(F^*)$. If $\mu(F^*)$ is finite we are done. Otherwise, let $B_n = \mathbb{B}_n(x_0)$, consider

$$F_k \cap (\overline{B}_n \setminus B_{n+1}) \underset{k\to\infty}{\nearrow} F^* \cap (\overline{B}_n \cap B_{n+1})$$

So there exists N such that$\mu(F_k \cap (\overline{B}_n \setminus B_{n+1})) > F^* \cap (\overline{B}_n \cap B_{n+1}) - \frac{\epsilon}{2^n}$ Then do the $\epsilon/2^n$ trick.

□

Proposition 3 (Regularity of Borel Measures that are Finite on Balls)**.** *Suppose μ is a measure on a metric space such that $\mu(\mathbb{B}_r(x)) < \infty$ for all $r < \infty$. Then for every Borel set E and for every $\epsilon > 0$ there exists an open $\mathcal{O}$ and closed F such that $F \subseteq E \subseteq \mathcal{O}$ and*

$$\mu(\mathcal{O} \setminus E) < \epsilon, \quad \mu(E \setminus F) < \epsilon$$

Proof. Let $C = \{E \subseteq X :$ the conclusion holds for $E\}$. We want to show that C contains all Borel sets. First we show that C is a σ-algebra. First we clearly have $\emptyset \in \mathbb{C}$, $E \in C \Rightarrow E^C \in C$. How about countable unions?

Let $E_j \in C$. So there exists $F_j \subseteq E_j \subseteq O_j$ with F_j closed and O_j open and

$$\mu(O_j \setminus E_j) < \epsilon/2^j$$

$$\mu E_j \setminus F_j) < \epsilon/2^j$$

Then

$$E := \bigcup E_j \subseteq \bigcup O_j =: O$$

$$\mu(O \setminus E) < \epsilon$$

Let $F^* = \bigcup F_j$. By the Observation we know that there exists $F \subseteq F^*$ such that $\mu(F^* \setminus F) < \epsilon$. Thus

$$\mu(E \setminus F) \leq \mu(E \setminus F^*) + \mu(F^* \setminus F) < 2\epsilon$$

Observe that

$$F_k := \{x \in O \mid d(x, O^C) \geq \frac{1}{k}\}$$

So we have

$$\bigcup F_k = O$$

Which is D_δ and thus $O \in C$.

□

3.4.1 Recall

Definition 17. *μ_* is a measure if*

1. $\mu_*(\emptyset) = 0$
2. *monotonicity*
3. *countable sub-additivity*

Definition 18.

$$\mathcal{M} = \{E \subseteq X \mid \forall A \subseteq X, \mu_*(A) = \mu_*(A \cap E) + \mu_*(A \setminus E)$$

Definition 19. *A **premeasure** μ_0 on an algebra $\mathcal{A} \subseteq 2^X$. ($\mathcal{A}$ is closed under complements and* finite *unions and intersections). Is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ satisfying*

1. $\mu_0(\emptyset) = 0$
2. *If $E = \bigcup_{j=1}^{\infty} E_j$, $E_j \in \mathcal{A}$ and $E \in \mathcal{A}$ then it holds that*

$$\mu_0(E) = \sum_{j=1}^{\infty} \mu_0(E_j)$$

Lemma 4. *Given a premeasure μ_0 on an algebra $\mathcal{A} \subseteq 2^X$, define the **exterior measure***

$$\mu_*(E) = \inf \left\{ \sum \mu_0 E_j \right\}, \qquad E_j \in \mathcal{A}, \quad E \subseteq \bigcup^{\infty} E_j$$

It holds that

1. *μ_8 is an exterior measure on X*
2. *$\mu_*(E) = \mu_0(E)$ for all $E \in \mathcal{A}$*
3. *μ is equal to the $\{$Carathéothdory- measurable sets for $\mu_*\}$, $\mu := \mu_*|_{\mathcal{M}}$ then*

$$\mu|_{\mathcal{A}} = \mu_0$$

Proof. First observe that (1) holds since $\mu_* : 2^X \to [0, \infty]$ thus it is well defined.

To prove the rest of (1) we have from the definition of a premeasure that $\mu_*(\emptyset) = 0$. Also, the definition of μ_* implies that monotonicity holds. Countable additivity uses a $\epsilon/2^j$ trick. We need to verigy

$$\mu_*\left(\bigcup E_j\right) \leq \sum \mu_*(E_j)$$

By definition of μ_* for every j we can find sets $A_{j,k}$ such that $A_{j,k} \in \mathcal{A}$ and $E_j \subseteq \bigcup_{k=1}^{\infty} A_{j,k}$. Using the definition of inf we can

$$\mu_*(E_j) > \sum \mu_0(A_{j,k} - \epsilon/2^j$$

Take

$$E := \bigcup E_j \subseteq \bigcup A_{j,k}$$

We then have

$$\mu_*(E) \leq \sum_{j,k} \mu_0(A_{j,k}) = \sum_{j=1} \left(\sum_{k=1}^{\infty} \mu_0(A_{j,k}) \right) < \sum_j \mu_*(E_j) + \epsilon$$

□

Theorem 10. *Suppose that $\mathcal{A}$ is an algebra of sets in X and μ_0 is a premeasure on $\mathcal{A}$. Let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then*

1. *There exists μ a measure on $\mathcal{M}$ which extends μ_0.*
2. *If μ_0 is σ-finite. (ie there exist countably many sets $\{E_j\}$ such that $E_j \in \mathcal{A}$ and $\mu_0(E_j) < \infty$ and $\bigcup E_j = X$. Then μ is the unique such extension. (See page 272)*

Proof. (1) We have μ_0 on $\mathcal{A}$. So by the lemma we have μ_* on S^X. But by Carth. Thm (1.1) we have $\mathcal{M}_*, \mu = \mu_*|_{\mathcal{M}_*}$. Observe that $\mathcal{M}_* \supset \mathcal{M}$ since $\mathcal{M}_* \supset \mathcal{A}$ by the lemma.

Remark: Usually $\mathcal{M}_* \neq \mathcal{M}$.

(2) To see uniqueness suppose ν is another measure on $\mathcal{M}$,

$$\nu|_{\mathcal{A}} = \mu_0 = \mu|_{\mathcal{A}}$$

Suppose $F \in \mathcal{M}$, $\mu(F) < \infty$. Claim that $\nu(F) = \mu(F)$. We already know that $\mu(F) = \mu_*(F)$. It then follows that

$$\nu(F) \leq \sum_{j=1}^{\infty} \nu(E_j) = \sum \mu_0(E_j)$$

Hence $\nu(F) \leq \mu(F)$.

Next up we observe that if $E = \bigcup^\infty E_j$ with $E_i \in \mathcal{A}$ then we can denote by A_σ a countable union of sets in $\mathcal{A}$. We have

$$\nu(E) =_{n\to\infty} \nu\left(\bigcup_{j=1}^n E_j\right) = \lim_{n\to\infty} \mu\left(\bigcup^n E_j\right) = \mu(E)$$

We know that $\nu - \mu$ on $\mathcal{A}_\sigma$ sets. Again, let $F \in \mathcal{M}$ and $\mu(F) < \infty$ Fix $\epsilon > 0$. We claim that there exists $E_j \in \mathcal{A}$ such that

$$F \subseteq \bigcup E_j = E$$

and

$$\mu(F) > \mu(E) - \epsilon$$

This is true because we know that

$$\mu(F) = \mu_*(F)$$

by definition of μ_* we have there exist $E_j \in \mathcal{A}$ with $F \subseteq \bigcup E_j$. Thus

$$\mu_*(F) > \sum \mu_0(E_j) - \epsilon$$

We know that

$$\mu(E) \leq \sum \mu(E_j) = \sum \mu_0(E_j)$$

So we have (because $\mu(F)$ is finite)

$$\mu(E \setminus F) = \mu(E) - \mu(F) < \epsilon$$

We can then write (since E is $\mathcal{A}_\sigma$)

$$\mu(F) \leq \mu(E) = \nu(E) = \nu(F) + \nu(E \setminus F) \leq \nu(F) + \mu(E \setminus F) < \nu(F) + \epsilon$$

Thus,

$$\mu(F) \leq \nu(F)$$

So we have proven it for $F \in \mathcal{M}$ with $\mu(F) < \infty$. Lastly, for any $F \in \mathcal{M}$ we have $F = \bigcup(F \cap E_j)$ (assuming WLOG that the E_j are disjoint) hence

$$\mu(F) = \sum \mu(F \cap E_j) = \sum \nu(F \cap E_j) = \nu(F)$$

□

3.4.2 Example

Let $X = \mathbb{Q}$. Suppose that $\mathcal{A}$ is all finite unions of $(a, b] \cap \mathbb{Q}$ with $a, b \in \mathbb{R}$. This is an algebra. Let

$$\mu_0(\emptyset) = 0$$

$$\mu_0(A) = \infty, \qquad A \neq \emptyset$$

Observe that the σ algebra generated by $\mathcal{A}$ is $2^{\mathbb{Q}}$. We could take the trivial extension. But there is more than one extension, the counting measure on $\mathbb{Q}$.

3.4.3 Example

1. Lebesgue-Stieltjes measures (6.3.3): Let $X = \mathbb{R}$ and $\mathcal{A}$ be finite unions on $(a, b]$. Take $F : \mathbb{R} \to \mathbb{R}$ with F increasing and right continuous ($F(x) = \lim_{t \to 0^+} F(x + t)$). take

$$\mu_0^F(a, b] = F(b) - F(a)$$

   If you let $F(x) = x$ then we get Lebesgue Measure.

2. Product measures

3.4.4 Remark

Let $\mu_0 \to \mu_* \to \mathcal{M}_*$ with $\overline{\mu} = \mu_*|_{\mathcal{M}_*}$. We have μ is an extension of $|mu_0$ from the Thm. Then (a) if μ is σ-finite then $\overline{\mu}$ is the completion of μ.

4 Developing the Integral

4.1 Measurable Functions

Definition 20. *$(X, \mathcal{M}, \mu)$ is a measure space and let $f : X \to \mathbb{R}$. We say that f is **measurable** if for all $a \in \mathbb{R}$ we have*

$$f^{-1}([-\infty, a)) = \{f < a\} = \{x \in X : f(x) < a\} \in \mathcal{M}$$

Definition 21. *f is **finite valued** if $f(x) \subseteq \mathbb{R}$.*

Lemma 5.

1. *f is measurable $\iff$ $\{f \leq a\} \in \mathcal{M}$*
2. *f is measurable $\iff$ $\{f \geq a\} \in \mathcal{M}$ $\iff$ $\{f > a\} \in \mathcal{M}$*
3. *If f is finite valued then f is measurable iff for all $a, b \in \mathbb{R}$ with $a < b$ we have*

$$f^{-1}((a, b)) \in \mathcal{M} \iff f^{-1}([a, b]) \in \mathcal{M}$$

Proof.

$$\{f \leq a\} = \bigcap \{f < a + \frac{1}{k}\}$$

Others are similar.

□

Corollary 2.

$$f \text{ is measurable} \iff -f \text{ is measurable}$$

Observation 9 (Observation 1)**.** *If f is finite valued then f is measurable $\iff$ $f^{-1}(\mathcal{O}) \in \mathcal{M}$ for all $\mathcal{O} \subseteq \mathbb{R}$.*

Observation 10 (Observation 2)**.** *:*

1. *Suppose X is a metric space and $\mathcal{M} \supset \mathcal{B}_X$ then $f : X \to \overline{\mathbb{R}}$ is continuous implies that f is measurable.*
2. *If f is measurable and finite valued and $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous then $\Phi \circ f$ is measurable.*

Proof. (1) Follows from previous observation and definition of continuity. □

Observation 11 (Observation 3)**.** *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on $(X, \mathcal{M})$ then*

$$\sup_n f_n(x), \inf_n f_n(x), \limsup f_n(x), \liminf f_n(x)$$

are all measurable.

Observation 12 (Observation 4)**.** *If $f_n(x) \to f(x)$ for all $x \in X$, (pointwise convergene). The f_n are measurable implies that f is measurable.*

Observation 13 (Observation 5)**.** *f, g are measurable implies that*

1. *for all $k \in \mathbb{Z}^+$, f^k is measurable.*
2. *$f + g$ is measurable if both are finite valued.*
3. *fg is measurable if both are finite valued.*

Definition 22. *A property holds **almost everywhere (μ-a.e.)** if it holds outside a set of measure 0.*

Observation 14 (Observation 6)**.** *if f is measurable and $f = g$ almost everywhere then g is measurable.*

4.2 Approximation of Functions by Simple Functions

Definition 23. *Let $E \subseteq \mathcal{M}$ and χ_E be a characteristic function (all characteristic functions are measurable). A **simple function** is of the form*

$$f(x) = \sum^{N} a_k \chi_{E_k}, \qquad E_k \in \mathcal{M}, \mu(E_k) < \infty$$

A step function in $(\mathbb{R}^d, \mathcal{M}, m)$ is of the form

$$f(x) = \sum a_k \chi_{R_k}, \qquad R_k - \text{ Rectangles}$$

Theorem 11 (Theorem 1)**.** *$(X, \mathcal{M}, \mu)$ is σ-finite.*

1. *$f : X \to [0, \infty]$ is measurable then there exist $\{\varphi_k\}$ simple functions such that $\varphi_k \nearrow f(x)$ for all x and $\varphi_k(x) \leq \varphi_{k+1}(x)$.*
2. *$f : X \to \overline{\mathbb{R}}$ is measurable then there exist $\{\varphi_k\}$ such that $|\varphi_k| \leq |\varphi_{k+1}|$ for all k and*

$$\varphi_k(x) \to f(x), \qquad \forall x$$

Proof. (1) is complicated see folland. To prove (2) first write $f(x) = f^+(x) - f^-(x)$, with $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. By (1) we can find a sequence $\varphi_k^+ \to f^+$ and $\varphi^- \to f^-$ (pointwise). If we take $\varphi_k = \varphi^+ - \varphi^- \to f$ for all x. Notice that $|\varphi_k(x)| = \varphi^+(x) + \varphi^-(x)$ since both are increasing to $f^+ + f^- = |f(x)|$ and thus $|\varphi_k| < |\varphi_{k+1}|$.

□

4.2.1 Remark

In (1) and (2) we can choose the φ_k to converge uniformly on every set where f is bounded.

Lemma 6. *For all simple functions φ on $\mathbb{R}^d$ and for all $\epsilon > 0$ there exists ψ a step function such that*

$$m(\{x : \psi(x) \neq \varphi(x)\}) < \epsilon$$

Proof. It is enough to take $\varphi = \psi_E$ with $m(E) < \infty$. Approximate E be an open set such that

$$m(\mathcal{O} \setminus E) < \epsilon$$

Write

$$\mathcal{O} = \bigcup_{j}^{\infty} Q_j$$

with the Q_j almost disjoint closed cubes. We know that $m(\mathcal{O}) = \sum m(Q_j)$ so there exists N such that

$$m\left(\sum_{N+1}^{\infty} m(Q_j)\right) < \epsilon$$

take

$$\psi(x) = \sum_{j}^{N} \chi_{Q_j}$$

It is clear that

$$\psi(x) = \chi_E(x), \qquad \forall x \in \left(\bigcup^N Q_j^\circ \cap E\right) \cup \mathcal{O}$$

So it is enough to show that

$$\{x : \psi(x) \neq \varphi(x)\} \subseteq E \triangle \bigcup^N Q_j^\circ$$

has measure less than 2ϵ. i.e. we want to show that

$$m(E \setminus \bigcup_j^N Q_j^\circ) \leq m(\mathcal{O} \setminus \bigcup_j^N Q_j^\circ)$$

$$m(\bigcup_j^N (Q_j \setminus E)) \leq m(\mathcal{O} \setminus E) < \epsilon$$

Bute we have

$$m(\mathcal{O} \setminus \bigcup_j^N Q_j^\circ 0 = m(\mathcal{O}) \setminus \bigcup^N m(Q_j^\circ) < \epsilon$$

□

Theorem 12. *In $\mathbb{R}^d$ we have for all $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ there exist ψ_k step functions such that $\psi_k \to f$ almost everywhere.*

Proof. By (1) there exist $\varphi_k(x) \to f(x)$ for all $x \in \mathbb{R}^d$. By the lemma we can find $\psi_k(x)$ such that

$$m(\underbrace{\{x : \psi_k(x) \neq \varphi_k(x)\}}_{E_k}) < \frac{1}{2^k}$$

Standard trick: is to take $\bigcap_{n=1}^\infty \bigcup_{k=n+1}^\infty E_k$. So we know that F is measurable and

$$m(F) \leq m(\bigcup_{k=n+1}^\infty E_k) \leq \sum m(E_k) = \sum_{k=n+1}^\infty \frac{1}{2^k} = \frac{1}{2^n}$$

We claim that $\psi_k(x) \to f(x)$ for all $x \in F^c$. Notice that

$$x \in \left(\bigcap \bigcup E_k\right)^c \implies x \in \bigcup \bigcap E_k^c$$

So, there exist n such that if

$$x \in \bigcap_{k=n+1}^\infty E_k^c \implies x \in E_k^c, \quad \forall k \geq n+1$$

So, we have $\psi_k(x) = \varphi_k(x)$ with $k \geq n+1$ this implies that $\psi_k(x) \to f(x)$.

□

Theorem 13 (Egorov's Theorem)**.** *A sequence of functions which converges a.e. converges nearly uniformly. More precisely,*

Given $(X, \mu, \mathcal{M})$, $\mu(X) < \infty$, f_n measurable and $f_n \to f$ a.e. This implies that for all $\epsilon > 0$ there exist A_ϵ such that $\mu(A_\epsilon) < \epsilon$ $f_n \to f$ uniformly on $X \setminus A_\epsilon$

Proof. WLOG assume that $f_n \to F$ on X. For $k, n \in \mathbb{N}$ define the set

$$E_n(k) := \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq \frac{1}{k}\}$$

Notice that $E_n(k) \searrow$ and so the set

$$\bigcap_{n=1}^{\infty} E_n(k)$$

is empty by our construction. Choose n_k so that

$$\mu(E_{n_k}(k) < \frac{\epsilon}{2^k}$$

Let

$$E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$$

we know that

$$\mu(E) \leq \epsilon$$

Claim that $f_n \to f$ uniformly on $X \setminus E$. Note that if $x \notin E$ this implies

$$x \in (E_{n_k}(k))^c, \quad \forall k$$

and thus

$$|f_n(x) - f(x)| < \frac{1}{k}, \qquad \forall n \geq n_k$$

□

Theorem 14 (Luzin)**.** *Let $f : E \to R$ with $E \subseteq \mathbb{R}^d, m(E) < \infty$ then for all $\epsilon > 0$ there exists compact set $F_\epsilon \subseteq E$ such that $m(F_\epsilon) \leq \epsilon$ and $f\Big|_{E \setminus F_\epsilon}$ is continuous.*

Remark We don't calim that f is continuous on F_ϵ^c, jus tthat $f|_{F_\epsilon}^c$ is continuous. For example,

$$\chi_{\mathbb{Q}} \text{ is discontinuous for all } x \in \mathbb{R}$$

$$\chi_{\mathbb{Q}}\Big|_{\mathbb{Q}^c} \equiv 0$$

Proof. By the step function approximation theorem we know that there exists $\{\psi_n\}$ step functions on $\mathbb{R}^d$. Note that

$$\psi(x) = \sum_{k=1}^{N} a_k^n \chi_{Q_k}$$

for some cube Q_k.

$$\psi_n \to f, \qquad \text{almost everywhere}$$

Using Egorov's Theorem we find $A_{\epsilon/2}$ such that

$$\psi_n \to f$$

uniformly on $A^c_{\epsilon/2}$ and $m(A_{\epsilon/2}) < \epsilon/2$. Observe that $\psi(x)$ is continuous on

$$\left(\bigcup_1^N (\partial Q_k^n)\right)^c$$

since it has full lebesgue measure. Let E_n be the set of discontinuities of ψ_n and $E_n = \bigcup \partial Q_k^n$ we have

$$m(E_n) = 0$$

and so

$$F' = A_{\epsilon/2} \setminus \bigcup_1^\infty E_n$$

Then

$$m(F') = m(A_{\epsilon/2}) < \epsilon/2$$

and the $\psi_n|_{(F')^c}$ are continuous for all n and thus converge to $f|_{(F')^c}$ uniformly. We still need a *compact* set. So we use inner regularity and take a compact $F \subseteq E \setminus F'$ such that $m(F) > m(E \setminus F') - \frac{\epsilon}{2}$. We are now done. □

There is an alternate proof.

Proof. Pick $\{V_n\}$ a countable basis for open sets in $\mathbb{R}$. For example

$$\{(r, s) : r, s \in \mathbb{Q}\}$$

By inner regularity there exists compact $K_n \subseteq f^{-1}(V_n)$ such that $m(f^{-1}(V_n) \setminus K_n) < \frac{\epsilon}{2^{n+1}}$. So there exists compact

$$K'_n \subseteq E \setminus f^{-1}(V_n) \text{ such that } m(E \setminus f^{-1}(V_n) \setminus K'_n) < \frac{\epsilon}{2^{n+1}}$$

We then have

$$m(E \setminus (K_n \cup K'_n)) < \epsilon/2^n$$

From our construction the following set is compact

$$K := \bigcap^\infty (K_n \cup K'_n)$$

additionally, from demorgan's and subadditivity we have

$$m(E \setminus K) < \epsilon$$

Claim that $f|_K$ is continuous. It is enough to show that

$$(f|_K)^{-1}(V_n) \text{ is relatively open in } K$$

Pick $x \in K$, $f(x) \in V_n$ let $\mathcal{O} = K_n^c$.

$$f(\mathcal{O} \cap K) \subseteq (K_n^c \cap (K_n \cup K'_n)) = f(K_n) \subseteq V_n$$

□

4.3 Integration

Note: This is all in Folland.

4.3.1 S &S

1. Simple Functions
2. Bounded functions a set of finite measure.
3. non-negative functions
4. general integrable

4.3.2 Folland

1. Simple Functions
2. non-negative functions
3. general integrable functions

4.3.3 Notation

$(X, \mathcal{M}, \mu)$, $L^+ =$ space of measurable functions $f : X \to [0, +\infty]$

4.4 Simple Functions

Definition 24. *Let $\varphi(x) = \sum^N a_j \chi_{E_j}$ be a simple function,*

$$\int \varphi d\mu = \sum_1^N a_j \mu(E_j)$$

Definition 25.

$$\int_A \varphi d\mu := \int (\varphi \chi_A) d\mu$$

$$\varphi \chi_A = \sum_j a_j \chi_{E_j \cap A}$$

Proposition 4. *Let φ, ψ be simple in L^+*

1. *$c \geq 0$ then $\int c\varphi = c \int \varphi$*
2. *$\int \varphi + \psi = \int \varphi + \int \psi$*
3. *$\varphi \leq \psi \Rightarrow \int \varphi \leq \int \psi$*
4. *$A \in \mathcal{M}, \varphi \in L^+$, associate $A \mapsto \int_A \varphi = \int \varphi \chi_A$. We have $\mu_\varphi(A) = \int_A \varphi$ is measurable on $\mathcal{M}$.*

Definition 26. *$\{E_j\}_1^n$ is a **measurable partition** of X if $\bigcup_1^n E_j = X$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.*

4.5 Official Definition of the Integral

Definition 27. *Let $f \in L^+$*

$$\int f = \sup\left\{\int \varphi : \varphi \leq f, \varphi \text{ simple}\right\}$$

4.5.1 Properties

1. f i simple $\Rightarrow \int f$ agrees with the previous definition.
2. $f \leq g \Rightarrow \int f \leq \int g$
3. $c \in [0, +\infty]$ then $\int cf = c \int f$

Theorem 15 (Monotone Convergence Theorem)**.** *Let $f_n \in L^+$ with $f_n \nearrow f$ (so $f \in L^+$) then*

$$\int f_n \nearrow \int f$$

Proof. Since $\int f_n \leq \int f_{n+1}$ we know that $\lim \int f_n$ exists. Also since $f_n \leq f$ for all n we have

$$\int f_n \leq \int f$$

and thus $\lim \int f_n \leq \int f$. It remains to prove the reverse inequality.

Fix $\alpha \in (0, 1)$, let φ be simple such that $0 \leq \varphi \leq f$. Let $E_n = \{x : f_n(x) \geq \alpha\varphi(x)\} \in \mathcal{M}$. Also, notice that $E_n \subseteq E_{n+1}$ by the fact that the f_n are monotone. Additionally, $\bigcup E_n = X$.

$$\begin{aligned}
\int f_n &\geq \int_{E_n} f_n \\
&= \int f_n \chi_{E_n} \\
&\geq \int \alpha\varphi\chi_{E_n} \\
&= \alpha \int_{E_n} \varphi
\end{aligned}$$

By Prop 1 (d) we have $\mu_\varphi(E_n) \to \mu_\varphi(X)$

$$\int_{E_n} \varphi \to \int_X \varphi$$

So,

$$\lim \int f_n \geq \alpha \lim \int_{E_n} \varphi = \alpha \int \varphi$$

□

5 Convergence Theorems

5.1 The Monotone Convergence Theorem Corollaries

Remark: For every $f \in L^+$ there exists $\varphi_n \nearrow f$ with the φ_n simple in L^+ and so by the MCT we have $\int_f = \lim \int \varphi_n$.

Corollary 3. *$f, g \in L^+$ implies that $\int (f+g) = \int f + \int g$.*

Corollary 4. *If $f_n \in L^+$ then*

$$\int \sum_1^\infty f_n = \sum_1^\infty \int f_n$$

5.1.1 In general the MCT doesn't hold...

- $(\mathbb{R}, m)$ Let $f_n = \chi_{[n,n+1]}$ notice that $f_n \to 0$ but $\int d_n dm = 1$ for all n.
- $((0,1), m)$ Take $f_n = n\chi_{(0,\frac{1}{n})} \to 0$, but $\int f_n = 1$ for all n.

Theorem 16 (Fatou's Lemma)**.** *$f_n \in L^+$ implies that*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Proof. Recall that

$$\liminf f_n = \lim_{k\to\infty} \left(\inf_{n\geq k} f_n \right) = \sup_k \left(\inf_{n\geq k} f_n \right)$$

So we have

$$\left(\inf_{n\geq k} f_n \right) \nearrow \liminf f_n$$

Thus by the MCT we have

$$\lim_{k\to\infty} \int \inf_{n\geq k} f_n = \int \liminf f_n$$

Notice that

$$\inf_{n\geq k} f_n \leq f_k$$

and so

$$\int \inf_{n\geq k} f_n \leq \int f_k$$

Taking the $\liminf$ of both sides we have

$$\liminf \int \inf_{n\geq k} f_n \leq \liminf \int f_k$$

But since the limit exists on the left side we have

$$\lim \int \inf_{n\geq k} f_n \leq \liminf \int f_k$$

□

Proposition 5. *$f \in L^+$ implies that $\int f = 0 \iff f = 0$ a.e.*

Proof. ($\Leftarrow$) If $f = 0$ a.e. then $\mu(\{x : f(x) > 0\}) = \mu(E) = 0$. We have

$$\int f = \int_E f + \int_{E^c} f = 0 + 0 = 0$$

($\Rightarrow$) Suppose that $\int f = 0$ and to show a contradiction suppose that $\mu(\{x : f(x) > 0\}) > 0$. Let

$$E_n = \{x : f(x) > \frac{1}{n}\}$$

We have

$$E = \bigcup E_n$$

With $E_n \nearrow E$. We wish to show that $\mu(E_n) = 0$. Observe that

$$0 = \int f \geq \int_{E_n} f > \int_{E_n} \frac{1}{n} = \frac{1}{n}\mu(E_n)$$

So we've done it. □

Corollary 5. *Take $f_n \in L^+$, $f \in L^+$ if $f_n \to f$ a.e. then $\int f \leq \liminf \int f_n$.*

5.1.2 Remark

Note that it is not true that $\int \limsup f_n \geq \limsup \int f_n$. Consider the example on $(\mathbb{R}, m)$. Let $f_n = \chi_{[n,n+1]}$ notice that $f_n \to 0$ but $\int d_n dm = 1$ for all n.

Also it is not true that $\int \limsup \leq \limsup \int f_n$. To see this consider the functions $f_{2n+1} = \chi_{[0,\frac{1}{2}]}$ and $f_{2n} = \chi_{[\frac{1}{2},1]}$. Observe that $\liminf f_n = 0$ and $\limsup f_n = \chi_{[0,1]}$. So $\int \limsup f_n = 1$.

5.2 Integrals over General Functions, $f \in L^1$

Definition 28. *$f : X \to \overline{\mathbb{R}}$ is measurable, $f = f^+ - f^-$ with $f^+, f^- \in L^+$. We then define*

$$\int f := \int f^+ - \int f^-, \qquad \text{if both are not equal to } \infty$$

Definition 29. *We say that f is **integrable** (i.e. $f \in L^1$) if*

$$\int f+ < \infty, \qquad \int f^- < \infty$$

5.2.1 Properties of L^1

- Linearity
- Real linear space
- $\left|\int f\right| \leq \int |f|$

5.2.2 Extending to Complex Functions

Suppose that $f : X \to \mathbb{C}$ with $f = Re(f) + iIm(f)$ We then have $f \in L^1(X, \mu)$ if

$$\int |f| < \infty$$

implies that

$$\int f = \int Re(f) + i \int Im(f)$$

We have $L^1(X, \mu)$ is a complex linear space and $\int$ is complex linear and $\left|\int f\right| \leq \int |f|$ as above.

Proposition 6. *:*

1. *If $f \in L^1$ then $\{x : f(x) \neq 0\}$ is σ-finite (a countable union of sets of finite measure).*
2. *If $f, g \in L^1$ then TFAE*
   (a) $\int_E f = \int_E g$ *for all* $E \in \mathcal{M}$
   (b) $\int |f - g| = 0$
   (c) $f = g$ *almost everywhere*

1. *Proof.* We know that

$$\{x : f(x) \neq 0\} = \{x : |f(x)| > 0\ \} = \bigcup_{n=1}^{\infty} \{x : |f(x)| > \frac{1}{n}\}$$

Well, we know that

$$\mu\left(\underbrace{\{x : |f(x) < \frac{1}{n}\}}_{E_n}\right)$$

So we have

$$\int |f| \geq \int_{E_n} |f| \geq \int_{E_n} \frac{1}{n} = \frac{1}{n}\mu(E_n)$$

So we have

$$\frac{1}{n}\mu(E_n) \leq \int |f| < \infty$$

and thus $\mu(E_n) < \infty$. □

5.3 Lebesgue Dominated Convergence Theorem

Theorem 17 (Lebesgue Dominated Convergence Theorem)**.** *Start with $(X, \mathcal{M}, \mu)$ and $f_n : X \to \mathbb{C}$ are measurable with $f_n \to f$ a.e. and there exists $g \in L^+$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$ then*

$$\lim_{n\to\infty} \int f_n = \int \lim f_n = \int f$$

Proof. First off, it is enough to prove that this holds for $f_n \in L^1 - \mathbb{R}$. We will rely on Fatou's Lemma. Let's start with $|f_n| \leq g$ implies that $f_n \leq g$ and $-f_n \leq g$. Hence

$$g - f_n \in L^+$$

$$g + f_n \in L^+$$

So we have by Fatou's Lemma

$$\int \liminf(f - f_n) \leq \liminf \int (g - f_n)$$

But since

$$\lim(g - f_n) = g - f = \liminf(g - f_n)$$

we have

$$\begin{aligned}\int (g - f) &= \int \liminf(f - f_n) \\ &\leq \liminf \int (g - f_n) = \liminf(\int g - \int f_n) = \int g + \liminf(-\int f_n) = \int g - \limsup \int f_n\end{aligned}$$

Now, observe that

$$\int g + \int f = \int \liminf(g + f_n) \leq \liminf \int (g + f_n) = \int g + \liminf \int f_n$$

this implies that

$$\liminf \int f_n \geq \int f \geq \limsup f_n$$

Thus,

$$\liminf \int f_n = \limsup \int f_n = \lim \int f_n = \int f$$

□

Corollary 6 (Bounded Convergence Theorem)**.** *Let $\mu(X) < \infty$ and suppose that f_n are measurable with $|f_n(x)| \leq C$ for all x and some $C \in \mathbb{R}$, then*

$$f_n \to f \quad a.e. \implies \int f_n \to \int f$$

Proof. Follows from LDCT. Take $g = C$ then $\int C = C\mu(X) < \infty$. □

5.3.1 Example

1. Consider

$$\lim_{n\to\infty} \int_{[0,1]} \frac{1 + nx^2}{(1 + x^2)^n} dm$$

First we have

$$f_n(x) = \frac{1 + nx^2}{(1 + x^2)^n} \underset{n\to\infty}{\longrightarrow} \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$$

So we let $f(x) = \lim f_n(x) = 0$ a.e. Notice that the binomial theorem gives us

$$\frac{1+nx^2}{(1+x^2)^n} < 1$$

So by BCT we have

$$\lim_{n\to\infty} \int_{[0,1]} \frac{1+nx^2}{(1+x^2)^n} dm = \int 0 = 0$$

5.4 Comparison with the Riemann Integral

Let $f : [a,b] \to \mathbb{R}$, f bounded, $|f(x)| < K$. Do business with upper and lower sums of a partition. Let $\overline{I}_a^b f$ be the inf of the lower sums and $\underline{I}_a^b f$ be the sup of the lower sums.

Definition 30. *f is said to be Riemann integrable if $\underline{I}_a^b f = \overline{I}_a^b f = \int_a^b f(x)dx$*

Theorem 18. *:*

1. *f is Riemann integrable implies that f is measurable, $\int_{[a,b]} f dm = \int_a^b f(x)dx$*
2. *f is bounded real-valued, then f is R integrable if and only if*

$$m\left(\{x : f \text{ is discontinuous at } x\}\right) = 0$$

Corollary 7. *Improper Riemann integrals agree with Lebesgue if they converge absolutely. Suppose*

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

and $\int_a^\infty |f(x)|dx < \infty$. Then

$$\int_a^\infty f(x)dx = \int_{[a,\infty]} f dm$$

Proof. We use the dominated convergence theorem. Let $g(x) = |f(x)|$. We first show that $g \in L^1(a,\infty)$. Let

$$f_n = f\chi_{[a,n]} \to f, \qquad \forall x$$

Also observe that

$$|f_n| \nearrow |f|$$

So we have

$$\int_{[a,\infty]} |f(x)|dm = \lim \int_{[a,n]} |f| = \lim \int_{[a,\infty]} |f_n| = \int_{[a,\infty]} \lim |f_n| = \int_{[a,\infty]} |f|dm < \infty$$

Now use dominated convergence theorem with

$$f_n \to f, \quad |f_n| \le |f|$$

□

5.4.1 Example

Let $f = \sum_{n=1}^\infty \frac{1}{n}(-1)^n \chi_{(n,n+1)}$. Notice that $\int_0^\infty f(x)dx$ exists but it does not converge absolutely.

5.4.2 Example

$$\lim \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} dx$$

Notice that for $x > 0$ we have

$$|\sin(x/n)| \leq x/n$$

So we have

$$|f_n(x)| \leq \frac{1}{1+x^2} \in L^1$$

So we can use the dominated converence theorem, hence

$$\lim \int f_n = \int \lim f_n$$

and since

$$\lim(\frac{n}{x} \sin \sin \frac{x}{n}) = 1$$

We have

$$\lim \int f_n = \int \lim f_n = \int_0^\infty \frac{dx}{1+x^2} = \arctan x \Big|_0^\infty = \frac{\pi}{2}$$

5.5 Differentiating under the Integral

Theorem 19. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \times [a,b] \to \mathbb{C}$. For each $t \in [a,b]$ we have*

$$f_t : X \to \mathbb{C}$$

is integrable. So

$$F(t) = \int_X f(x,t) d\mu(x), \qquad F : [a,b] \to \mathbb{C}$$

Suppose there exists $\frac{\partial f}{\partial t}$ and there exists $g \in L^1(\mathcal{M})$ with $g \geq 0$ and

$$\left| \frac{\partial f}{\partial x}(x,t) \right| \leq g(x)$$

Then

$$F'(t) = \int \frac{\partial f}{\partial t}(x,t) d\mu(x)$$

5.5.1 Example

A special case of the Γ function:

$$\int_0^\infty x^n e^{-x} dx = n!, \qquad n \geq 1$$

Proof. We have

$$\int_0^\infty e^{-tx} dx = \frac{1}{t}$$

So we can differentiate with respect to t and then take $t = 1$.

□

5.6 L^1 Space

Definition 31. *Given $(X, \mathcal{M}, \mu)$ we let*

$$||f||_1 = \int |f| d\mu$$

Remark;

Definition 32. *We say $f \sim g$ if $f = g$ a.e. So*

$$L^1(X, \mu) = L^1(\mu)$$

is the set of equivalence classes of integrable functions of $(X, \mathcal{M}, \mu)$ inherits the vector space structure.

Definition 33. *We can define a metric on L^1 with $\rho(f, g) = ||f - g||_1$.*

Definition 34. *A function $x \to ||x|$ is called a **semi-norm** if*

1. *For all $a \in \mathbb{R}$ we have $||x|| = |a|||x||$*
2. *$||x + y|| \leq ||x|| + ||y||$*

Definition 35. *A complete normed vector space is called a **Banach Space**.*

Definition 36. *A series $\sum_1^\infty x_n \subseteq X$ converges to x if $s_n = \sum_1^N x_n \to x$, and converges absolutely if $\sum_1^\infty ||x_n||$ converges.*

Theorem 20. *A normed vector space X is complete if and only if every absolutely convergent series converges.*

Corollary 8. *$L^1(X, \mu)$ is complete.*

Proof. Need to check that absolutely convergent series converges take $\{f_n\}_1^\infty$ with $f_n \in L^1(X, \mu)$ and suppose that $\sum_1^\infty ||f_n||_1 < \infty$. First off we have

$$\sum^\infty \int |f_n| d\mu = \int \sum_1^\infty |f_n| d\mu$$

Let $g = \sum_1^\infty |f_n| \in L^1$. We want to use the dominated convergence theorem. So we look at the partial sums

$$\left| \sum_1^N f_n \right| \leq \sum_1^N |f_n| \leq g$$

so g dominates all of the partial sums. We can now consider the limit

$$\lim \sum_1^N \int f_n d\mu = \int \lim \sum_1^N f_n = \int f d\mu$$

We know that

$$\int \sum_1^\infty |f_n| d\mu < \infty$$

and so

$$\sum_1^\infty |f_n(x)| < \infty \qquad \text{a.e.}$$

This implies by completeness of $\mathbb{C}$ that

$$\sum_1^\infty f_n(x) < \infty \qquad \text{a.e}$$

We can finally compute

$$\begin{aligned} ||f - \sum_1^N f_n||_1 &= ||\sum_{N+1}^\infty f_n|| \\ &= \int \left| \sum_{N+1}^\infty f_n(x) \right| d\mu \\ &\leq \int \sum_{N+1}^\infty |f_n(x)| d\mu \\ &= \sum_{N+1}^\infty ||f_n||_1 \to 0 \end{aligned}$$

□

Theorem 21. *Consider $(\mathbb{R}^d, m)$. The following families are dense in $L^1(\mathbb{R}^d)$:*

1. *Simple Functions*
2. *Step Functions*
3. *Continuous functions of compact support*

Proof. (1) Enough to consider f real valued, $f = f^= - f^-$, WLOG $f \geq 0$. We know there exist simple functions $\varphi_k \nearrow f \geq 0$ so by the MCT we have

$$\int \varphi_k dm \to \int f dm$$

Thus

$$||f - \varphi_k||_1 \to 0$$

So we have

$$\int |f - \varphi_k| dm = \int (f - \varphi_k) dm$$

(2) It is enough to approximate χ_E with $m(E) < \infty$. By Theorem 4.3 there exists a finite family of almost disjoint rectangles $\{R_j\}_1^M$ such that

$$m\left(\bigcup_1^M R_j \triangle E\right) < \epsilon$$

This implies that

$$||\chi_E - \sum_1^M \chi_{R_j}|_1 < \epsilon$$

Where we use the fact that

$$||\chi_E - \chi_F||_1 = m(E \triangle F)$$

(3) It is enough to approximate the χ_R by continuous functions. Also, suffices to consider functions in $\mathbb{R}$ since we can take the product. □

Corollary 9. *$L^1(\mathbb{R}^d)$ is separable (i.e. there exists a countable dense set).*

Definition 37. *f is integrable means that $\int |f| < \infty$. We say $f \in L^1(\mu)$.*

5.7 Invariance Properties of the Lebesgue Integral

Recall that m is invariant under translation, that is $m(E) = m(E+h) = m(E_h)$.

Proposition 7. *If f is integrable on $\mathbb{R}^d$ then $f_h(x) = f(x-h)$ is also integrable and*

$$\int f dm = \int f_h dm \qquad (*)$$

Proof. Let $f = \chi_E$ with $m(E) > 0$. First notice that

$$(\chi_E)_h = \chi_{E_h}$$

Then we have

$$\int f dm = m(E) = m(E_h) = \int \chi_{E_h} dm = \int f_h \, dm$$

So, by linearity he have (*) holds for simple functions.

Given $f \in l^1(\mathbb{R}^d)$, there exist $\varphi_k \xrightarrow{L^1} f$, with φ_k simple. Recall that

$$\varphi_k \xrightarrow{L^1} f \quad \text{means that} \quad ||\varphi_k - f||_1 \to 0$$

First let us assume that $f \in L^+$ and so there exist $\varphi_k \nearrow f$ with φ_k simple in L^+, then by MCT

$$\int \varphi_k dm \to \int f dm < \infty$$

But we have

$$\int (\varphi_k)_h dm = \int \varphi_k dm \to \int f dm < \infty$$

□

Proposition 8. *If $f \in L^1(\mathbb{R}^d)$ then for $\delta > 0$ we have $f(\delta x) \in L^1(\mathbb{R}^d)$ and $f(-x) \in L^1(\mathbb{R}^d)$ and*

$$\delta^d \int_{R^d} f(\delta x) dm = \int_{R^d} f(x) dm$$

$$\int f(-x) dm = \int f(x) dm$$

Corollary 10 (Convolutions)**.** *Suppose that f, g are measurable on $\mathbb{R}^d$ and $y \mapsto f(x-y)g(y)$ is integrable. Then $y \mapsto f(y)g(x-y)$ is also integrable and*

$$(f * g)(x) := \int f(x-y)g(y)dm = \int f(y)g(x-y)dm$$

(Note that the above integrals are being integrated with respect to y.)

Proof. Let $h(y) = f(x-y)g(y)$ we use translation and reflection invariance

$$h_{-x}(y) = f(-y)g(y-x)$$
$$h(-y) = f(x+y)g(-y)$$
$$h_{-x}(-y) = f(y)g(x-y)$$

□

Proposition 9. *Let $f \in L^1(\mathbb{R}^d)$, this implies that*

$$||f_n - f|_{L^1} \to 0, \qquad h \to 0$$

So we have

$$||f_h - f||_1 = \int_{\mathbb{R}^d} |f_h(x) - f(x)|dm = \int_{\mathbb{R}^d} (f(x-h) - f(x))dm \to 0$$

Proof. First let f be continuous and compactly supported. Then $f(x-h) \to f(x)$ for all x. Hence

$$|f(x-h) - f(x)| \leq 2|f(x)| \in L^1$$

So $||f_h - f||_1 \to 0$ by DCT.

Next, take arbitrary $f \in L^1(\mathbb{R}^d)$, we have there exists $g \in C_0(\mathbb{R}^d)$ (note that C_0 means continuous of compact support) satisfying

$$||f - g||_1 < \epsilon$$

Then

$$f - f_h = (f - g) + (g - g_h) + (g_h - f_h)$$

So

$$||f - f_h||_1 \leq ||f - g||_1 + ||g - g_h|| + ||(f - g)_h|| < 3\epsilon$$

□

5.8 Modes of Convergence Summary

Proposition 10. *The following holds concerning convergence*

1. *If $f_n \to f$ in L^1 then $f_n \xrightarrow{\mu} f$.*
2. *$f_n \to f$ in L^1 then there exist $f_{n_k} \to f$ a.e.*
3. *$f_n \xrightarrow{\mu} f$ then there exist $f_{n_k} \to f$ a.e.*
4. *$f_n \to f$ a.e. on a finite measure space then $f_n \overset{\mu}{f}$.*

Corollary 11. *Suppose $g_j \in L^1$ and $\sum |\mathfrak{g}_j||_1 < \infty$. Then $\sum g_j$ converges almost everywhere and*

$$\int \sum_1^\infty g_j = \sum_1^\infty \int g_j$$

6 Fubini's Theorem

6.1 Product Measure

Remark: This will mostly follow Folland and the textbook that was emailed to us.

Definition 38. *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathbb{N}, \nu)$ are two measure spaces we define the* ***product σ-algebra*** *as the σ-algebra generated by*

$$\mathcal{M} \otimes \mathbb{N} = \{A \times B : A \in \mathcal{M}, B \in \mathbb{N}\} = \textit{Set of measurable Rectangles}$$

Lemma 7. *The set of finite disjoint unions of rectangles in $\mathcal{M} \otimes \mathbb{N}$ is an algebra, $\mathcal{A}_0$, in $X \times Y$.*

Definition 39. *We can define a premeasure π on $\mathcal{A}_0$ with*

$$\pi(A \times B) = \mu(A)\nu(B)$$

$$\pi(\bigcup_{j=1}^{n} A_j \times B_j = \sum_{1}^{n} \mu(A_j)\nu(B_j)$$

Proposition 11. *:*

1. *This π is well defined, i.e. does not depend on the representation.*
2. *π is a premeasure.*

Definition 40. *The* ***product measure*** *denoted $\mu \times \nu$ is the of extension the premeasure π to the σ-algebra $\mathcal{M} \otimes \mathbb{N}$.*

Note: *If μ, ν are σ-finite then this extension is unique.*

6.1.1 Remarks

1. *You don't need μ, ν to be complete.*
2. *Even if μ, ν are complete it is usually the case that $\mu \times \nu$ is not complete.*

Definition 41. *Let $E \subseteq X \times Y$ we have x-slice of E is $E_x = \{y : (x, y) \in E\}$ and y-slice of E is $E^y = \{x : (x, y) \in E\}$. And also $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$.*

Proposition 12. *.*

1. *if $E \subseteq \mathcal{M} \otimes \mathbb{N}$ then $E_x \in \mathbb{N}$ for all x and $E^y \in \mathcal{M}$ for all y*
2. *If $f(x, y)$ is $(\mathcal{M} \otimes \mathbb{N})$-measurable then f_x is $\mathbb{N}$-measurable for all x and f^y is $\mathcal{M}$ measurable for all y.*

Proof. (1) Let $\mathcal{R} = \{E \subseteq X \times Y$ which satisfy all properties in (1)$\} = \{E \subseteq X \times Y : E_x \in \mathbb{N} \forall x, E^y \in \mathcal{M} \forall y\}$. Observe that

- $\mathcal{R} \supset A \times B$ for all $A \in \mathcal{M}$ and $B \in \mathbb{N}$.
- $\mathcal{R}$ is closed under countable unions and complements.
- $\mathcal{R}$ is a σ-algebra. Since $\mathcal{M} \otimes \mathbb{N}$ is the smallest σ-algebra containing all rectangles and so $\mathbb{R} \supset \mathcal{M} \otimes \mathbb{N}$.

(2) Observe that $(f_x)^{-1}(\mathcal{U}) = (f^{-1}(\mathcal{U}))_x$ (to see this just unpack the definitions). So we have the result.

□

WARNING $m \times m \neq m_2$ rather m_2 is the completion of $m \times m$.

6.2 Monotone Class Lemma

Note that this exposition comes from Rich Bass's textbook.

Definition 42. *$\mathcal{M} \subseteq 2^x$ is a **monotone class** in X if it is closed under countable increasing unions and countable decreasing unions.*

Notice that every σ-algebra is a monotone class.

Theorem 22 (Monotone Class Lemma)**.** *If $\mathcal{A}_0$ is an algebra then the monotone class $\mathcal{M}$ generated by $\mathcal{A}_0$ coincides with the σ-algebra $\mathcal{A}$ generated by $\mathcal{A}_0$.*

Proof. (Step 1) Show that $\mathbb{N}_1 = \{A in \mathcal{M} : A^c \in \mathcal{M}\} = \mathcal{M}$

$$\mathcal{A}_0 \subseteq \mathbb{N}_1 \subseteq \mathcal{M}$$

and so $\mathcal{M}$ is closed under complements.

(Step 2) Show that $\mathbb{N}_2 = \{A \in \mathcal{M} : A \cap B \in \mathcal{M}, \forall B \in \mathcal{A}_0\}$

$$\mathcal{A}_0 \subseteq \mathbb{N}_2 \subseteq \mathcal{M}$$

is a monotone class and thus $\mathbb{N}_2 = \mathcal{M}$. So we have $\mathcal{M}$ is closed under intersections.

(Step 3) Show that $\mathbb{N}_3 = \{A \in \mathcal{M} : A \cap B \in \mathcal{M}, \forall B \in \mathcal{M}\}$

$$\mathcal{A}_0 \overset{*}{\subseteq} \mathbb{N}_3 \subseteq \mathcal{M}$$

(* holds by Step 2) is a monotone class and so $\mathbb{N}_3 = \mathcal{M}$ and $\mathcal{M}$ is closed under intersections.

(Step 4) We know that $\mathcal{M}$ is closed under complements and intersections. So if $A_i \in \mathcal{M}$ then $\bigcap_1^n A_i \searrow \bigcap_1^\infty A_i$ and we are done.

□

6.3 Fubini's Theorem

Theorem 23 (Fubini for Sets)**.** *Suppose $(X, \mathcal{M}, \mu), (Y, \mathbb{N}, \nu)$ are σ-finite, $E \subseteq \mathcal{M} \otimes \mathbb{N}$, then*

$$x \mapsto \nu(E_x), \qquad \textit{is } \mu\textit{-measurable}$$

$$y \mapsto \mu(E^y), \qquad \textit{is } \nu\textit{-measurable}$$

$$\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y)$$

Proof. (Sketch)
Define the set $\mathcal{R} = \{E \subseteq X \times Y$ for which the conclusion holds$\}$

(Step 1) Show holds for $A \times B$.

(Step 2) Show holds for the algebra.

(Step 3) Show holds for the monotone class and then apply the monotone class lemma.

□

Theorem 24 (Tonelli-Fubini)**.** *Let $(X, \mathcal{M}, \mu), (Y, \mathbb{N}, \nu)$ be σ-finite and $f(x, y)$ is $\mathcal{M} \otimes \mathbb{N}$ measurable.*

(a) $f \in L^+(X \times Y)$ then $x \mapsto \int f_x(y) d\nu(y)$ is μ-measurable. and $y \mapsto \int f^y(x) d\mu(x)$ is ν-measurable. Also, we have

$$\int f(x, y) d(\mu \times \nu)(x, y) = \int \int f_x(y) d\nu(y) d\mu(x) = \int \int f^y(x) d\mu(x) d\nu(y)$$

(b) $f \in L^1(X \times Y)$ then $f_x \in L^1(Y)$ and $x \mapsto \int f_x(y) d\nu(y)$ is in L^1.

Proof. (a) Start with simple functions $\varphi = \sum a_j \chi_{E_j}$ with $E_j \in \mathcal{M} \otimes \mathbb{N}$. For χ_E this is just Fubini for Sets for E. Next, if we take any $f \in L^+(X \times Y)$ then there exist $\varphi_k \in L^+$ such that $\varphi_k \nearrow f$ seal the deal by using MCT.

(b) Follows from (a) by considering real and imaginary parts and f^+ and f^-.

□

6.3.1 Example: Failure of Theorem for non-σ-finite measure spaces

Let $X = Y = [0, 1]$ and $\mathcal{M} = \mathbb{N} = \mathcal{B}_{[0,1]}$ Define $\mu = m$ and $\nu =$ the counting measure. Let $D = \{(x, x) : x \in [0, 1]\}$. We have

$$\int \int \chi_D(x, y) d\mu(x) d\nu(y)$$

Observe that $D^y = \{y\}$ which has measure 0 in X so the above integral is 0. However, $D^x = \{x\}$ has measure 1 in Y so

$$\int \nu(D_x) d\mu) x) = \int 1 dm = 1$$

Also we have $\mu \times \nu(D) = \infty$ (since any covering of D by rectangles has infinite measure in the product measure because any interval $[c, d]$ contains infinitely many points). Thus, the integral is also ∞.

Lemma 8. *X, Y separable metric spaces implies that $\mathcal{B}_{X \times Y} = \mathcal{B}_X \times \mathcal{B}_Y$.*

Theorem 25 (Fubini-Tonelli for Complete Measures)**.** *Fix $(X, \mathcal{M}, \mu), (Y, \mathbb{N}, \nu)$ complete and σ-finite. Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathbb{N}, \mu \times \nu)$. if f is $\mathcal{L}$-measurable and either (a) $f \geq 0$ or (b) $f \in L^1(\lambda)$ then f_x is $\mathbb{N}$ measurable for a.e. x and f^y is $\mathcal{M}$-measurable for a.e. y. Also,*

$$x \mapsto \int f - x d\nu, y \mapsto \int f^y d\mu, \qquad \text{are measurable}$$

and

$$\int f d\lambda = \int \int f(x, y) d\mu(x) d\nu(y) = \int \int f(x, y) d\nu(y) d\mu(x)$$

Proof. (Sketch)

1. First take $E \in \mathcal{M} \otimes \mathbb{N}$ and let $\mu \times \nu(E)$ implies $\nu(E_x) = 0$ a.e. x and $\mu(E^y) = 0$ for a.e. y. By Fubini,

$$\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = 0 \implies \nu(E_x) = 0 \qquad \text{a.e. } x$$

2. Same holds for $E' \in \mathcal{L}$ with $\lambda(E') = 0$ we have

$$E' \subseteq E \in \mathcal{M} \otimes \mathbb{N}, \mu \times \nu(E) = 0 \implies \nu(E_x) = 0 \quad \text{a.e.} x$$

Hence using that ν is compact

$$E'_x \subseteq E_x \implies \nu(E'_x) = 0$$

$\square$

7 Topics in Integration

7.1 Rotation Invariance of Lebesgue Measure

Lemma 9. *Every $T \in GL(n, \mathbb{R})$ is a product of finitely many elementary transformations. Scaling, Translation, Interchanging Rows.*

Theorem 26. *Let $T \in GL(n, \mathbb{R})$. If f is measurable in $\mathbb{R}^n$ then $f \circ T$ is Lebesgue measurable. If $f \geq 0$ or $f \in L^1(m)$ then*

$$\int f(z)dm(x) = |\det T| \int (f \circ T)(x)dm(x)$$

Proof. By the lemma enough to show holds for elementary transformations. We first show for Borel sets in a straightforward manner. Next show that if N has $m(N) = 0$ then $m(T(N)) = 0$ by using the fact that $N \subseteq B^N$ for some Borel set $m(B^N) = 0$. Use the fact that E is lebesgue measurable then $E = B \cup N$ and so $T(E) = T(B) \cup T(N)$ and we are done. From sets we can go to simple functions and from simple functions to nonnegative measurable functions then we get general functions.

□

Corollary 12. .

1. *For all $E \in \mathcal{L}^n$ we have $T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det E| m(E)$.*
2. *m is rotation invariant.*

7.2 Integration in Polar Coordinates

Definition 43. *In $\mathbb{R}^n$ we have $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. For $x \in \mathbb{R}^n \setminus \{0\}$ the **polar coordinates of** x are*

$$r = |x|, \qquad z' = \frac{x}{|x|}$$

We have $\Phi : \mathbb{R}^n \setminus \{0\} \to (0, \infty) \times S^{n-1}$

$$\Phi(x) = (r, x'), \qquad \Phi^{-1}(r, x') = rx'$$

7.2.1 Construction

Let m_* be the Borel measure on $(0, \infty) \times S^{n-1}$ induced by Φ from m.

$$m_*(E) := m(\Phi^{-1}(E))$$

On $(0, \infty)$ define $\rho = \rho_n$

$$\rho(E) = \int_E r^{n-1} dm(r)$$

Theorem 27. *There exists a unique measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that*

$$m_* = \rho \times \sigma$$

So that if f is Borel-measurable on $\mathbb{R}^n$ and $f \geq 0$ or $f \in L^1(m)$ then

$$\int f dm = \int_0^\infty \int_{S^{n-1}} f(rx')r^{n-1} d\sigma(x')dr$$

7.2.2 Potpouri

Proposition 13. *(Stein) Let $f \in L^1(X, \mathcal{M}, \mu)$. Then there exists $\epsilon > 0$ and there exists $\delta ?0$ such that*

$$E \in \mathcal{M}, \qquad \mu(E) < \delta \implies \int_E |f| d\mu < \epsilon$$

7.3 Borel Measures on $\mathbb{R}$ and Lebesgue-Stieltjes Integrals

We use Stein's Exposition.

Definition 44. *A function F on $\mathbb{R}$ is **right-continuous** at x_0 if*

$$F(x_0) = \lim_{x \to x_0^+} F(x) = F(x_0^+)$$

Theorem 28. *Let F be an increasing function on $\mathbb{R}$ and assume that F is right continuous. Then,*

1. *There exists a unique measure $\mu = dF$ on $\mathcal{B}_{\mathbb{R}}$ such that*

$$\mu((a, b]) = F(b) - F(a), \quad a < b$$

2. *Conversely If μ is a Borel measure on $\mathbb{R}$ and μ is finite on compact sets define*

$$F(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < \infty \end{cases}$$

then F is increasing and right continuous with $\mu = dF$.

7.3.1 Remarks

1. *We need finite on compact intervals in (2) because Hausdorff measures are an example of a measure that fails this and creates problems.*

8 Review for 524

8.1 Review For Final Questions

(6) Compute

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n \log(2+\cos\frac{x}{n})dx$$

We know that

$$\log(1-y) = -\sum_1^\infty \frac{y^n}{n} \leq -y$$

And so

$$\log(1-\frac{x}{n}) \leq -\frac{x}{n}$$

This means

$$(1-\frac{x}{n})^n = e^{n\log(1-\frac{x}{n})} \leq e^{n(-\frac{x}{n})} = e^{-x} \in L^1(0,\infty)$$

So the integrand is bounded by $\log 3$. We can apply the dominated convergence theorem and write

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n \log(2+\cos\frac{x}{n})dx = \int_0^\infty \lim_{n\to\infty}(1-\frac{x}{n})^n \log(2+\cos\frac{x}{n})dx = \int_0^\infty e^{-x}\log 2dx$$

8.2 Review of Theoretical Questions

8.2.1 Caratheodory Thm

If X is a space and μ_* is an outer measure then we have

$$\mathcal{M} = \{\mu_* - \text{measurable sets}\} = \{A \subseteq X : \forall E \subseteq X \text{ it holds that } \mu_*(E) = \mu_*(E\cap A) + \mu_*(E \cap A^c)\}$$

Theorem 29 (Caratheodory Thm)**.** *1. $\mathcal{M}$ is a σ-algebra.*

2. $\mu_*|_{\mathcal{M}}$ us a measure.

Proof. (1) Step 1: Show that $\mathcal{M}$ is an algebra. ie.e prove that $A \in \mathcal{M}$ implies $A^c \in \mathcal{M}$ and $A, B \in \mathcal{M}$ implies $A \cup B \in \mathcal{M}$.

Step 2: Show that $\mathcal{M}$ is a σ-algebra. Let $A = \bigcup_1^\infty A_i$ with A_i disjoint (can do this since any countable union can be made into a disjoint).

□

8.2.2 Regularity of Borel Measures

Definition 45. *A **Borel Measure** is a measure on $\mathcal{B}_X$.*

Proposition 14 (Regularity of Borel Measures)**.** *Assume that (X,d) is a metric space and μ is Borel with $\mu(\mathbb{B}_r(x)) < \infty$ for all $r > 0$. Then for all $E \in \mathcal{B}_X$ and $\epsilon > 0$ there exists a closed set $F \subseteq E \subseteq \mathcal{O}$ with $\mathcal{O}$ open such that*

$$\mu(O \setminus E) < \epsilon$$

$$\mu(E \setminus F) < \epsilon$$

Proof. Let $\mathcal{C}$ be the class of all sets $E \subseteq \mathcal{B}_X$ which satisfy the proposition. We want to show that $\mathcal{C}$ is a σ-algebra that contains the open sets. A key lemma is that if

$$F_* = \bigcup_1^\infty F_k$$

then there exists closed $F \subseteq F_*$ such that $\mu(F_* \setminus F) < \epsilon$. Can assume that

$$F_k \nearrow F_*$$

this implies that

$$\mu(F_k) \nearrow \mu(F_*)$$

So if we let $X = \bigcup_1^\infty \overline{B}_n$ and $\overline{B}_n \overline{\mathbb{B}(x,n)}$. We have

$$F_* = \bigcup_n F_* \cap (\overline{B}_n \setminus B_{n-1})$$

Use a $\epsilon/2^n$ argument. □

sectionLecture One: 1/4

8.2.3 Question 1

Suppose f is integrable wrt Lebesgue measure. and $f : [a,b] \to \mathbb{R}$. Define

$$F(x) := \int_a^x f(t)dt$$

Q: Can we say that $F'(x) = f(x)$ a.e ?

8.2.4 Question 2

Given $F(x)$ on $[a,b]$. When is F differentiable almost everywhere and

$$\int_a^b F'(t)dt = F(b) - F(z)$$

8.2.5 The Derivative

Definition 46. *Let $F : \mathbb{R} \to \mathbb{R}$. We have (if it exists)*

$$F'(x) = \lim_{\epsilon \to 0} \frac{F(x+\epsilon) - F(x)}{\epsilon}$$

8.2.6 Averaging Problem

We can write

$$\frac{1}{\epsilon} \int_x^{x+\epsilon} f(t)dt = \frac{1}{|I|} \int_I f, \qquad I = [x, x+\epsilon]$$

and ask when is for $x \in I$

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_I f = f(x)$$

8.2.7 Generalization to $\mathbb{R}^d$

We can replace I with a ball $B \ni x$ and ask the question:

> Suppose f is integrable on $\mathbb{R}^d$. Is it true that for almost every $x \in B$
> $$\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B f(y)dy = f(x)$$

Lemma 10. *If f is continuous at x, then*

$$\lim \frac{1}{m(B)} \int_B f(y)dy = f(x)$$

Proof. We have

$$\frac{1}{m(B)} \int_B f(y)dy - f(x) = \frac{1}{m(B)} \int_B (f(y)dy - f(x))\, dy$$

So we can make inequalities

$$\left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| = \left| \frac{1}{m(B)} \int_B (f(y)dy - f(x))\, dy \right| \leq \epsilon$$

Where the last inequality follows from continuity.

□

We will try and approach the averaging question by using the fact that $Co(\mathbb{R}^d)$ = the set of continuous compactly supported functions is dense in $L^1(\mathbb{R}^d)$.

8.3 §1.1 Hardy-Littlewood Maximal Function

Definition 47. *Let $f \in L^1(\mathbb{R}^d)$ and let*

$$f^*(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)|dy$$

*f is called the **Hardy-Littlewood Maximal Function**.*

Lemma 11 (Chebyshev Inequality)**.** *If $g \in L^1(\mathbb{R}^d)$ then*

$$m(\{x : |g(x| > \alpha\}) \leq \frac{1}{\alpha} ||g||_1$$

Lemma 12 (Covering Lemma)**.** *Let $\mathbb{B} = \{B_1, \ldots, B_N\}$ a collection of balls in $\mathbb{R}^d$. Then there exists a disjoint subcollection $\{B_{i_1}, \ldots, B_{i_k}\}$ such that*

$$\bigcup_{j=1}^{k} 3B_{i_j} \supset \bigcup_{i=1}^{N} B_i, \qquad 3B(x,r) = B(x,3r)$$

Proof. Choose a ball of maximal radius. Call this B_{i_1}. Notice that if $B_i \cap B_j \neq \emptyset$ and $r_j \leq r_i$ then $3B_i \supset B_j$. So we have $3B_{i_1}$ covers all ball with nonempty intersection with B_{i_1}. We take B_{i_2} to be a maximal radius ball which does not intersect $3B_{i_1}$. Continue in this manner.

□

Theorem 30 (Hardy-Littlewood Maximal Theorem)**.** *Let $f \in L^1(\mathbb{R}^d)$.*

1. *f^* is measurable.*
2. *$f^*(x) < \infty$ almost everywhere.*
3. *$m(\{x \in \mathbb{R}^d : f^*(x) < \alpha\}) \leq \frac{A}{\alpha}||f||_1$. This is called the **weak-type** estimate where $A = 3^d$.*

Proof. (1) We will show that $\{f^*(x) > \alpha\}$ is open. If $f^*(\overline{x}) > \alpha$ this implies that there exists an open ball $B \ni \overline{x}$ such that

$$\frac{1}{m(B)} \int |f(y)|dy > \alpha$$

Then we have $f^*(x) > \alpha$ for any other $x \in B$. And so the set is open and thus measurable.

(2) It is clear that (3) $\Rightarrow$ (2). So it remains to prove (3).

(3) Let $E_\alpha = \{x : f^*(x) > \alpha\}$ recall that $m(E_\alpha) = \sup\{m(K) : K \subseteq E_\alpha, K \text{ compact}\}$. So for all $x \in E_\alpha$ there exist $B_x \ni x$ such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|dy > \alpha$$

Fix a compact set $K \subseteq E_\alpha$. We have $K \subseteq \bigcup x \in KB_x$ is an open cover which implies that there exists a finite subcover $\{B_1, \ldots, B_N\}$. Using the Covering Lemma we have

$$K \subseteq \bigcup_{l=1}^{N} B_l \subseteq \bigcup_{j=1}^{k} 3B_{i_j}$$

We then have

$$\begin{aligned}
m(K) &\leq m\left(\bigcup_1^N B_l\right) \\
&\leq m\left(\bigcup_1^k 3B_{i_j}\right) \\
&= 3^d \sum_1^k m(B_{i_j}) \\
&< \frac{3^d}{\alpha} \sum_1^k \int_{B_{i_j}} |f| \\
&= \frac{3^d}{\alpha} \int_{\bigcup B_{i_j}} |f| \\
&\leq \frac{3^d}{\alpha} \int |f| \\
&= \frac{3^d}{\alpha} ||f||
\end{aligned}$$

□

9 Differentiation

9.1 Lebesgue Differentiation Theorem

Theorem 31 (Lebesgue Differentiation Theorem)**.** *If we have $f \in L^1(\mathbb{R}^d)$ then for all $B \ni x$ we have for almost every $x \in \mathbb{R}^d$ s*

$$\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B f(y)dy = f(x)$$

Proof. For $\alpha > 0$ define $E_\alpha = \left\{x : \limsup_{m(B)\to 0} \left|\frac{1}{m(B)} \int f(y)dy - f(x)\right| > \alpha\right\}$ We will have the averages converge at x iff $x \in \bigcap_{r>0, r\in\mathbb{Q}} E_\alpha^c$ where we have

$$\bigcup_{\alpha>0} E_\alpha = \bigcup_{\alpha\in\mathbb{Q},\alpha>0} E_\alpha, \qquad \text{will have } m\text{-measure zero}$$

Fix $\epsilon > 0$. Find g continuous with compact support so that

$$||f - g||_1 < \epsilon$$

We then have for fixed x

$$\frac{1}{m(b)} \int_B f(y)dy - f(x) = \frac{1}{m(B)} \int_B (f(y) - g(y))dy + \frac{1}{m(B)} \int_B g(y)dy - g(x) + (g(x) - f(x))$$

Putting in absolute values gives

$$\left|\frac{1}{m(b)} \int_B f(y)dy - f(x)\right| \leq \frac{1}{m(B)} \int_B |(f(y) - g(y))|\, dy + \left|\frac{1}{m(B)} \int_B g(y)dy - g(x)\right| + |(g(x) - f(x))|$$

So if we take the limsup and use continuity of g we obtain

$$\limsup_{m(B)\to 0} \left|\frac{1}{m(B)} \int f(y)dy - f(x)\right| \leq (f - g)^*(x) + 0 + |f(x) - g(x)|$$

Also, since $E_\alpha \subseteq F_{\alpha/2} \cup G_{\alpha/2}$ we must have

$$F_{\alpha/2} = \{x : (f - g)^* > \alpha/2\}, \quad G_{\alpha/2} = \{x : |f(x) - g(x)| > \alpha/2\}$$

Monotonicity implies

$$m(E_\alpha) \leq m(F_{\alpha/2}) + m(G_{\alpha/2})$$

by the Maximal Theorem,

$$m(F_{\alpha/2}) \leq \frac{A}{\alpha/2} ||f - g||_1 \leq \frac{A\epsilon}{\alpha}$$

$$m(G_{\alpha/2}) \leq \frac{1}{\alpha/2} ||f - g||_1 \leq \frac{2\epsilon}{\alpha}$$

As ϵ is arbitrary we have

$$m(E_\alpha) = 0$$

□

Corollary 13. *$|f(x)| \leq f^*(x)$ for a.e x.*

Proof. We apply differentiation theorem to $|f|$ and obtain

$$|f(x)| = \lim_{m(B)\to 0} \frac{1}{m(B)} \int_B |f(y)dy \leq f^*(x), \qquad \text{for a.e. } x$$

□

9.1.1 Remark

It is enough to assume that f is locally integrable.

Definition 48. *A function f is **locally integrable** denoted by $f \in L^1_{loc}(\mathbb{R}^d)$ if $f \cdot \chi_{B_R} \in L^1$ for all $R > 0$.*

Corollary 14 (Lebesgue Density Theorem)**.** *Let $E \in \mathcal{M}$ with $m(E) > 0$. Almost every $x \in E$ is a density point and almost every $x \notin E$ is not a density point of E.*

Proof. We have

$$\int_B \chi_E(y)dy = \int \chi_E \chi_B(y) = \int \chi_{E\cap B}(y) = m(E \cap B)$$

So by the differentiation theorem we have

$$\lim_{m(B)\to 0} \frac{m(E \cap B)}{m(B)} = \chi_E(x), \qquad \text{for almost every } x$$

□

9.2 Lebesgue Sets

Definition 49. *Suppose f is locally integrable. The **Lebesgue set** of f is*

$$L_f := \left\{ \overline{x} \in \mathbb{R}^d : |f(\bar{)}| \neq \infty \text{ and } \lim_{m(B)\to 0} \frac{1}{m(B)} \int |f(y) - f(\overline{x})|dy = 0 \right\}$$

Note that the triangle inqequality implies that averages converge to $f(\overline{x})$ for all $x \in L_f$.

Corollary 15. *f is locally integrable implies that $m(L_f^c) = 0$.*

Proof. Let $r \in \mathbb{Q}$ and apply the differentiation theorem to $|f(y) - r|$. We know that there exists $E_r, m(E_r) = 0$ such that for all $x \notin E_r$ we have

$$\frac{1}{m(B)} \int_B |f(y) - r|dy \underset{m(B)\to 0}{\longrightarrow} |f(x) - r|$$

Let

$$E := \bigcup_{r\in\mathbb{Q}} E_r, \qquad m(E) = 0$$

We would like to show that $\overline{x} \notin E$ and $f(\overline{x})$ is finite implies $\overline{x} \in L_f$. Fix $\epsilon > 0$, there exists $r \in \mathbb{Q}$ such that $f(\overline{x}) - r| < \epsilon$. We also have

$$\frac{1}{m(B)} \int_B |f(y) - f(\overline{x})|\, dy \leq \frac{1}{m(B)} \int_B |f(y) - r|\, dy + |f(\overline{x}) - r|$$

Our choice of r tells us that $|f(\overline{x}) - r| < \epsilon$. Then if we take the limsup we have the whole expression is less than 2ϵ.

□

Definition 50. *A family $\mathcal{B}$ of sets in $\mathbb{R}^d$ **shrink regularly** to $\overline{x}$ if there exists a $c > 0$ such that for all $\overline{x} \in U_\alpha$ there exists a ball $B \supseteq U_\alpha$ satisfying*

$$m(U_\alpha) \geq cm(B)$$

We also need for all $\epsilon > 0$ there exists $U_\alpha \ni \overline{x}$ such that $m(U_\alpha) < \epsilon$.

Corollary 16. *If $\{U_\alpha\}$ shrinks nicely to $\overline{x}$ and $\overline{x} \in L_f$ then*

$$f(\overline{x}) = \lim_{m(U_\alpha)\to 0} \frac{1}{m(U_\alpha} \int_{U_\alpha} f(y)dy$$

Proof. Consider

$$\left| \frac{1}{m(U_\alpha)} \int_{U_\alpha} f(\overline{y}) - f(\overline{x})dy \right| \leq \frac{1}{m(U_\alpha)} \int_{U_\alpha} |f(\overline{y}) - f(\overline{x})|\, dy$$

Since there exist $B \subseteq U_\alpha$ with $m(U_\alpha) \geq cm(B)$ and $\frac{1}{m(U_\alpha)} \leq \frac{1}{cm(B)}$ we can write

$$\frac{1}{m(U_\alpha)} \int_{U_\alpha} |f(\overline{y}) - f(\overline{x})|\, dy \leq \frac{1}{cm(B)} \int_B |f(\overline{y}) - f(\overline{x})| dy \longrightarrow 0$$

□

9.3 Differentiability

9.3.1 Question

For a function $F(x)$ when do we have

$$\int_a^b F'(x) = F(b) - F(a)$$

Two possible problems are that not every F is differentiable and that F' may exist but need not be integrable. However, we know that if $f \in L^(a, b)$ and $F(x) = \int_a^x f(y)dy$ then $F'(x) = f(x)$ almost everywhere. Additionally $\int_a^b F'(x)dx = F(b) - F(a)$. In general how can we characterize all F that satisfy this.

9.3.2 Rectifiable Curves

Definition 51. *Let γ be a curve in $\mathbb{R}^d$ (γ is a continuous function $\gamma : [a, n] \to \mathbb{R}^d$). Given a partition*

$$a = t_0 < t_1 < t_2 < \cdots < t_N = b$$

We let

$$L(\gamma) = \sup \left(\sum_1^N |\gamma(t_j) - \gamma(t_{j-1}| \right), \qquad \text{over all partitions}$$

*A curve, γ, is called **rectifiable** if $L(\gamma) < \infty$.*

Definition 52. *Let $f : [a, b] \to \mathbb{C}$ (f need not be continuous). Let $\mathcal{P} = \{a = t_0 < t_1 < t_2 < \cdots < t_N = b\}$. The **variation of** f on $\mathcal{P}$ is*

$$\mathrm{Var}(f, \mathcal{P}) = \sum_1^N |f(t_j) - f(t_{j-1})|$$

*We say that f **is of bounded variation**, written $f \in BV([a, b])$ if there exists $M < \infty$ such that*

$$\mathrm{Var}(f, \mathcal{P}) \leq M, \qquad \forall \mathcal{P}$$

Observation 15. *We have for $\mathcal{P}_2$ a refinement, $\mathcal{P}_2 \supset \mathcal{P}_1 \implies \mathrm{Var}(F, \mathcal{P}_2) \geq \mathrm{Var}(F, \mathcal{P}_1)$.*

Definition 53. *The notation $F \nearrow$ means that F is **increasing**. i.e. $F(x) \leq F(y)$ if $x < y$. Similarly for $F \searrow$.*

9.3.3 Examples

1. If F is monotonic then $F \in BV$ since $\mathrm{Var}(F, \mathcal{P}) = |F(b) - F(a)|$.
2. IF F is differentiable then for all $x \in (a, b)$ we have $|F'(x)| \leq M$ and so by the mean value theorem we have
$$|F(t_j) - F(t_{j-1}| = |F'(c_j)(t_j 0 t_{j-1})| \leq M|t_j - t_{j-1}| \leq M|b - a|$$
3. (Homework) Consider
$$F(x) = \begin{cases} x^a \sin(x^{-b}), & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$
Then $F \in BV([0, 1] \iff a > b$.

Definition 54. *The **total variation** of F on $[a, x]$ is*
$$T_F(a, x) := \sup \sum_1^N |F(t_j) - F(t_{j-1})|, \qquad \textit{over all partitions of [a,x]}$$

Definition 55. *Suppose that F is real valued. We define the **positive total variation***
$$P_F(a, x) = \sup_{\mathcal{P}} \sum_+ F(t_j) - F(t_{j-1}), \qquad F(t_j) \geq F(t_{j-1})$$
*Similarly, we define the **negative total variation***
$$N_F(a, x) = \sup_{\mathcal{P}} \sum_- F(t_{j-1}) - F(t_j)), \qquad F(t_j) \leq F(t_{j-1})$$

Lemma 13. *Suppose that F is real valued and $BV[a, b]$, then for every $x \in [a, b]$ we have*
$$F(x) - F(a) = P_F(a, x) - N_F(a, x)$$
$$T_F(a, x) = P_F(a, x) + N_F(a, x)$$

Proof. Fix x. We know that $P_F, N_F < \infty$. So for all $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that
$$\left| P_F - \sum_+ (F(t_j) - F(t_{j-1})) \right| < \epsilon$$
$$\left| N_F - \sum_- -(F(t_j) - F(t_{j-1})) \right| < \epsilon$$
Notice that
$$F(x) - F(a) = \sum_1^N (F(t_j) - F(t_{j-1})) = \sum_+ - \sum_- \implies |F(x) - F(a) - (P_f - N_F)| < 2\epsilon$$
For the second part observe that
$$\mathrm{Var}(F, \mathcal{P}) = \sum_+ + \sum_-$$
So it holds that
$$T_F \leq P_F + N_F + 2\epsilon \implies T_F \leq P_F + N_F$$
Since we know that $T_F \geq P_F + N_F$ we have equality. □

Theorem 32. *For all $F \in BV[a,b]$ there exist F_1, F_2 both increasing such that $F(x) = F_1(x) - F_2(x)$.*

Proof. Let $F_1(x) = P_F(a,x) + F(a)$ and $F_2(x) = N_F(a,x)$. By the lemma we have

$$F(x) = F_1(x) - F_2(x), \qquad F_1, F_2 \nearrow$$

And we have the result.

□

9.4 Covering Lemma's and Differentation of BV

Lemma 14 (Elementary Covering Lemma)**.** *For any finite collection of ball (open or closed, non-trivial) $\{B_l\}_1^N$, there exists a disjoint subcollection $B_{l_1}, \ldots, B_{l_k}$ such that*

$$\sum_1^k m(B_{l_j}) > 3^{-d} \sum_1^n m(B_l)$$

Definition 56. *A collection of open or closed balls $\mathcal{B}$ (with nonempty interior) is a **Vitali covering** (sometimes called a fine covering) of a set E if for all $x \in E$ and $\epsilon > 0$ there exists an $B \in \mathcal{B}$ such that $x \in B$ and $m(B) < \epsilon$.*

Lemma 15 (Vitali Covering Lemma)**.** *Suppose that $m(E) < \infty$ and $\mathcal{B}$ is a Vitali covering. Then*

1. *For all $\delta > 0$ there exist $B_1, \ldots, B_N \in \mathcal{B}$ such that the B_i are disjoint and*

$$\sum_1^N m(B_i) \geq m(E) = \delta$$

2. *We can make sure that $m\left(\setminus \bigcup_1^N B_i\right) < 2\delta$.*

Proof. (1) Fix δ and assume that $M(E) > \delta$. There exist a compact set $E' \subseteq E$ with $m(E') \geq \delta$. And so for all $x \in E'$ theer exists $B_x \ni x$, $B_x \in \mathcal{B}$ such that

$$\{\overset{\circ}{B}_x\}_{x \in E'}, \qquad \text{is an open cover of E'}$$

This means that there is a finite subcover, $\overset{\circ}{B}_1, \ldots, \overset{\circ}{B}_N$. We use the elementary covering lemma and so there exist disjoint $B_1, \ldots, B_{N_1}$ with

$$\sum_1^{N_1} m(B_i) \geq 3^{-d} m(E')$$

Case 1: If $\sum_1^{N_1} m(B_i) \geq m(E) - \delta$ then we are done by taking $N = N_1$.

Case 2: If $\sum_1^{N_1} m(B_i) < m(E) - \delta$. Then we let $E_2 := E \setminus \bigcup_1^{N_1} \overline{B}_i$ with $m(E_2) > \delta$. Now repeat the argument and so there exists compact $E_2' \subseteq E_2$ with $m(E_2') \geq \delta$. The balls in $\mathcal{B}$ which are disjoint from $B_1, \ldots, B_{N_1}$ form a Vitali covering of E_2'. Now we find a finite cubcover by these open balls and choose $B_{N_1+1}, \ldots, B_{N_2}$ a disjoint cubcollection such that

$$\sum_{N_1+1}^{N_2} m(B_i) \geq 3^{-d} m(E_2)$$

We repeat again with *Case 1:* $\sum_1^{N_2} m(B_i) \geq m(E) - \delta$ and *Case 2:* $\sum_1^{N_2} m(B_i) < m(E) - \delta$. Since the case two increases by at least a fixed increment this is a finite process with no more steps then

$$\frac{m(E)}{3^{-d}\delta} < \infty$$

(2) There exist open $\mathcal{O} \supset E$ such tht $m(\mathcal{O}) < m(E) + \delta$. We can restrict the construction to balls in $\mathcal{O}$. We then have

$$m\left(E \setminus \bigcup_1^N B_i\right) \leq m(\mathcal{O}) - m\left(\bigcup_1^N B_i\right) \leq m(E) + \delta - m(E) + \delta = 2\delta$$

□

Definition 57. *Given a monotone function f on $[a, b]$ we define the* ***Dimi derivatives***

$$\begin{aligned}
D^+ f(x) &= \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \\
D_+ f(x) &= \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \\
D^- f(x) &= \limsup_{h \to 0^+} \frac{f(x) - f(x-h)}{h} \\
D_- f(x) &= \liminf_{h \to 0^+} \frac{f(x) - f(x-h)}{h}
\end{aligned}$$

Observation 16. *It is clear that*

$$D^+ f(x) \geq D_+ f(x), \qquad D^- f(x) \geq D_- f(x)$$

If all are equal and not $\pm\infty$ then we say that f ***is differentiable at*** *x and $f'(x)$ is the common value.*

Theorem 33 (Lebesgue, 1904)**.** *f is monotone on $[a, b]$ implies that*

1. *f' exists almost everywhere*
2. *f' is measurable and is not $\pm\infty$ almost everywhere.*
3. *f' is integrable and $\int_1^b f'(x)dx \leq f(b) - f(a)$.*

Proof. There are multiple proofs. The one in Stein uses the 'rising sun lemma'. Folland's proof uses the Radon-Nicodemus Theorem. We will use the proof from Royden.

The proof will utilize Dimi derivatives. To prove (1) is enough to show that

$$m\left(\left\{x : D^+ f(x) > D_- f(x)\right\}\right) = 0$$

Since this implies all of the Dimi derivatives are equal. We define

$$E = \{x : D^+ f(x) > D_- f(x) = \bigcup_{u,v \in \mathbb{Q}} E_{uv}, \quad E_{uv} = \left\{x : D^+ f(x) > u > v > D_- f(x)\right\}$$

Fix u, v. Let $\epsilon > 0$. There exists an open $\mathcal{O} \supseteq E_{uv}; m(E_{uv}) > m(\mathcal{O}) - \epsilon$. By definition of E_{uv} for every $x \in E_{uv}$ there exist arbitrarily small $[x-h, x] \subseteq \mathcal{O}$ with $h = h_x$ which satisfy

$$f(x) - f(x-h) < vh$$

Then we have the collection of all such intervals for all $x \in E_{uv}$ is a Vitali covering of E_{uv}. Vitali's Covering Lemma tells us that there exists a disjoint subcollection of $\{I_1, \ldots, I_n\}$ such that

$$m\left(E_{uv} \cap \bigcup_1^N \overset{\circ}{I}_n\right) > m(E_{uv}) - \epsilon$$

Notational note: $\overset{\circ}{X} = Int(X)$. When we sum over these intervals we get

$$I_n = [x_n - h_n, x_n] \sum_1^N [f(x_n) - f(x_n - h_n)] < v \sum_1^n h_n = vm\left(\bigcup_1^N I_n\right) \leq vm(\mathcal{O}) < v(m(E_{uv})) + \epsilon$$

Now work with $A = E_{uv} \cap \bigcup_1^N \overset{\circ}{I}_n$. For all $y \in A$ there exist k arbitrarily small so that

$$[y, y+k] \subseteq \overset{\circ}{I}_n, \qquad \text{for some } n$$

and

$$f(y+k) - f(y) > uk$$

All these $[y, y+k]$ form a Vitali covering of A. By Vitali's Lemma there exists a disjoint subcollection $\{J_1, \ldots, J_M\}$ satisfying $J_i = [y_i + k_i]$ and

$$m\left(A \cap \bigcup_1^m J_n\right) > m(A) - \epsilon > m(E_{uv}) - 2\epsilon$$

Notice that M will probably be much bigger than N. Next, we sum over the intervals. If we consider

$$\sum_1^M [f(y_i + k_i) - f(y_i)] > u \sum_1^M k_i \geq u \cdot m\left(A \cap \bigcup_1^M J_i\right) > u\left(m(E_{uv}) - 2\epsilon\right)$$

Also, since f is increasing

$$\sum_{J_i \subseteq I_n} (f(y_i + k) - f(y_i)) \leq f(x_n) - f(x_n - h_n) = \sum_n \sum_{J_i \subseteq I_n} (f(y_i + k) - f(y_i))$$

$$\sum_{J_i \subseteq I_n} (f(y_i + k) - f(y_i)) = \sum_n \sum_{J_i \subseteq I_n} (f(y_i + k) - f(y_i)) \leq \sum_1^N]f(x_h_n) - f(x_n)] < v\left(m(E_{uv}) + \epsilon\right)$$

Putting it all together then gives

$$u(mm(E_{uv}) - 2\epsilon) < v(m(E_{uv}) + \epsilon)$$

$$um(E_{uv}) \leq vm(E_{uv})$$

Thus when we have $u > v \implies m(E_{uv}) = 0$ and (1) is proven.

(2) is easy to see since f' is the limit of a measurable function. More precisely we have

$$f'(x) = g(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Define the sequence

$$g_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} \longrightarrow g(x), \qquad a.e$$

Set $f(x) = f(v)$ for all $x > b$. Since f is increasing we know that f is measurable and so the g_n are measurable and so $g = f'$ is measurable. The last part of (2) will follows from (3). So, we turn our attention to (3).

To prove (3) we will use Fatou's Lemma. As $f \nearrow$ the $g_n \geq 0$. Fatou tells us that

$$\begin{aligned}
\int_a^b g(x)dx &\leq \liminf \int_a^b g_n(x)dx \\
&= \liminf_{n\to\infty} n \int_a^b f(x+\frac{1}{n}) - f(x)]dx \\
&= \liminf_{n\to\infty} \left[n \int_b^{b+\frac{1}{n}} f(x) - n \int_a^{a+\frac{1}{n}} f(x)dx \right]
\end{aligned}$$

We know that

$$\lim_{n\to\infty} \left(n \int_b^{b+\frac{1}{n}} f(x)dx \right) = f(b)$$

And by monotonicty we kow that

$$\liminf_{n\to\infty} \left(-n \int_a^{a+\frac{1}{n}} f(x)dx \right) \leq -f(a)$$

So we have

$$\int_a^b g(x)dx \leq \liminf \int_a^b g_n(x)dx \leq f(b) - f(a)$$

□

Corollary 17. *Every BV function is differentiable almost everywhere.*

9.5 The Cantor-Lebesgue Function

There is a continuous function $F : [0,1] \to [0,1]$ with $F(0) = 0, F(1) = 1,\ F'(x) = 0$ almost everywhere. This implies that

$$\int_0^1 F'(x)dx = 0 \neq F(1) - F(0) = 1$$

Let $\mathcal{C} = \bigcap_0^\infty C_k$ be the standard Cantor set. We let $F_0(x) = x$. We want $F_k(x)$ to be constant on the complement of C_k and increasing linearly on C_k so it is continuous. We have

$$|F_{k+1}(x) - F_k(x)| \leq \frac{1}{2^k} \implies F_k \to F, \text{ uniformly}$$

So F is continuous and increasing, but constant on $\mathcal{C}^c$.

9.6 Absolute Continuity

Definition 58. *A function $F : [a, b] \to \mathbb{R}^d$ is **absolutely continuous** if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all disjoint $\{(a_k, b_k)\}$ if*

$$\sum_1^N (b_k - a_k) < \delta \implies \sum_1^N |F(b_k) - F(a_k)| < \epsilon$$

Definition 59 (Abstract Definition from Ch. 6)**.** *If μ, ν are measures on $(X, \mathcal{M})$ then μ is **absolutely continuous** with respect to ν if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $E \in \mathcal{M}$ then*

$$\nu(E) < \delta \implies \mu(E) < \epsilon$$

This is denoted by $\mu \ll \nu$.

Proposition 15. *Let m denote Lebesgue measure and μ_F be the Lebesgue-Stiltjes measure with $\mu_F([a, b]) = F(b) - F(a)$. It holds that*

$$F \text{ is absolutely continuous} \iff \mu_F \ll m$$

9.6.1 Remarks

1. If F is absolutely continuous then F is uniformly continuous. (Take $N = 1$).

2. If F is absolutely continuos then $F \in BV[a, b]$.

   Proof. Suppose without loss of generality that $[a, b] = [0, 1]$. Take $\epsilon = 1$ and choose corresponding δ. Let $\frac{1}{N} < \delta$ and let $\mathcal{P}$ be any partition. Consider the refinement $\mathcal{P} \cup \{\frac{k}{N} : k = 0, 1, \ldots, N-1\}$. We have $TV_F[\frac{k-1}{N}, \frac{k}{N}] \leq \epsilon = 1$. Thus $TV_F[0, 1] \leq \sum_1^N TV_F[\frac{k-1}{N}, \frac{k}{N}] \leq N$ and so $F \in BV[0, 1]$. □

3. $f(x)$ is integrable implies that $F(x) = \int_a^x f(y)dy$ is absolutely continuous.

   Proof. We know from a proposition from last quarter that for any integrable f and all $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $m(E) < \delta$ it holds that $\int_E |f| < \epsilon$. So we can write

$$\sum_1^N |F(b_k) - F(a_k)| = \sum \left| \int_{a_k}^{b_k} f \right| \leq \sum_1^N \int |f| = \int_E |f| < \epsilon$$

□

Theorem 34. *If F is absolutely continuous on $[a, b]$ and $F'(x) = 0$ almost everywhere then $F(x)$ is constant.*

Proof. It is enough to prove that $F(b) = F(a)$ since we could apply the same proof to the interval $[a, x]$. Define the set

$$E = \{x \in (a, b) : F'(x) = 0\}, \qquad m(E) = b - a$$

Let $\epsilon > 0$, we have

$$\lim_{h \to 0} \left| \frac{F(x + h) - F(x)}{h} \right| = 0, \qquad \forall x \in E$$

This implies that for all $\eta > 0$ there exists $I_x \subseteq [a,b]$ with $I_x \ni x$ such that the interval $I_x = [a_x, b_x]$ satisfies $|b_x - a_x| < \eta$ and $|F(b_x) - F(a_x)| \leq \epsilon(b_x - a_x)$. These intervals form a Vitali cover of E. By Vitali's covering lemma there exists a disjoint family $I_1, \ldots, I_N, I_i = (a_i, b_i)$ such that

$$\sum_1^N m(I_i) = \sum_{i=1}^N (b_i - a_i) \geq m(E) - \delta = (b-a) - \delta$$

Where $\delta = \delta(\epsilon)$ from the definition of absolute continuity. We then have

$$\sum |F(b_i) - F(a_i)| \leq \sum \epsilon(b_i - a_i) \leq \epsilon(b-a)$$

So if we consider

$$[a,b] \setminus \bigcup_{j=1}^N I_j = \bigcup_{k=1}^M [\alpha_k, \beta_k], \qquad \sum_1^M (\beta_k - \alpha_k) < \delta$$

Therefore we have by absolute continuity it holds that

$$\sum_1^M |F(\beta_k) - F(\alpha)| < \epsilon$$

This implies

$$|F(b) - F(a)| \leq \epsilon(b-a) + \epsilon$$

Since ϵ is arbitrary this gives $F(b) = F(a)$. □

Theorem 35. *If F is absolutely continuous on $[a,b]$ this implies that F' exists almost everywhere and is integrable. Moreover,*

$$F(x) - F(a) = \int_a^x F'(Y)dy, \qquad \forall x \in [a,b]$$

Conversely, if f is integrable on $[a,b]$ then ther exists an absolutely continuous function F such that $F'(x) = f(x)$ almost everywhere and

$$F(x) = \int_a^x f(y)dy$$

Proof. The only part of the theorem that we don't already know is the statement

$$F(x) - F(a) = \int_a^x F'(y)dy$$

To prove this consider $G(x) = \int_a^x F'(y)dy$. We know that $G'(x) = F'(x)$ almost everywhere from the Lebesgue Differentiation Theorem. Therefore we have $(G(x) - F(x))' = 0$ almost everywhere. Since $G - F$ is absolutely continuous it follows that $G - F$ is constant. Let $G - F = F(a)$. It follows that

$$F(x) - F(a) = G(x) = \int_a^x f(y)dy$$

□

10 Signed Measures

(This material comes from Ch. 6 of Folland)

10.1 Absolute Continuity of Measures

Definition 60. *Let $(X, \mathcal{M})$ be a measurable space. The function $\nu : \mathcal{M} \to [-\infty, \infty]$ is a* ***signed measure on*** *X if it is countably additive. i.e. for $\{E_j\} \subseteq \mathcal{M}$ disjoint sets it holds that $\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu(E_j)$.*

10.1.1 Example

1. If f is measurable and real-valued on $(X, \mathcal{M}, \mu)$. Since we can always write $f = f^+ - f^-$ assume that $\int f^- d\mu < \infty$. Then we can take
$$\nu(E) := \int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$$
and ν is a signed measure.

2. If μ_1, μ_2 are measures then $\nu = \mu_1 - \mu_2$ is a signed measure.

Definition 61. *The* ***total variation*** *of ν on $\mathcal{M}$ is*
$$|\nu|(E) = \sup_{\mathcal{P}} \sum_{j=1}^{\infty} |\nu(E_j)|, \qquad \mathcal{P} \text{ a partition of } E \text{ into disjoint sets } E_j$$

Proposition 16. *$|\nu|$ is a measure.*

Proof. See Stein and Shakarchi. □

Definition 62. *Define the functions ν^+, ν^- by*
$$\nu^+ := \frac{\nu + |\nu|}{2} \geq 0$$
$$\nu^- := \frac{|\nu| - \nu}{2} \geq 0$$
The ***Jordan Decomposition*** *is*
$$\nu = \nu^+ - \nu^-, |\nu| = \nu^+ + \nu^-$$

Definition 63. *Let ν and μ be signed measures on $(X, \mathcal{M})$. We say that ν, μ are* ***mutually singular*** *if there exist $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ such that $\mu(A^c) = 0$, $\nu(B^c) = 0$. We denote this by*
$$\nu \perp \mu$$

Definition 64. *We say that a signed measure ν is* ***supported*** *on A if*
$$\forall E \in \mathcal{M}, \nu(E) = \nu(E \cap A)$$
Don't confuse this with $\operatorname{supp}(\nu) = \bigcap F$ *such that F is closed and $\mu(F^c) = 0$.*

Definition 65. *A measure ν is absolutely continuous with respect to μ if $E \in \mathcal{M}$ has $\mu(E) = 0$ then $\nu(E) = 0$. We will call this property $(*)$ and denote it by*
$$\nu \ll \mu$$

10.1.2 Example

If $f \in L^1(X, \mu)$ then $\nu(E) = \int_E f d\mu$ is a signed measure and $\nu \ll \mu$. We write

$$d\nu = f d\mu$$

Proposition 17. *Let (**) be the property that for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta) \implies \nu(E) < \epsilon$.*

1. $(**) \implies (*)$.
2. *If $|\nu|$ is finite then $(*) \implies (**)$.*

Proof. Suppose not. Then there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$ there exists E_n satisfying $\mu(E_n) \leq \frac{1}{2^n}$ but $\nu(E_n) \geq \epsilon$. Consider $\limsup E_n = \bigcap \bigcup E_n$. We have

$$\mu\left(\bigcup_{n \geq k} E_n\right) \leq \frac{1}{2^{k-1}}$$

This implies hat $\mu(E) = 0$. So we have

$$\nu(E) = \lim_{k \to \infty} \nu\left(\bigcup_{n \geq k} E_n\right) \geq \epsilon > 0$$

□

10.2 Radon-Nikodym Theorem

Theorem 36 (Radon-Nikodym)**.** *Suppose that μ is a σ-finite positive measure on $(X, \mathcal{M})$ and ν is a σ-finite signed measure on $(X, \mathcal{M})$. Then there exist unique signed measures ν_{ac} and ν_{sing} on $\mathcal{M}$ such that $\nu = \nu_{ac} + \nu_{sing}$ with*

$$\nu_{ac} \ll \mu, \qquad \nu_{sing} \perp \mu$$

Moreover, $d\nu_{ac} = f d\mu$.

Proof. We delay the proof until we know Hilbert Spaces. □

Corollary 18. *If $\nu \ll \mu$ then there exists f such that $d\nu = f d\mu$. f is called the Radon-Nikodym derivative of ν with respect to μ. We write*

$$f = \frac{d\nu}{d\mu}$$

10.2.1 Digression

1. A special case is ν-positive on $\mathbb{B}_{\mathbb{R}}$ and $\nu \ll m$. Consider $F(x) = \nu((-\infty, x])$ note that F is absolutely continuos and so ν satisfies $(**)$. This implies that there exists F' a.e. and $F(x) = \int_0^x F'(y)dy$. Thus $d\nu = F'dx$.
2. Consider ν a measure on $\mathcal{B}_{\mathbb{R}}$ and write $\nu = \nu_{ac} + \nu_{sing}$, where singular means singular with respect to Lebesgue by default. Small digression - if $\nu \perp m$ then we can write $\nu = \nu_{sc} + \nu_d$ with

$$\nu_{sc}(\{y\}) = 0, \qquad \forall t \in \mathbb{R}$$

$$\nu_d = \sum_1^\infty b_i \delta_x$$

10.2.2 Example

1. Let $F(x)$ be the Cantor-Lebesgue function and define $\nu_F((a,b]) = F(b) - F(a)$ and extend so that $\nu_F \perp m$.

2. There exists singular measures whose compact support is $\mathbb{R}$ or $[0,1]$. For $p \in [0,1]$ consider the measure μ_p a measure on $[0,1]$ where we look at probabilities on disjoint interval and let the intervals shrink in a binary method. It can be shown that for $p \neq q$ it holds that $\mu_p \perp \mu_q$.

11 Hilbert Spaces

Definition 66. *A **pre-Hilbert space**, $\mathcal{H}$, is a complex vector space over $\mathbb{C}$ has an inner product*

$$\mathcal{H} \times \mathcal{H} \to \mathbb{C}, \qquad (x, y) \mapsto \langle x, y \rangle \quad \textit{(denoted by } (x, y)\textit{)}$$

which satisfies

1. *$x \mapsto (x, y)$ is linear for all $y \in \mathcal{H}$.*
2. *$(y, x) = \overline{(x, y)}$*
3. *$(x, x) \in (0, \infty)$ for all $x \neq 0$.*

Let

$$\|x\| := (x, x)^{1/2}$$

.

Lemma 16 (Cauchy-Schwartz Inequality)**.** *It holds that*

$$|(x, y)| \leq \|x\| \|y\|$$

Proof. If $(x, y) = 0$ we are done. Let $(x, y) \neq 0$. So, $y \neq 0$. Let

$$\alpha = \operatorname{sign}(x, y) = \frac{(x, y)}{|(x, y)}$$

and define $z = \alpha y$. We then have $(x, y) = \alpha |(x, y)|$ and so

$$(x, z) = (x, \alpha y) = \overline{\alpha}(x, y) = \overline{\alpha}\alpha |(x, y)| = |(x, y)| = (z, x)$$

Define the function $Q : \mathbb{R} \to \mathbb{R}$ by

$$Q(t) = \|x - tz\|^2$$

We have

$$0 \leq Q(t) = (x - tz, x - tz) = \|x\|^2 - t(z, x) - (x, tz + t^2 \|z\|^2 = \|x\|^2 - 2t|(x, y)| + \|y\|^2 \geq 0$$

Because of simple facts about parabolas. And equality holds if and only if $x = tz$ if and only x and y are linearly dependent. □

Lemma 17. *$\| \cdot \|$ is a norm on $\mathcal{H}$ and so $\mathcal{H}$ is a metric space.*

Proof.

1. We have $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
2. $\|\lambda x\| = |\lambda| \, \|x\|$ since
$$\|\lambda x\|^2 = (\lambda x, \lambda x) = \lambda \overline{\lambda}(x, x) = |\lambda|^2 \, \|x\|^2$$
3. The triangle inequality holds because
$$\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2 \, \|x\| \, \|y\|$$

□

Definition 67. *A pre-Hilbert space is a **Hilbert Space** if it is complete with respect to the metric in the previous lemma.*

11.0.3 Remark

The following containment is proper: {Hilbert Spaces} $\subsetneq$ {Banach Spaces}.

11.0.4 Example

The space $\mathbb{C}^n$ is a Hilbert space. But the golden example is $L^2(X, \mathcal{M}, \mu)$.

11.1 $L^2(X, \mathcal{M}, \mu)$

Definition 68. *Let* $L^2(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} \mid \int |f|^2 < \infty\}$ *We denote*

$$\|f\|_2 = \|f\|_{L^2} = \left(\int |f|^2\right)^{1/2}$$

Definition 69. *We define* $(f, g) := \int f\overline{g}d\mu$ *so indeed* $\|f\|_2 = (f, f)^{1/2}$.

Lemma 18. L^2 *is a pre-Hilbert space. That is* L^2 *satisfies*

(i) L^2 *is a vector space*

(ii) (f, g) *is an inner product*

Proof. Left to reader. □

Theorem 37. L^2 *is complete and thus a Hilbert space.*

Proof. Recall that a normed space is complete if and only if every absolutely convergent sequence converges. By definition

$$f_n \to f \iff \|f_n - f\|_2 \to 0 \iff \|f_n - f\|_2^2 \to 0 \iff \int |f_n - f|^2 d\mu \to 0$$

Given a series $\sum_1^\infty f_k$ absolute convergence mean that $\sum_1^\infty \|f_k\|_2^2 < \infty$. Denote

$$G_n(x) = \sum_1^n |f_k(x)|, \quad G = \sum_1^\infty |f_k(x)|, \quad G_n \geq 0, G_n^2 \nearrow G^2 \quad \text{a.e.} x$$

We have

$$\|G_n\|_2 \leq \sum_1^n \|f_k\|_2 \leq B$$

By the monotone convergence theorem it holds that

$$\|G_n\|_2^2 = \int G_n^2 \to \int G^2 = \|G\|_2^2 \leq B$$

Hence $G \in L^2$. So we have

$$\int G^2(x)d\mu < \infty \implies G(x) < \infty \text{ a.e. } x \implies \sum_1^\infty f_k(x) \text{ converges for a.e. } x$$

It remains to show that the series converges in L^2. We have

$$\left|F - \sum_1^n f_k\right|^2 \leq (2|G(x)|)^2 \in L^1$$

Now, using dominated convergence we have

$$\left|S(x) - \sum_1^n f_k(x)\right|^2 \to 0 \quad a.e. \implies \int \left|F(x) - \sum_1^n f_k(x)\right|^2 \to 0$$

Which is precisely what we wanted to prove. □

Proposition 18. *The set of simple functions* $\left\{\sum_1^n a_j \mathbb{1}_{E_j} : \mu(E_j) < \infty, a_j \in \mathbb{C}\right\}$ *is dense in* L^2.

Clearly the simple functions are in L^2. If E_j are disjoint then

$$\left\|\sum_1^n a_j \mathbb{1}_{E_j}\right\|^2 = \sum_1^n |a_j|^2 \mu(E_j) < \infty$$

Let $f \in L^2$, there exist simple $f_n \nearrow f$ almost everywhere. This implies that $f_n \in L^2$ and so $|f_n - f|^2 \leq (2|f|)^2 \in L^1$ And this implies from the dominated convergence theorem that $\|f_n - f\| \to 0$. note that all f_n have $\mu(E_j) < \infty$.

11.1.1 Example

Look in $L^2(\mathbb{R}^d)$ and define the spaces $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$ with $\mathbb{Z}$ or $\mathbb{N}$ with the counting measure as

$$\ell^2(\mathbb{Z}) = \{(a_j)_{j\in\mathbb{Z}} : a_j \in \mathbb{C}, \|a\|_2^2 \sum_{j\in\mathbb{Z}} |a_j|^2 < \infty$$

11.2 Orthogonality

Let $x, y \in \mathcal{H}$. We know that $x \perp y$ if $(x, y) = 0$.

Proposition 19. *If* $x \perp y$ *then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Proposition 20. *We know that*

$$\|x + y\|^2 = (x + y, x + y) = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(x, y)$$

But $\mathrm{Re}(x, y) = 0$ *so we are done.*

Definition 70. *The set* $\{e_j\} \subseteq \mathcal{H}$ *is an* ***orthonormal*** *set if* $(e_i, e_j) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Corollary 19. *If* $\{E_j\}^n$ *is orthonormal then*

$$\left\|\sum_1^N a_j e_j\right\|^2 = \sum_1^N \|a_j e_j\|^2 = \sum_1^N |a_j|^2$$

Definition 71. *Let $\{e_j\}_{j\geq 1}$ be finite or countable infinite is an* ***orthonormal basis*** *for $\mathcal{H}$ if the set of finite linear combinations of the e_i's are dense in $\mathcal{H}$.*

$$\overline{\text{Span}\,\{e_j\}_{j\geq 1}} = \mathcal{H}$$

Theorem 38. *Suppose that $\mathcal{H}$ is separable. Let $\{e_j\}_{j\geq 1}$ be an orthonormal set. The following are equivalent*

(i) $\{e_j\}_{j\geq 1}$

(ii) $(f, e_j) = 0$ *for all* $j \implies f = 0 \in \mathcal{H}$.

(iii) $S_N(f) := \sum_1^N (f, e_j)e_j \to f$ *in* $\mathcal{H}$ *for all* $f \in \mathcal{H}$.

(iv) $\forall f \in \mathcal{H}$, $a_j = (f, e_j)$ *satisfy* $\sum |a_j|^2 = \|f\|^2$. *(Parseval's identity).*

Proof. It is clear that $(iii) \implies (i)$. We will prove that

$$(i) \implies (ii) \implies (iii) \implies (iv) \implies (iii)$$

$\boxed{(i) \implies (ii)}$ Let $f \in \mathcal{H}$ with $(f, e_j) = 0$ for all $j \geq 1$. So we have $(f, g) = 0$ for all $g \in \text{Span}\{e_j\}_{j\geq 1}$. Thus for all $\epsilon > 0$ ther exists $g \in \text{Span}\{e_j\}_{j\geq 1}$ such that $\|f - g\| < \epsilon$. We know that $(A, g) = 0$ So we have

$$\|f\|^2 = (f, f) = (f, f) - (f, g) = |(f, f - g)| \leq \|f\|\,\|f - g\| \leq \epsilon\,\|f\|$$

If $\|f\| \neq 0$ we get $\|f\| < \epsilon$ for all ϵ.

$\boxed{(ii) \implies (iii)}$ Observe that

$$(f - S_N(f), S_N(f)) = \left(f - \sum_1^n (f, e_j)e_j, \sum_1^N (f, e_k)e_k\right) = 0$$

Since $(f - S_N(f), e_k) = 0$ for all $k \leq N$ we know that the above holds. Next observe that

$$\|f\|^2 = \|f - S_N\|^2 + \|S_N(f)\|^2 = \|f - S_N(f)\|^2 + \sum_1^N |a_j|^2 \geq \sum_1^N |a_j|^2$$

This implies that as $N \to \infty$

$$\sum_1^\infty |a_j| \leq \|f\|^2 < \infty$$

The above inequality is called Bessel's inequality. First we prove that $\{S_N(f)\}_1^\infty$ is Cauchy. This is true since

$$\|S_n(f) - S_M(f)\|^2 = \left\|\sum_{j=M+1}^N a_j e_j\right\|^2 = \sum_{M+1}^N |a_j|^2 \to 0$$

Since $\mathcal{H}$ is compete we know that $S_N(f) \to g \in \mathcal{H}$. Observe that

$$(f - S_N(f), e_j) = 0, \qquad \forall N \geq j$$

But this is equal to

$$(f, e_j) - (f, e_j)$$

Whence

$$(f-g, e_j) = 0, \qquad \forall j$$

This gives $f - g = 0$ by (ii) the implies that $S_n(f) \to f$.

Note: We used that fact that $x_n \overset{\mathcal{H}}{\to} x$ implies that for all $y \in \mathcal{H}$ it holds that $(x_n, y) \to (x, y)$ we have

$$|(x_n, y) - (x, y)| = |(x_n - x, y)| \leq \|x_n - x\| \, \|y\| \to 0$$

$\boxed{(iii) \implies (iv)}$ We proved that

$$\|f\|^2 = \|S_N f - f\|^2 + \sum_1^N |a_j|^2$$

So if we let $N \to \infty$ we obtain

$$\|f\|^2 = \sum_1^\infty |a_j|^2.$$

$\boxed{(iv) \implies (iii)}$ But this follows immediately from the proof of $(iii) \implies (iv)$ since the statements are iff. □

Theorem 39. *Any separable Hilbert space has an orthonormal basis.*

Proof. The finite dimensional case is covered by Graham-Schmidt. Start with $\{h_k\}_1^\infty$ a countable dense subset of $\mathcal{H}$. Take $h_{k_1} \neq 0$ and let h_{k_2} be such that $\{h_{k_1}, h_{k_2}\}$ is linearly independent. Keep going to obtain $\{h_{k_i}\}_{i=1}^N$ which are linearly independent in $\mathcal{H}$ and $\text{Span}\{h_{k_i}\}_1^N$. We orthogonalize each finite set with Graham-Schmidt so that $\{h_{k_i}\}_1^\infty$ are linearly independent and $\overline{\text{Span}\{h_{k_i}\}} = \mathcal{H}$. □

11.2.1 Example

Let $\mathcal{H} = L^2([-\pi, \pi], m)$ with $(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$. We have

$$\{h_k\}_1^\infty = \{e^{inx}\}_{n \in \mathbb{Z}}$$

is an orthonormal basis. So for all $f \in L^2$ we know that $f \sim \sum_{n \in \mathbb{Z}} a_n e^{inx}$ where $a_n = (f, e^{inx})$. And so Fourier series converge in L^2 for all $f \in l^2$ and

$$\|f\|^2 = \sum_{n \in \mathbb{Z}} |a_n|^2$$

.

11.3 Unitary Opertors

Definition 72. *An **operator** is a linear transformation.*

Definition 73. *A **unitary operator** is a map $U : \mathcal{H}_1 \to \mathcal{H}_2$ which is*

(i) Linear

(ii) Bijective

(iii) Isometry. i.e. for all $x \in \mathcal{H}_1$, $\|Ux\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}$

11.3.1 Remarks

- U is unitary implies that U^{-1} is unitary.
- U is unitary implies that $(Ux, Uy)_{\mathcal{H}_2} = (x, y)_{\mathcal{H}_1}$. This follows from *polarization* which says

$$\forall F, G \in \mathcal{H}, (F, G) = \frac{1}{4}\left[\|F+G\|^2 - \|F_G\|^2\right] + i\left[\|F/i+G\|^2 - \|F/i-G\|^2\right]$$

Definition 74. *$\mathcal{H}$ and $\mathcal{H}'$ are isomorphic or unitarily equivalent if $\exists U : \mathcal{H} \to \mathcal{H}'$ such that U is unitary.*

Corollary 20. .

- *A finite dimensional Hilbert space is isomorphic to $\mathbb{C}^N$ for some $N \in \mathbb{N}$.*
- *Any two infinite dimensional separable Hilbert spaces are isomorphic. (So $\ell^2(\mathbb{Z})$ models all infinite Hilbert spaces).*

Proof. For the infinite dimensional case we let $\{e_j\}$ and $\{e'_j\}$ be orthogonal bases. Define

$$U(e_j) = e'_j$$

We know that every $f \in \mathcal{H}$ can be written $f = \sum a_j e_j$ where $a_j = (f, e_j)_{\mathcal{H}}$. So, U extends to be

$$U(f) = \sum a_j e'_j$$

which is clearly linear and bijective. Moreover, we know that U is an isometry since

$$\|U(f)\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$$

from Parseval's identity which says that $\|f\|^2_{\mathcal{H}} = \sum |a_j|^2$. □

11.4 Completion of pre-Hilbert Space

Let $\mathcal{H}_0$ be a pre-Hilbert space. For example, $\mathcal{H}_0 = \{$polynomials on $[-\pi, \pi]\}$ or $\{$Riemann integrable functions on [-

Proposition 21. *Suppose $\mathcal{H}_0$ is a pre-Hilbert space. Then there exists $\mathcal{H}$ a Hilbert space with an inner product such that*

(i) $\mathcal{H}_0 \subseteq \mathcal{H}$

(ii) $(f, g) = (f, g)_0$

(iii) *$\mathcal{H}_0$ is dense in $\mathcal{H}$¿*

Proof. (sketch) Let $\mathcal{H}$ be the set of equivalence classes of Cauchy sequences $\{f_n\}$. We have

$$\{f_n\} \sim \{g_n\} \iff f_n - g_n \to 0 \quad \text{in } \mathcal{H}_0$$

We embed $\mathcal{H}_0$ with $f \mapsto \{f, f, \ldots, \}$. The hard part is seeing that $\mathcal{H}$ is complete. We have $F_k \to F$ in $\mathcal{H}$ where $F = [\{f^{(n)}_{N(n)}\}]$ and $f_n = f^{(n)}_{N(n)}$ with for all $j \geq N(n)$

$$\left|f^{(n)}_{N(n)} - f^{(n)}_j\right| \leq \frac{1}{n}$$

□

11.5 Closed Subspaces and Orthogonal Projections

Definition 75. *$\mathcal{S} \subseteq \mathcal{H}$ is a **closed subspace** if it is a subspace and for any sequence $\{f_n\} \to f$ with $\{f_n\} \subseteq \mathcal{S}$ it holds that $f \in \mathcal{S}$.*

Lemma 19. *Suppose § is a closed subspace of $\mathcal{H}$ and $f \in \mathcal{H}$. Then*

1. *There exists a unique $g_0 \in \mathcal{S}$ such that*

$$\|f - g_0\| = \inf_{g \in S} \|f - g\| .$$

2. *$(f - g_0, g) = 0$ for all $g \in \mathcal{S}$.*

Proof. If $f \in \mathcal{S}$ we are done. So assume that $f \notin$ §. Let $d = \inf_{g \in \mathcal{S}} \|f - g\|$. Note that d is positive since $\mathcal{S}$ is closed. Also, d is finite since the norm is finite. This means that there exists a sequence $g_n \in \mathcal{S}$ such that

$$\|f - g_n\| \to d$$

We will show that g_n is Cauchy with limit g_0. The identity which will prove this is know as the **parallelogram law** which states that

$$\|A + B\|^2 + \|A - B\|^2 = 2\left(\|A\|^2 + \|B\|^2\right), \qquad \forall A, B \in \mathcal{H}$$

Let $A = f - g_n$ and $B = f - g_m$ we have

$$\|2f - g_n - g_m\|^2 + \|g_m - g_n\|^2 = 2\left[\|f - g_n\|^2 + \|f - g_m\|^2\right]$$

This means that $\frac{g_n + g_m}{2} \in \mathcal{S}$ which implies

$$\|2f - (g_n + g_m)\| = 2\left\|f - \frac{g_n + g_m}{2}\right\| \geq 2d$$

Hence

$$\|g_m - g_n\|^2 \leq 2\left[\|f - g_m\|^2 + \|f - g_n\|^2\right] - 4d^2 \longrightarrow 0, \quad m, n \to \infty$$

This means that $\{g_n\}$ is Cauchy. So there exists $g_0 = \lim g_n \in \mathcal{S}$ and $\|f - g_0\| = \lim \|f - g_m\|$. Uniqueness of g_0 follows from orthogonality in (2) which we will now prove.

Now we prove (2) by perturbing. For al ϵ with $|\epsilon|$ small consider $g_0 - \epsilon g$. So we have

$$\|f - (g_0 - \epsilon f)\|^2 \geq \|f - g\|^2$$

But the left side is equal to

$$\|f - g_0\|^2 + \epsilon^2 \|g\|^2 + 2\epsilon \operatorname{Re}(f - g_0, g)$$

Therefore

$$2\epsilon \operatorname{Re}(f - g_0, g) + \epsilon^2 \|g\|^2 \geq 0$$

Which means that

$$\operatorname{Re}(f - g_0, g) = 0.$$

Similarly, we could perturb in the complex direction with $g - i\epsilon g_0$. In this case we get

$$\operatorname{Im}(f - g_{,}g) = 0.$$

□

Definition 76. *Define the* ***orthogonal projection*** *of f to $\mathcal{S}$ to be the g_0 from the previous lemma, denote*

$$P_{\mathcal{S}} f = g_0$$

Definition 77. *Let $\mathcal{S} \subseteq \mathcal{H}$, and let $\mathcal{S}^{\perp} = \{f \in \mathcal{H} : (f, g) = 0, \forall g \in \S\}$. We call $\mathcal{S}^{\perp}$ the* ***orthogonal complement****.*

11.5.1 Properties of Projections

1. $P_{\mathcal{S}}$ is linear: $f \mapsto P_{\mathcal{S}} f$ is a linear transformation
2. If $f \in \mathcal{S}$ this implies that $P_{\mathcal{S}} f = f$.
3. $P_{\mathcal{S}} f = 0 \iff f \in \mathcal{S}^{\perp}$
4. $\|P_{\mathcal{S}} f\| \leq \|f\|$
5. If $\{e_k\}$ is an orthonormal set in $\mathcal{H}$ then $f \mapsto_{k \geq 1} \sum (f, e_k) e_k$ is the orthogonal projection to the closed span of $\{e_k\}$.

11.5.2 Properties

1. $\mathcal{S} \perp$ is a subspace
2. $\mathcal{S} \cap \S^{\perp} = \{0\}$
3. $\mathcal{S}^{\perp}$ is closed.

Proposition 22. *Let $\mathcal{S}$ be closed and write $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$ which means by definition $\forall f \in \mathcal{H}, \exists! g \in \mathcal{S}, h \in \mathcal{S}^{\perp}$ such that $f = g + h$.*

Proof. Let $f \in \mathcal{H}$ and let $g = P_{\mathcal{S}} f$, then $h = f - g \in \mathcal{S}^{\perp}$, uniqueness follows from Property 2. □

11.6 Linear Transformations

11.6.1 Examples

Both unitary transformations and projections are linear transformations.

Definition 78. *Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be linear. T is* ***bounded*** *if there exists M such that*

$$\|Tf\| \leq M \|f\|$$

Definition 79. *We define the* ***norm of T*** *as*

$$\|T\| = \sup_{f \neq 0} \frac{\|Tf\|}{\|f\|}$$

Observation 17. *We have*

$$\|T\| = \sup\{\|Tf\| : \|f\| = 1\}$$

The observation is true because for $g \neq 0$ we can write $g = \|g\| \left(\frac{g}{\|g\|}\right)$ and then

$$\|Tg\| = \|g\| \left\| T\left(\frac{g}{\|g\|}\right) \right\|$$

11.6.2 Note:

$$\|Tf\| \leq \|T\| \, \|f\|$$

11.6.3 Example

The zero operator , $\mathcal{O} : f \mapsto 0$, and the identity operator $I : f \mapsto f$ make

$$\mathcal{L}(\mathcal{H}) = \{\text{bounded operators from } \mathcal{H} \to \mathcal{H}\}$$

is an algebra with the unit I.

Lemma 20.

$$\|T\| = \sup\{(Tf, g) : \|f\| \leq 1, \|g\| \leq 1\}$$

Proof. We have

$$|(Tf, g)|| \leq \|Tf\| \, \|g\| \leq \|T\| \, \|f\| \, \|g\| \leq \|T\|$$

For $\geq$ take

$$g = \frac{Tf}{\|Tf\|}$$

□

Definition 80. *A linear operator is* ***continuous*** *means that if $f_n \to f$ then $T(f_n) \to T(f)$.*

Lemma 21. *T is bounded if and only if T is continuous.*

Proof. ($\Rightarrow$) Let $f_n \to f$, this means that $\|f_n - f\| \to 0$. Hence

$$\|Tf_n - Tf\| = \|T(f_n - f)\| \leq \|T\| \, \|f_n - f\| \to 0$$

($\Leftarrow$) Suppose not bounded. Then there exists f_n, $\|f_n\| = 1$ but $\|Tf_n\| \geq n$. So if we consider

$$\frac{f_n}{n} \to 0, \quad \text{but} \quad \left\| T\left(\frac{f_n}{n}\right) \right\| = \frac{\|Tf_n\| \, n}{\geq} 1$$

Hence $T\left(\frac{f_n}{n}\right) \not\to 0$, which means T is not continuous. □

11.7 Linear Functionals

This material comes from Stein and Shakarchi III.

Definition 81. *A* ***linear functional*** *is any linear function $\ell : \mathcal{H} \to \mathbb{C}$.*

11.7.1 Example

$\ell(f) : \mathcal{H} \to \mathbb{C} : g \mapsto (f, g)$ is a bounded linear functional since

$$|(f, g)| \leq \|f\| \, \|g\| \implies \|\ell(f)\| \leq \|g\|$$

Theorem 40 (Riesz)**.** *For every continuous functional $\ell : \mathcal{H} \to \mathbb{C}$ there exists a unique $g \in \mathcal{H}$ such that $\ell(f) = (f, g)$ for every $f \in \mathcal{H}$. Moreover, $\|l\| = \|g\|$.*

Proof. Given ℓ, consider the nullspace of ℓ. let $\mathcal{S} = \{f : \ell(f) = 0\}$, which is a closed subspace. First, if $\mathcal{S} = \mathcal{H}$ then $\ell = 0$. Take $g = 0$. Otherwise if $\mathcal{S}^\perp \neq \{0\}$ then pick $h \in \mathcal{S}^\perp$ with $\|h\| = 1$. We choose g to be

$$g = \overline{\ell(h)} h$$

and it is an easy verification from here.

□

11.8 Adjoints

Definition 82. *Let $T : \mathcal{H} \to \mathcal{H}$ be bounded and linear. Then there exists a unique map $T^* : \mathcal{H} \to \mathcal{H}$ which is bounded, linear and satisfies*

(i) $(Tf, g) = (f, T^*g), \qquad \forall f, g \in \mathcal{H}$

(ii) $\|T^*\| = \|T\|$

(iii) $(T^*)^* = T$.

*We call T^* the **adjoint** to T.*

Proof. (i) Fix $g \in \mathcal{H}$ and consider $\ell(f) = (Tf, g)$, is a linear functional $\ell : \mathcal{H} \to \mathbb{C}$. This is bounded since

$$|\ell(f)| = |(Tf, g)| \leq \|Tf\| \, \|g\| \leq \|T\| \, \|f\| \, \|g\|$$

So let $M = \|T\| \, \|g\|$.

By Riesz Representation Theorem there exists $h \in \mathcal{H}$ such that $\ell(f) = (f, h)$. Define $T^*g := h$, then by definition

$$(Tf, g) = \ell(f) = (f, g) = (f, T^*g)$$

(ii) We had a lemma that

$$\begin{aligned} \|T\| &= \sup\{|(Tf, g)| : \|f\| \leq 1, \|g\| \leq 1\} \\ &= \sup\{|(f, T^*g)| : \|f\| \leq 1, \|g\| \leq 1\} \\ &= \|T^*\| \end{aligned}$$

(iii) We know that $(T^*)^* = T$ and so $(Tf, g) = (f, T^*g)$. Rest follows quickly. □

11.8.1 Remarks

1. If $T : \mathbb{C}^N \to \mathbb{C}^N$ has a matrix A then T^* has the matrix $\overline{A}^t$ (transpose complex conjugate).
2. T is said to be **symmetric** (alternatively **self-adjoint**) if $T^* = T$.

   (It is true that if $T = T^*$ then $\|T\| = \sup\{|(Tf, f)| : \|f\| \leq 1\}$.)
3. $(TS)^* = S^*T^*$

11.9 Radon-Nikodym Theorem

Theorem 41 (Radon-Nikodym Theorem)**.** *(I) Let μ be a σ-finite positive measure on $(X, \mathcal{M})$ and ν a σ-finite signed measure on $(X, \mathcal{M})$. Then there exist unique signed measures ν_a, ν_s on $\mathcal{M}$ such that*

$$\nu_a = \nu + \nu_s, \qquad \nu_a \ll \mu, \nu_s \perp \mu$$

Moreover, $d\nu_a = f d\mu$ for some extended μ-integrable f. i.e.

$$\nu_a(E) = \int_E f d\mu, \qquad E \in \mathcal{M}$$

Proof. The main part is to assume that μ and ν are both positive and finite (then we use σ-finiteness and signed measure on ν).

Let $\rho = \mu + \nu$. Consider the Hilbert space $\mathcal{H} = l^2(X, \rho)$. Recall that

$$(f, g) = \int_X f\overline{g}$$

Define the functional

$$\ell(\psi) = \int \psi d\nu$$

We claim that $\ell(\psi)$ is a bounded linear functional. Linearity is clear by linearity of integrals. Boundedness follows from the following estimate

$$\left|\int \psi d\nu\right| \leq \int |\psi| d\nu \leq \int |\psi| d\rho = (|\psi|, 1)_{L^2(\rho)} \leq \|\psi\|_{L^2(\rho)} \|1\|_{L^2(\rho)} = \|\psi\|_{L^2(\rho)} \cdot \rho(X)^{1/2}$$

So by Riesz we know that there exists $g \in L^2(\rho)$ with

$$\ell(\psi) = (\psi, g)$$

We obtain the following formula

$$\int \psi(x) d\nu(x) = \int \psi(x)\overline{g}(x) d\rho(x), \quad \forall \psi \in L^2(X, \rho) \tag{1}$$

Let us take $\psi(x) = \mathbb{1}_E(x)$ for fixed $E \in \mathcal{M}$. We have

$$\nu(E) = \int_E \overline{g}(x) d\rho(x), \qquad \forall E$$

Since ρ is a positive measure it follows that $\overline{g}(x) \geq 0$ a.e. with respect to ρ. We conclude that $g(x)$ is real and nonnegative ρ-a.e. We write

$$\nu(E) = \int_E g(x) d\rho(x) \leq \rho(E0$$

This implies that

$$\int (1 - g(x)) d\rho(x) \geq 0, \qquad \forall E \implies g(x) \leq 1, \rho\text{-a.e.}$$

Therefore

$$0 \leq g(x) \leq 1, \rho\text{-a.e}$$

So we can rewrite (1) as

$$\int \psi d\nu = \int \psi g d\rho = \int \psi g d\mu + \int \psi g d\nu$$

This gives us formula

$$\int \psi(1 - g) d\nu = \int \psi g d\mu \tag{2}$$

Define the sets

$$A := \{x \in X : 0 \leq g(x) < 1\}, \qquad B = \{x \in X : g(x) = 1\}$$

We have

$$A \cap B = \emptyset, A \cup B = X$$

We can now define two measures

$$\nu_a := \nu|_A, \qquad \nu_s := \nu|_B$$

More precisely,

$$\nu_a(A) := \nu(A \cap E)$$
$$\nu_s(E) = \nu(b \cap E)$$

We claim that

$$\nu_s \perp \mu$$

So take $\psi = \psi_B$ in (2). We then have

$$0 = \int \psi|_B d\mu = \mu(B)$$

We can then say by definition

$$\mu(B) = 0 \implies \nu_s \perp \mu.$$

The next step is to work with ν_a. Define the function in (2)

$$\psi := \mathbb{1}_E(1 + g + g^2 + \cdots + g^n)$$

This gives by (2)

$$\int_E (1 - g^{n+1}) d\nu = \int_E g(1 + g + \cdots + g^n) d\mu \tag{3}$$

If we let $n \to \infty$ then

$$x \in B \implies 1 - g^{n+1}(x) = 0$$
$$x \in A \implies 1 - g^{n+1}(x) \to 1, \qquad \forall x$$

By the dominated convergence theorem we have

$$\int_E (1 - g^{n+1}) d\nu \longrightarrow \int_E \mathbb{1}_A d\nu = \nu(A \cap E)$$

So the left hand side converges to $\nu(A \cap E)$. While the right hand side

$$g(1 + g + \cdots g^n) \to \frac{g}{1-g}, \qquad \forall x \in A$$

Since everything is positive we can apply the monotone convergence theorem and conclude that the right hand side converges to $\int_E \frac{g}{1-g} d\mu$. Observe that since $\mu(A^c) = 0$ we can write

$$\int_E g(1 + \cdots + g^n) d\mu = \int_{E \cap A} f(1 + \cdots + g^n) d\mu$$

We conclude that (letting $f = \frac{g}{1-g} \in L^1(\mu)$)

$$\nu(A \cap E) = \int_{A \cap E} f d\mu$$

So this proves the main step. To finish the job we would next consider (II) the σ-finite positive case by writing

$$X = \bigcup_j E_j, E_j \in \mathcal{M}, \mu(E_j) < \infty, \nu(E_j) < \infty$$

and defining the measures $\mu_j = \mu|_{E_j}$ and $\nu_j = \nu|_{E_j}$ and applying (I).

Next we would consider (III) wher ν is signed and work with $\nu = \nu^+ - \nu^-$.

The last item to verify is uniqueness. Suppose we can write

$$\nu = \nu_a + \nu_s, \qquad \nu_a \ll \mu, \nu_s \perp \mu$$

and

$$\nu = \nu'_a + \nu'_s, \qquad \nu'_a \ll \mu, \nu'_a \perp \mu.$$

Then we have

$$\nu_a - \nu'_a = \nu'_s - \nu_s$$

if a measure is $\ll \mu$ and $\perp \mu$, then it is zero.

$\square$

12 Point Set Topology

(Foland Ch. 4) Our main goal is to prove the Stone-Weirstrauss Theorem and Arzelá-Ascoli Theorem.

Definition 83. *Let $X \neq \emptyset$. A **topology** on X is a collection $\mathcal{T} \subseteq P(X)$ such that $\emptyset, X \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions and finite intersections.*

Definition 84. *$K \subseteq X$ is **closed** if $K = U^c$ for some $U \in \mathcal{T}$.*

$$\overline{A} = \bigcap_{K \supset A} K$$

$$A^\circ = \bigcup_{U \subseteq A} U$$

Definition 85. *$A \subseteq X$ is **dense** in X if $\overline{A} = X$. $A \subseteq X$ is **nowhere dense** if $(\overline{A})^\circ = \emptyset$.*

Definition 86. *U is a **neighborhood of** x if $x \in U^\circ$. (Note that U need not be open).*

Definition 87. *x is an **accumulation point** (denoted by $x \in A'$) if every neighborhood U of x satisfies $A \cap (U \setminus \{x\}) \neq \emptyset$.*

Proposition 23. *$\overline{A} = A \cup A'$.*

Definition 88. *If $\mathcal{E} \subseteq P(X)$, then there exists a coarsest topology $\mathcal{T}(\mathcal{E})$ containing $\mathcal{E}$. Called **the topology generated by** $\mathcal{E}$.*

Definition 89. *Suppose that $\mathcal{T}$ is a topology on X. A **neighborhood base for** $\mathcal{T}$ **at** $x \in X$ is a family $\mathbb{N} \subseteq \mathcal{T}$ satisfying*

(i) For all $V \in \mathbb{N}$ we have $x \in V$.

(ii) For all $U \in \mathcal{T}$ such that $x \in U$ there exist $V \in \mathbb{N}$ such that $x \in V \subseteq U$.

Definition 90. *A **base** for $\mathcal{T}$ is a set $\mathcal{B} \subseteq \mathcal{T}$ which contains a neighborhood base for each $x \in X$.*

Proposition 24. *Let $\mathcal{T}$ be a topology on X and $\mathcal{E} \subseteq \mathcal{T}$. Then $\mathcal{E}$ is a base for $\mathcal{T}$ if and only if for all $U \in \mathcal{T}$, $U \neq 0$ we have $U = \bigcup_\alpha E_\alpha$ with $E_\alpha \in \mathcal{E}$.*

Proposition 25. *Let $\mathcal{E} \subseteq P(X)$. For $\mathcal{E}$ to be a base for a topology on X it is necessary and sufficient that*

(a) For all $x \in X$ there exists $V \in \mathcal{E}$ such that $x \in V$

(b) For all $U, V \in \mathcal{E}$ such that $x \in U \cap V$ there exists $W \in \mathcal{E}$ such that $x \in W$ and $W = U \cap V$.

Proof. Necessity is clear. To see sufficiency let

$$\mathcal{T} := \{U \subseteq X : \forall x \in U \exists V \in \mathcal{E} \text{ such that } x \in V \subseteq U\}.$$

We need to check that $\mathcal{T}$ is a topology. Property (a) implies that $X \in \mathcal{T}$. Also, $\emptyset \in \mathcal{T}$ trivially. Also, it is clear for unions. Property (b) guarantees that the intersection of two sets belongs to $\mathcal{T}$ and thus $\mathcal{T}$ is closed under finite unions.

□

Proposition 26. *Let $\mathcal{E} \subseteq P(X)$ then $\mathcal{T}(\mathcal{E}) = \{\emptyset, X\} \cup \{$unions of finite intersections of elements of $\mathcal{E}\}$*

Proof. Check that $\mathcal{T}(\mathcal{E})$ satisfies the properties of a topology. □

12.1 More Point Set Topology

Definition 91. *$(X, \mathcal{T})$ is **first countable** if it has a countable base at each point. It is **second countable** if it has a countable base. It is **separable** if it has a countable dense subset.*

Definition 92. *We say that x_n **converges to** x, denoted by $x_n \to x$ in $(X, \mathcal{T})$. If for all $u \in \mathcal{T}$ we have $U \ni x$ then there exists N satisfying for all $n \geq N$ it holds that $x_n \in U$¿*

Proposition 27. *X is first countable and $A \subseteq X$ then $x \in \overline{A} \iff \exists \{x_n\} \subseteq A : x_n \to x$.*

12.1.1 Separation Axioms

- T_0: For every $x \neq y$ either exists $U \ni x$ such that $U \not\ni y$ or $U \ni y, U \not\ni x$.
- T_1: Points are closed.
- T_2: Hausdorff
- T_3: Regular; $\forall A \subseteq X$ closed and $y \notin A, \quad \exists U \supset A, V \ni x : U \cap V = \emptyset$.
- T_4: Normal; If A, B are disjoint closed sets then they can be separated by disjoint open sets.

12.1.2 Example

- (T_1 but not T_2) Let X be an infinite set with cofinite topology.
- (T_2 but not T_3) Let $\mathcal{T} = \{U \cup (V \cap \mathbb{Q})\}$ with $U, V \subseteq \mathbb{R}$ open.

12.2 Continuity

Definition 93. *The function $f : X \to Y$ is **continuous** if $f^{-1}(V)$ is open for all open $V \subseteq Y$. A function $f : X \to Y$ is **continuous at** $x \in X$ if for all $V \ni f(x)$ there exists $U \ni x$ such that $f(U) \subseteq V$.*

Proposition 28. .

(i) Composition of continuous maps is continuous.

(ii) $f : X \to Y$ is continuous if and only if f is continuous for all $x \in X$.

(iii) If $\mathcal{T}_Y$ is generated by $\mathcal{E}$ then $f : X \to Y$ is continuous if and only if

$$f^{-1}(V) \in \mathcal{T}_X, \qquad \forall V \in \mathcal{E}$$

12.3 Weak Topology

Definition 94. *Suppose that $\{f_\alpha\}_\alpha$ is a family of maps $f_\alpha : X \to Y_\alpha$ with X a set and Y_α a topological space. The **weak topology** generated by $\{f_\alpha\}$ is the smallest topology so that all of the f_α are continuous. This topology is generated by $\{f_\alpha^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{T}_{Y_\alpha}\}$.*

Definition 95. *Let $\{X_\alpha\}$ be a collection of topological spaces. The **product topology** is defined on $X = \prod_\alpha X_\alpha$ and is the weak topology for the family of projections $\pi_\alpha : X \to X_\alpha$.*

Lemma 22. *A base for the product topology is*

$$\bigcap_1^n \pi_{\alpha_j}^{-1}(U_{\alpha_j}), \qquad n \in \mathbb{N}, U_{\alpha_j} \text{ open in } X_\alpha$$

Alternatively we can write this as

$$\prod_\alpha U_\alpha, \text{ with } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha_j$$

Proposition 29. *We have X_α is T_2 for all α implies that $X = \prod_\alpha X_\alpha$ is T_2.*

Proof. Let $x \neq y$ in X. Then there exists α such that $\pi_\alpha(x) \neq \pi_\alpha(y)$. There must then exist U_α with $V_\alpha \subseteq X_\alpha$ separating the images of x and y. We then have $\pi_\alpha^{-1}(U_\alpha), \pi_\alpha^{-1}(V_\alpha)$ separate x and y. □

Proposition 30. *A function $f : Y \to \prod X_\alpha$ is continuous if and only if $\pi_\alpha \circ f$ is continuous for all α.*

Proposition 31. *Let X be a topological space and $A \neq \emptyset$. Let $\{f_n\} \subseteq X^A$ (meaning $f_n : A \to X$). It holds that $f_n \to f$ in the product topology if and only if $f_n(\alpha) \to f(\alpha)$ for all $\alpha \in A$.*

Proof. ($\Rightarrow$) Choose $U \ni f(\alpha)$. Introduce the notation $N(U) = \pi_\alpha^{-1}(U)$. We then must have for all $n \geq N$ it hold that $f_n \in N(U) \implies f_n(\alpha) \in U$.

($\Leftarrow$) Fix a base in X^A. We let $n(U_1, \ldots, U_k) = \bigcap_1^k \pi_\alpha^{-1}(U_j)$. Then use that $f_n(\alpha_j) \to f(\alpha_j)$. □

12.3.1 Notation

We will denote $B(X, \mathbb{R}), B(X, \mathbb{C})$ to be bounded functions and $C(X, \mathbb{R}), C(X, \mathbb{C})$ to be continuous functions. $BC(X, F) = B(X, F) \cap C(X, F)$ are bounded continuous functions. By default $F = \mathbb{C}$, so we write $B(X)$ an $C(X)$.

Observation 18. *$B(X)$ and $C(X)$ are complex vector spaces.*

Definition 96. *The uniform norm $\|f\|_u = \sup\{|f(x)| : x \in X\}$. If $f \in B(X)$ then $\rho(f, g) = \|f - g\|_u$ is a metric.*

Proposition 32. *$B(X)$ is complete in $\|\cdot\|_u$.*

Proof. Let $\{f_n\}$ be Cauchy in $\|\cdot\|$ this implies that $\{f_n(x)\}$ is Cauchy for all x. This means that there exists $f(x) = \lim f_n(x)$, it is the uniform limit. □

Proposition 33. *Let X be a topological space and let $BC(X) \subseteq B(X)$ be closed in $\|\cdot\|_u$. Hence $BC(X)$ is complete.*

Proof. We know that $f_n \in BC(X)$ and $f_n \to f$ uniformly. We would like to show that f is continuous. (use a 3ϵ argument) Fix $\epsilon > 0$. So there exist N such that $\|f_n - f\|_u < \epsilon$ for all $n \geq N$. Next, fix $x \in X$. We want to show that there exists a neighborhood $U \ni x$ such that $|f(y) - f(x)| < \epsilon$ for every $y \in U$. We have

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(y)| + |f_n(x) - f(x)| < \epsilon/3 + \epsilon/3 + \epsilon/3$$

So there exist U by continuity of f_n. □

12.4 Urysohn's Lemma

12.4.1 Example

There exists a regular topological space with no nonconstant continuous functions.

Theorem 42 (Uryohn's Lemma)**.** *Suppose that X is normal. Then for all A, B closed disjoint sets there exists $f \in C(X, [0,1])$ such that $f|_A \equiv 0, f|_B \equiv 1$.*

To prove the theorem we first need a preparatory lemma.

Lemma 23. *There exists a family U_r such that for $r \in \Delta = \{$dyadic rationals in (0,1)$\} = \{\frac{k}{2^n} : n \in \mathbb{N}, 0 < k < 2^n\}$ the U_r are open and*

$$A \subseteq U_r \subseteq B^c$$

for all r and $\overline{U}_r \subseteq U_s$ for $r < s$.

Proof. See Folland or Munkres. □

Proof of Theorem

Proof. For $x \in X$ define

$$f(x) = \inf\{r \in \Delta : x \in U_r\}.$$

Let $U_1 := X$. First off note that this is well defined. Next we check that if $x \in A$ we have $f(x) = 0$. Since $A \subseteq U_r$ for all $y \in \Delta$. If $x \in B$ then $f(x) = 1$ since then $x \in U_1 \setminus \bigcup_{r<1} U_r$. It remains to check that f is continuous. It is enough to show that $\{x : f(x) < \alpha\}$ is open for all α and $\{x : f(x) > \alpha\}$ is open for all α.

We have the following equivalence

$$f(x) < \alpha \iff \exists r < \alpha \text{ such that } x \in U_r \iff x \in \bigcup_{r<\alpha} U_r (\text{which is an open set})$$

Also we have

$$f(x) > \alpha \iff \exists r > \alpha \text{ such that } x \notin U_r \iff \exists s > \alpha \text{ such that } x \notin \overline{U}_s \iff x \in \bigcup (\overline{U}_s)^c (\text{open})$$

□

Theorem 43 (Tietze Extension Theorem)**.** *Let X be normal and $A \subseteq X$ with $f \in C(A, [a,b])$. Then there exists $F \in C(X, [a,b])$ which extends f (i.e. $F|_A \equiv f$).*

Proof. Since we could consider $f_1(x) = (f(x) - a)/(b - a)$, it suffices to consider the interval $[a, b] = [0, 1]$. We claim that there exists a sequence $g_n \in C(X, [0,1])$ satisfying $0 \le g_n \le \frac{2^{n-1}}{3^n}$ on X and

$$0 \le f - \sum_1^n g_j \le \frac{2^n}{3^n}, \qquad x \in A.$$

Let $B = f^{-1}$, $C = f^{-1}([2/3, 1])$. We can use Urysohn's lemma to find a function $g_1 \in C(X, [0, 1/3])$ such that $g_1|_B \equiv 0$ and $g_1|_C \equiv 1/3$. So we have $0 \le g_1 \le 1/3$ on X and thus

$$0 \le f - g_1 \le 2/3, \qquad x \in A.$$

Continue inductively... This means that we assume $g_1, \ldots, g_{n-1}$ are constructed. We denote

$$f - \sum_1^{n-1} g_j =: F_{n-1} \leq \left(\frac{2}{3}\right) n - 1$$

We can let

$$B_n = \{x : F_{n-1}(x) \leq \frac{2^{n-1}}{3^n}\}, \qquad C_n = \{x : F_{n-1}(x) \geq \left(\frac{2}{3}\right)^n\}$$

There exists $g_n \in C(X, [0, \frac{2^{n-1}}{3^n}])$ with

$$g_n(x) = \begin{cases} 0, & x \in B_n \\ \frac{2^{n-1}}{3^n}, & x \in C_n \end{cases}$$

Then it is easy to verify that

$$F_n = f - \sum_1^n g_j \in \left[0, \left(\frac{2}{3}\right)^n\right], \qquad x \in A$$

So such a sequence exists. Let $G = \sum_{n=1}^{\infty} g_n \in C(X, [0,1])$ since it converges uniformly by a proposition from last lecture. Lastly, notice that for $x \in A$ we now have

$$0 \leq f - G \leq \left(\frac{2}{3}\right)^n \quad \forall n \implies f = G, \quad x \in A$$

□

Corollary 21. *If X is normal and $A \subseteq X$ is closed then $f \in C(A)$ implies that there exist $f \in C(X)$ such that $F|_A = f$.*

Proof. Without loss of generality suppose that $f : A \to \mathbb{R}$. Let $g = \frac{f}{1+|f|} \in C(A, (-1,1))$. So there exists $G \in C(X, [-1,1])$ with $G|_A = g$. We want to define

$$F := \frac{G}{1 - |G|}$$

Let $B = G^{-1}(\{1\})$ which is closed in X and disjoint from A. This means that there exists $h \in C(X, [0,1])$ such that $h = 1$ on A and $h = 0$ on B. Then $hG = G$ on A and $|hG| < 1$ on X. So we officially define

$$F := \frac{hG}{1 - |hG|}$$

□

12.5 Locally Compact Hausdorff Spaces

Definition 97. *A space $(X, \mathcal{T})$ is **compact** if every open cover of X has a finite subcover. A subset $A \subseteq X$ is compact in the relative topology. A set $A \subseteq X$ is **precompact** if $\overline{A}$ is compact.*

Definition 98. *The **finite intersection property** is that if $\{F_\alpha\}_{\alpha \in A}$ is a collection of closed sets and for all $B \subseteq A$ finite we have $\bigcap_{\alpha \in B} F_\alpha \neq 0$ then $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$.*

Proposition 34. *X is compact if and only if it has the finite intersection property.*

Proposition 35. *All closed subsets of a compact space are compact.*

12.5.1 Example

In general compact does not imply closed. For example consider the trivial topology where all subsets are compact, but most subsets are not closed.

Proposition 36. *If X is Hausdorff and $F \subseteq X$ is compact then*

$$x \notin F \implies \exists U, V \text{ open, disjoint such that } x \in U, F \subseteq V$$

Corollary 22. *If X is Hausdorff and $F \subseteq X$ is compact then F is closed.*

Corollary 23. *In Hausdorff spaces we have an arbitrary intersection of compact sets is compact.*

Proposition 37. *If X is Hausdorff and compact then X is normal.*

Proof. By Proposition 3 for all $x \in E$ there exists U_x, V_x open disks with $U_x \ni x, V_x \supset F$. Consider $U_{x \in X} \supset E$ and take a finite subcover. □

Proof. If X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact. □

Corollary 24. *If X is compact then $C(X) = BC(X)$.*

Proposition 38. *If X is compact and Y is Hausdorff with $f : X \to Y$ continuous and bijective then f is a homeomorphism.*

Proof. We need to check that f^{-1} is continuous. This is equivalent to proving that f is a closed map. Let $E \subseteq X$ be closed. Since X is compact we have E is compact and thus $f(E)$ is compact and closed since Y is Hausdorff. □

Definition 99. *X is **sequentially compact** if for all $\{x_n\} \subseteq X$ have a convergent subsequence.*

Definition 100. *$(X, \mathcal{T})$ is locally compact if every $x \in X$ has a compact neighborhood.*

Definition 101. *We will let **LCH** stand for locally compact and Hausdorff.*

Proposition 39. *If X is LCH and $U \subseteq X$ is open with $x \in U$, then there exists a compact set K such that $x \in K^\circ \subseteq K \subseteq U$*

Proof. Without loss of generality assume that $\overline{U}$ is compact. By proposition there exists disjoint relatively open $V, W \in \overline{U}$ such that $x \in V$ and $\partial U \subseteq W$. Take $K := \overline{V}$. Can easily check that this works. □

Proposition 40 (Proposition2)**.** *Suppose that X is LCH and $K^{cpct} \subseteq U \subseteq X$ with U open. Then there exists a precompact set V such that*

$$K \subseteq V \subseteq \overline{V} \subseteq u$$

Proposition 41 (Urysohn for LCH)**.** *Let K be compact and U be open such that $K \subseteq U \subseteq X$, then there exists $f \in C(X, [0,1])$ such that $f \equiv 1$ on K and $f \equiv 0$ outside a compact subset of U.*

Proof. Take V as in proposition 2 then there exists $finC(\overline{V}, [0,1])$ which can be extended to be continuous and take the appropriate values. □

Proposition 42 (Tietze, LCH version)**.**

Proof. exercise. □

Definition 102. *Let $f \in C(X)$. We define* $\operatorname{supp} f := \overline{\{x : f(x) \neq 0\}}$. *We say that f is* ***compactly supported*** *and denote by $f \in C_c(X)$. This means $f \in C_c(X) \iff \operatorname{supp} f$ is compact. $f \in C(X)$ is said to* ***vanish at*** *∞ if for all $\epsilon > 0$ the set $\{x : |f(x)| \geq \epsilon\}$ is compact. We let*

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at } \infty\}$$

$$C_c(X) \subseteq C_0(X) \subseteq BC(X)$$

Proposition 43. *If X is LCH then $C_0(X) = \overline{C_c(X)}^u$.*

Proof. We will prove both containments. Let $f_n \to f$ uniformly, $f_n \in C_c(X) \implies f \in C_0(X)$. So for all $\epsilon > 0$ there exists $n : \|f_n - f\| < \epsilon \implies |f(x)| < \epsilon$ for all $x \notin \operatorname{supp} f_n$. So we have $\overline{C_c(X)}^u \subseteq C_0(X)$.

Conversely given $f \in C_0(X)$ let $K_n = \{x : |f(x)| \geq \frac{1}{n}\}$ is compact. By Urysohn's LCH version we can find $g_n \in C_c(X)$ such that $0 \leq g_n \leq 1$ and $g_n \equiv 1$ on K_n. We then take $f_n = g_n f \to f$ which converges to f uniformly (to prove this observe that $\|f_n - f\| \leq 1/n$). □

12.6 One Point Compactification

Definition 103. *X is LCH, $X^* = X \cup \{\infty\}$ and*

$$\mathcal{T}^* := \{U \subseteq X^* : \text{ either } U \subseteq \mathcal{T} \text{ or } U = \{\infty\} \cup K^c, K\text{-compact in } X\}$$

Proposition 44. .

(i) $(X^*, \mathcal{T}^*)$-compact Hausdorff and $\iota : X \to X^*$ is an embedding. (HW)

(ii) $f \in C(X)$ extends to a function in $C(X^*) \iff f(x) = g(x) + c, g \in C_0(X)$.

12.7 Spaces of Functions

Let $\mathbb{C}^X$ be the space of all functions $f : X \to \mathbb{C}$. There are several topologies we can take on $C(X) \subseteq \mathbb{C}^X$.

1° : product topology = topology of pointwise convergence.

2° : Topology of uniform convergence on compact sets has neighborhood base $K \subseteq X, n \in \mathbb{N}, \{g : \sup_K |f(x) - g(x)| < 1/n\}$.

3° : Uniform convergence: $f \in \mathbb{C}^X$, neighborhood base: $\{g : \|f - g\|_u < 1/n\}$.

Lemma 24. *Let X be LCH and $E \subseteq X$ closed $\iff E \cap K$ is closed for all K compact.*

Proposition 45. *If X is LCH then $C(X)$ is a closed subspace of $\mathbb{C}^X$ in the topology 2°.*

Proof. Let $f_n \in C(X)$ and $f_n \to f$ in 2°. The function f is continuous at $x \in X$ since x has a compact neighborhood $N \ni x$. This means that $f|_N$ is in $\overline{\mathbb{C}(N)} \implies f|_N$ is continuous. □

Definition 104. *X is* ***σ-compact*** *if $X = \bigcup_1^\infty K_n$ with K_n compact.*

Proposition 46. *X is σ-compact and LCH then there exists $\{U_n\}$ open and precompact such that $\overline{U}_n \subseteq U_{n+1}$ for all n and $\bigcup_1^\infty U_n = X$.*

Proposition 47. *X is σ-compact then for all $f \in \mathbb{C}^X$ we have $\{g \in \mathbb{C}^X : \sup_{g \in \overline{U}_n} |f() - f(x)| < 1/m\}$ forms first countable neighborhood base for 2°.*

12.8 Compactness Theorems

Theorem 44 (Tychonoff's Theorem)**.** *$\{X-\alpha\}_{\alpha\in A}$ is compact implies that $\prod_{\alpha\in A} X_\alpha$ is compact in the product topology.*

Definition 105. *$\mathcal{F}$ is **equicontinuous at** x if for all $\epsilon > 0$ there exists open $U \ni x$ such that for all $f \in \mathcal{F}$ $|f(y) - f(x)| < \epsilon$ for all $y \in U$.*

Definition 106. *$\mathcal{F}$ is **equicontinuous** on X if and only if $\mathcal{F}$ is equicontinuous for all $x \in X$.*

Theorem 45 (Arezelá-Ascoli Theorem)**.** *(Compact Version) Let X be compact Hausdorff. Suppose that $\mathcal{F} \subseteq C(X)$ with uniform topology. If $\mathcal{F}$ is pointwise bounded ($\forall x \in X, f \in \mathcal{F}$ it holds that $|f(x)| < M$) and $\mathcal{F}$ is equicontinuous then $\mathcal{F}$ is totally bounded.*

Proof. Fix $\epsilon > 0$. Fora ll $x \in X$ there exists $U_x \ni x$ such that $|f(y) - f(x)| < \epsilon/4$. For all $y \in U_x$ and for all $f \in \mathcal{F}$ we know that $\{U_x\}_{x\in X}$ covers X. By compactness there exists $x_1, \ldots, x_n : X = \bigcup_{j=1}^n U_{x_j}$. Because

$$Q = \bigcup_{j=1}^n \{f(x_j) : f \in \mathcal{F}\} \qquad \text{is bounded in } \mathbb{C}$$

So there exists an $\epsilon/4$-net in $\mathbb{C}$ for Q. i.e. there exist $\{z_1, \ldots, z_m\} \subseteq \mathbb{C}$ such that

$$\bigcup_1^m B(z_i, \epsilon/4) \supset \bigcup_1^n \{f(x_j) : f \in \mathcal{F}\}$$

Define the sets

$$A = \{x_1, \ldots, x_n\}, \qquad B = \{z_1, \ldots, z_m\}$$

For all $\varphi : A \to B$ we define the set

$$\mathcal{F}_\varphi := \{f \in \mathcal{F} : |f(x_j) - \varphi(x_j)| < \epsilon/4\}$$

We conclude that from our construction we have

$$\bigcup_{\varphi\in B^A} F_\varphi = \mathcal{F}.$$

It remains to show that $\operatorname{diam}(\mathcal{F}_\varphi) < \epsilon$. Let $f, g \in F_\varphi$, we wish to prove that $\|f - g\|_n < \epsilon$. We can make the following bounds

$$\begin{aligned}
|f(x) - g(x)| &\le |f(x) - f(x_j)| + |f(x_j) - g(x_j)| + |g(x_j) - g(x)| \\
&< \epsilon/2 + |f(x_j) - g(x_j)| \\
&\le \epsilon/2 + |f(x) - \varphi(x_j)| + |\varphi(x_j) - g(x_j)| \\
&< \epsilon/2 + \epsilon/2
\end{aligned}$$

□

12.8.1 Two Questions

- $\mathcal{F} \subseteq C(X)$ is precompact $\overset{?}{\Longrightarrow}$ $\mathcal{F}$ is equicontinuous.
- $\mathcal{F}$ totally bounded $\overset{?}{\Longrightarrow}$ $\mathcal{F}$ is equicontinuous.

Theorem 46 (LCH version)**.** *Let X be σ-compact, $\{f_n\}$ equicontinuous and pointwise bounded in $C(X)$ implies that there exist $f \in C(X)$ and $f_{n_k} \to f$ uniformly on compact sets.*

12.9 Stone-Weierstrauss Theorem

Theorem 47 (Stone-Weierstrauss)**.** *Let X be a compact Hausdorff space and $\mathcal{A}$ a closed subalgebra of $C(X,\mathbb{R})$. If $\mathcal{A}$ separates points then either $\mathcal{A} = C(X,\mathbb{R})$ or there exists $x_0 \in X$ such that $\mathcal{A} = \{f \in C(X,\mathbb{R}), f(x_0) = 0\}$.*

Lemma 25. *Suppose that $X = \{a,b\}$ is a two-point space. We have $\mathbb{C}(X,\mathbb{R}) \simeq \mathbb{R}^2$. The only subalgebras of $\mathbb{R}^2$ are $\mathbb{R}^2, \{0\}, \{(x,0)\}, \{(x,x)\}$.*

Proof. Suppose $(x_1, r_1) \neq (0,0$ in $\mathcal{A}$. Then $(x_1,y_1)(x_1,y_1) = (x_1^2, y_1^2)$. If $\{(x_1, y_1, (x_1^2, y_2^2)\}$ is linearly independent that $\mathcal{A} = \mathbb{R}^2$. Otherwise $(x_1^2, y_1^2) = c(x-1, y_1)$ go through cases. □

Lemma 26. *For all $\epsilon > 0$ there exists a polynomial p on $\mathbb{R}$ with*

$$||x| - p(x)| < \epsilon, \qquad \forall x \in [-1,1,]$$

Proof. Start with

$$(1-t)^{1/2} = 1 - \frac{1}{2}t + \frac{\frac{1}{2}(-1/2)}{2!}(-t)^2 + \cdots = 1 - \sum_1^\infty c_n t^n$$

converges for $|t| > 0$. Notice that

$$\lim_{t \nearrow 1} \sum_1^\infty c_n t^n = \sum_1^\infty c_n = 1$$

therefore $1-(1-t)^{1/2} \to 1$. From this it follows that $1 - \sum_1^\infty c_n t^n \to (1-t)^{1/2}$ on $[-1,1]$ uniformly. We now have a partial sum $Q(t)$ which satisfies

$$\Big||1-t|^{1/2} - Q(t)\Big| < \epsilon/2, \qquad t \in [-1,1,]$$

Letting $t = 1 - x^2$ implies that

$$|(x^2)^{1/2} - Q(1-x^2)| < \epsilon/2, \qquad x \in [-1,1]$$

So letting R denote the remainder we have

$$||x| - R(x)| < \epsilon/2$$

When $x = 0$ this means that $|R(0)| < \epsilon/2$. Take $p(x) = R(x) - R(0)$. This gives

$$||x| - p(x)| < \epsilon.$$

□

Definition 107. *A **lattice** is a set of real functions $\mathcal{F}$ such that $f, g \in \mathcal{F} \implies \min(f,g) \in Ł, \max(f,g) \in Ł$.*

Lemma 27. *Suppose $\mathcal{A}$ is a closed subalgebra of $C(X,\mathbb{R})$ then*

(i) $f \in \mathcal{A}$ implies that $|f| \in \mathcal{A}$.

(ii) $\mathcal{A}$ is a lattice.

Proof. First notice that (ii) follows from (i). We have $\max(f,g) = \frac{f+g+|f-g|}{2}$ and $\min(f,g) = \frac{f+g-|f-g|}{2}$. So we only need prove (i). Let $f \in \mathcal{A}$. If $f = 0$ we are done, so suppose that $f \neq 0$. Define the function $h = \frac{f}{\|f\|_u}$. We have $h : X \to [-1,1]$. By the second lemma we know for all $\epsilon > 0$ there exists a polynomial p such that $p(0) = 0$, $p \circ h \in \mathcal{A}$ (this is true for any p) and

$$\||h| - p \circ h\|_u < \epsilon$$
$$||x| - p(x)| < \epsilon, \qquad \forall x \in [-1,1]$$

□

Lemma 28. *Let $\mathcal{A}$ be a closed lattice in $C(X,\mathbb{R})$ and $f \in C(X,\mathbb{R})$ be such that for all $x \neq y$ in X there exists $g_{xy} \in \mathcal{A}$ with*

$$g_{xy}(x) = f(x), \qquad g_{xy}(y) = f(y)$$

then $f \in \mathcal{A}$.

Proof. Fix $\epsilon > 0$. For $x, y \in X$ let

$$U_{xy} = \{z \in X : f(z) < g_{xy}(z) + \epsilon\}, \qquad V_{xy} = \{z \in X : f(z) > g_{xy} - \epsilon\}.$$

Notice that both sets are always open and contain x and y.

Fix y and consider the neighborhoods

$$\{U_{xy} : x \in X\} \supset X \implies \exists \{U_{x_j y}\}_{j=1}^n \supset X$$

Define the function $g_y := \max\{g_{x_1 y}, \ldots, g_{x_n y}\}$ then

$$f < g_y + \epsilon$$

We also know that $f > g_y - \epsilon$ on the open set $V_y = \bigcap V_{x_j y} \ni x$. We now run the same argument again and let $\{V_y\}_{y \in Y}$ be an open cover of X which has the refinement

$$\{V_{y_k}\}_1^m \supset X$$

Let $g = \min\{g_{y_j}\}_1^m \in \mathcal{A}$. Then

$$f > g - \epsilon, \qquad f < g + \epsilon$$

Therefore

$$\|f - g\|_u < \epsilon$$

□

Proof of Stone Weierstrauss

Proof. For $x \neq y$ let $\mathcal{A}_{xy} = \{(f(x), f(y)) : f \in \mathcal{A}\}$. Notice that $\mathcal{A}_{xy}$ is a subalgebra of $\mathbb{R}^2$. Either it is $\mathbb{R}^2$ for all x, y then Lemma 3 and Lemma 4 imply that $\mathcal{A} = C(X,\mathbb{R})$. Otherwise there exist x, y such that $\mathcal{A}_{xy} = \{(a,0)\}$ or $\mathcal{A} = \{(0,b)\}$. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$ then $\mathcal{A} = \{f \in C(x,\mathbb{R}) : f(x_0) = 0\}$.

□

Corollary 25. *Let $\mathcal{B}$ be a sub-algebra that separates points. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{B}$ then $\mathcal{B}$ is dense in $\{f \in C(X,\mathbb{R}) : f(x_0) = 0\}$. Otherwise $\overline{\mathcal{B}} = C(X,\mathbb{R})$.*

Proof. Note that $\overline{\mathcal{B}}$ is an algebra. □

12.9.1 Example

1. $X \subseteq \mathbb{R}^n$ compact. And $\mathcal{B} = \{\text{polynomials in } X\}$. Then $\overline{\mathcal{B}} = C(X, \mathbb{R})$.

(ii) $f \in C(\mathbb{R})$ is 2π-periodic then f can be approximated uniformly by

$$\varphi(x) = a_0 + \sum_1^N a_n \cos(nx) + b_n \sin(nx)$$

Theorem 48. *Suppose that $\mathcal{A} \subseteq C(X)$ is a closed subalgebra that separates points and is closed under $f \mapsto \overline{f}$. Then the same conclusion of Stone-Weierstrauss holds.*

Proof. Let $\mathcal{A}_{\mathbb{R}} := \{f \in \mathcal{A} : f \text{ is real valued}\}$. This is clearly a subalgebra that satisfies

$$\mathcal{A}_{\mathbb{R}} \subseteq \mathcal{A} \subseteq C(X)$$

$$\mathcal{A}_{\mathbb{R}} \subseteq C(X, \mathbb{R})$$

Recall that $\operatorname{Re} f = \frac{f+\overline{f}}{2} \in \mathcal{A}$ and $\operatorname{Im} f = \frac{f-\overline{f}}{2i} \in \mathcal{A}_{\mathbb{R}}$. So $\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}}$. It is then enough to work with just $\mathcal{A}_{\mathbb{R}}$. It is not hard to see that $\mathcal{A}_{\mathbb{R}}$ is closed. Moreover, $\mathcal{A}_{\mathbb{R}}$ separates points. This true since for $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$ then either $\operatorname{Re} f$ or $\operatorname{Im} f$ separates x and y.

□

Theorem 49 (LCH Version of Stone-Weierstrauss)**.** *Let $\mathcal{A} \subseteq C_0(X, \mathbb{R}) := C_0(X) \cap C(X, \mathbb{R})$ is a closed subalgebra that separates points then either $\mathcal{A} = C_0(X, \mathbb{R})$ or there exists $x_0 \in X$ such that $\mathcal{A} = \{f \in C_0(X, \mathbb{R}), f(x_0) = 0\}$.*

Proof. Homework Problem. □

13 L^p Spaces and Banach Spaces

13.0.2 Introduction

We will primarily use Stein and Shakarchi IV Chapter 1. But also pull from Folland Chapter 6 and Chapter 5.

Definition 108. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $0 < p < \infty$ and f be measurable. We define the L^p* ***norm*** *to be*

$$\|f\|_p := \left(\int |f|^p d\mu\right)^{1/p}.$$

We define L^p ***space*** *to be*

$$L^p(X, \mathcal{M}, \mu) := \{f : X \to \mathbb{C} \mid f \text{ is measurable } \|f\|_p < \infty\}.$$

This is often denoted by $L^p(X)$, $L^p(\mu)$ and L^p.

13.0.3 Remarks

- Note that we are really working with equivalence classes defined up to sets of measure zero.
- If A is an set with counting measure then we let $\ell^p(A) = L^p(A, \text{ counting measure})$.

$$\ell^p(\mathbb{Z}), \ell^p(\mathbb{N}) = \{(a_n)_1^\infty : \left(\sum |a_n|^p\right)^{1/p} = \|a_n\|_p < \infty\}$$

- L^p is a vector space.

13.0.4 Example

Consider $L^p_{\mathbb{R}}, (\ell^p_{\mathbb{R}}(X)) := \{\text{Real valued functions in } L^p \text{ (or } \ell^p)\}$. For example we have $\ell^p_{\mathbb{R}}(\{0,1\})$ consists of functions $f : \{0,1\} \to \mathbb{R}$. We let $f(0) = x_1, f(1) = x_2$. Observe that $\{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_p \leq 1\}$ goes to a square as $p \to \infty$ and bows in more and more to approach the axes as $p \to -\infty$.

13.0.5 Is $\|\cdot\|_p$ a norm?

We need to check three things. The first two conditions hold, however the triangle inequality does no necessarily hold.

(i) $\|f\|_p = 0 \iff f = 0$

(ii) $\|cf\|_p = |c| \, \|f\|_p$

(iii) However for $p < 1$ the triangle inequality fails. Fore example let $f = \mathbb{1}_E, g = \mathbb{1}_F$ with $E \cap F \neq \emptyset$ and $0 < \mu(E), \mu(F) < \infty$. We have

$$\|f\|_p = \mu(E)^{1/p}, \qquad \|g\|_p = \mu(F)^{1/p}$$

But since $p < 1$ we know that $(a+b)^{1/p} > a^{1/p} + b^{1/p}$.

$$\|f+g\|_p = \mu(E \cup F)^{1/p} == (\mu(E) + \mu(F))^{1/p} > \|f\|_p + \|g\|_p$$

We let $L^\infty = \{f : \|f\|_\infty < \infty\}$.

13.1 Hölder's Inequality

Theorem 50 (Hölder's Inequality)**.** *Let $p > 1$ and let q be the conjugate. Meaning $\frac{1}{q} + \frac{1}{p} = 1$. We let $q = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$. Let $f \in L^p, g \in L^q$ then $gh \in L^1$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q .$$

Lemma 29 (Generalized arithmetic geometric mean inequality (AGM))**.** *Let $\theta \in [0,1]$ it holds that*

$$A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B.$$

Note that when $\theta = 1/2$ we get $\sqrt{AB} \leq \frac{A+B}{2}$.

Proof. If $B = 0$ the true. Let $B > 0$ and divide by B. We then have

$$\left(\frac{A}{B}\right)^\theta \leq \theta \left(\frac{A}{B}\right) + (1-\theta)$$

Letting $t = \frac{A}{B}$ we have

$$t^\theta \leq \theta t + (1-\theta)$$

Assuming that $t \geq 0$ we can define the function f

$$f(t) = t^\theta - \theta t - (1-\theta)$$

Notice that $f(1) = 0$ and that

$$f'(t) = \theta(t\theta - 1 - 1) > 0, \qquad t < 1$$

$$f'(t) = \theta(t\theta - 1 - 1) < 0, \qquad t > 1$$

We then have $f(t) \leq 0, \qquad \forall t.$ □

Proof of Hölder

Proof. If $\|f\|_p$ or $\|g\|_q$ is zero then it is clear that $f(x)g(x) = 0$ almost everywhere and we are done. Otherwise we can normalize and consider $f/\|f\|_p$, $g/\|g\|_p$. So it suffices to assume that $\|f\|_p = 1 = \|g\|_p$. We would like to show that

$$\|fg\|_1 \leq 1.$$

Observe that

$$|f(x)g(x)| = (|f(x)|^p)^{1/p}(|g(x)|^q)^{1/q}$$

Letting $\theta = 1/p$ we apply the lemma and write the following pointwise inequality.

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|0 + \frac{1}{q}|g(x)|^q$$

□

13.2 Minkowski Inequality

Theorem 51 (Minkowski)**.** *It holds that* $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. We have

$$\begin{aligned}
\int |f+g|^p &= \int |f+g||f+g|^{p-1} \\
&\leq \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&\leq \|f\|_p \left(\int \left(|f+g|^{p-1}\right)^q \right)^{1/q} + \|g\|_p \left(\int \left(|f+g|^{p-1}\right)^q \right)^{1/q} \\
&= (\|f\|_p + \|g\|_p) \left(\int |f+g|^p \right)^{1/q}
\end{aligned}$$

So we have

$$\|f+q\|_p = \left(\int |f+g|^p \right)^{1-1/q} \leq \|f\|_p + \|g\|_p$$

□

Theorem 52. *$L^p(X)$ is a Banach space.*

Proof. We need only prove completeness. Very similar to proving L^1 is complete. See Folland. □

Theorem 53. *Simple functions of the form $\sum a_j \mathbb{1}_{E_j}$ form a dense family.*

13.3 Defining L^∞

Definition 109. *Let $f : X \to \mathbb{C}$ is measurable. The* ***infinity norm*** *$\|f\|_\infty$ is defined as*

$$\|f\|_\infty := \operatorname{esssup} |f| = \inf\{a > 0 : \mu)\{x : |f(x)| > a\}) = 0$$

Lemma 30. *$f \in L^\infty \iff \exists g$ measurable and bounded such that $f = g$ almost everywhere.*

Theorem 54 (Folland 6.8)**.** .

(a) $f \in L^1, g \in L^\infty$ then $\|fg\|_1 \leq \|f\| \, \|g\|_\infty$.

(b) $\|\cdot\|_\infty$ is a norm in L^∞.

(c) $f_n \to f$ in $L^\infty \iff \exists E \in \mathcal{M}, \mu(E^c) = 0$ with $f_n \to f$ uniformly on E.

(d) L^∞ is a Banach space.

(e) Simple functions are dense in L^∞.

13.4 Inclusions

Proposition 48. *On $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$ we have $p_0 < p_1 \implies L^{p_1} \subseteq L^{p_0}$*

Proof. Let $f \in L^{p_1}(X)$. We have

$$\int |f|^{p_0} = \int |f|^{p_0} \cdot 1 \leq \left(\int (|f|^{p_0})^{p_1/p_0} \right)^{p_0/p_1} \left(\int 1^q \right)^{1/q} = \mu(X)^{1/q} \left(\int (|f|^{p_0})^{p_1/p_0} \right)^{p_0/p_1}$$

It follows that

$$\|f\|_{p_0} \leq \|f\|_{p_1} \mu(X)^{\frac{1}{qp_0}} = \mu(X)^{\left(\frac{1}{p_0} - \frac{1}{p_1}\right)} \|f\|_{p_1}$$

□

Proposition 49. *Consider the counting measure with $\ell^p(\mathbb{N})$. It holds that $p < q \implies \ell^p(\mathbb{N}) \subseteq \ell^q(\mathbb{N})$.*

Proof. See Folland. □

Proposition 50. *Let $f \in L^\infty$ and $\mu(\{x : f(x) = 0\}) < \infty$ then $f \in L^p$ for all $p < \infty$ and $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.*

Proof. See Stein. □

13.5 Linear Functionals and Dual Banach Space

Definition 110. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A **linear functional** is an operator $T : \mathcal{B} \to \mathbb{R}$ which satisfies $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$.*

Definition 111. *A linear operator is **bounded** if there exists M such that $|T(f)| \leq M \|f\|$ for all $f \in \mathcal{B}$.*

Lemma 31. *A linear functional is bounded if and only if it is continuous.*

Proof. T is continuous on $\mathcal{B}$ if and only if T is continuous at $0 \in \mathcal{B}$. □

Definition 112. *Let $\mathcal{B}$ be a Banach space. We define the **dual** to be $\mathcal{B}^*$ to be the set of all bonded linear functionals on $\mathcal{B}$.*

13.5.1 Remark

Notice that $\mathcal{B}^*$ is a vector space and a normed space with

$$\|T\| = \sup_{\|f\| \leq 1} |T(f)| = \sup_{\|f\| = 1} |T(f)| = \sup_{f \neq 0} \frac{|T(f)|}{\|f\|}$$

Theorem 55. *$\mathcal{B}^*$ is a Banach space.*

Proof. The only thing to check is that $\mathcal{B}^*$ is complete. Let $\{T_n\} \subseteq \mathcal{B}^*$ be Cauchy. So for all $f \in \mathcal{B}$ it holds that $\{T_n(f)\} \subseteq \mathbb{R}$ is Cauchy. Hence there exists $\lim_{n\to\infty} T_n(f) =: T(f)$, which is clearly linear and bounded. Lastly, we must prove that $\|T_n - T\| \to 0$. This follows from the triangle inequality. □

Theorem 56. *Let $(X, \mathcal{M}, \mu)$ be σ-finite measure space. For all $T : L^p \to \mathbb{R}$ which are bounded and linear there exists $g \in L^q$ (where $1/p + 1/q = 1$) such that*

$$T(f) = \int f(x)g(x)d\mu(x), \qquad \forall f \in L^p$$

Moreover, $\|T\|_{(L^p)^*} = \|g\|_q$.

Lemma 32. *Let $1 \le p, q \le \infty$ with $1/p + 1/q = 1$.*

(i) For all $g \in L^q$ we have $\|g\|_q = \sup_{\|f\|_p<1} \left|\int fg\right|$.

(ii) If g is integrable on all sets of finite measure and

$$\sup\left\{\left|\int fg\right| : f \text{ is simple and } \|f\|_p \le 1\right\} = M < \infty$$

Then $g \in L^q$ and $\|g\|_q = M$.

Proof. (i) We know that $\left|\int fg\right| \le \int |fg| \le \|f\|_p \|g\|_q \le \|g\|_q$. So let $f(x) = \operatorname{sgn} g(x)$ (where sgn denotes the signum of g). Since we can writ $\int fg = \int |g|$ we conclude that

$$1 < p, q < \infty, \qquad f(x) = \operatorname{sgn} g(x) \left(\frac{|g(x)|}{\|g\|_q}\right)^{q-1}$$

We can then write

$$\int fg = \frac{1}{\|g\|_q^{q-1}} \int |g(x)|^q = \|g\|_q$$

and also

$$\|f\|_p^p = \int \frac{|g(x)|^{(q-1)p}}{\|g\|_q^{(q-1)p}} = \frac{\|g\|_q^q}{\|g\|_q^q} = 1$$

In the case $q = \infty$ and $p = 1$ we know for each $\epsilon > 0$ there exists E such that $0 < \mu(E) < \infty$ (we used σ-finite). It follows that $|g(x)| \ge \|g\|_\infty - \epsilon$. Assume that $\|g\|_\infty > 0$. We have

$$f(x) = \operatorname{sgn} g(x) \frac{\mathbb{1}_E(x)}{\mu(E)}, \|f\| \le 1$$

$$\left|\int fg\right| = \frac{1}{m(E)} \int_E |g| \ge \|g\|_\infty - \epsilon$$

(ii) Let $g_n \overset{a.e}{\to} g$ and $|g_n(x)| \le |g(x)|$. For $1 < p, q < \infty$ we can take

$$f_n(x) = \operatorname{sgn} g_n(x) \frac{|g_n(x)|^{q-1}}{\|g_n\|_q^{q-1}}, \qquad \|f_n\|_p = 1$$

Also notice that

$$\mu(E) < \infty \implies \int_E |g(x)| < \infty \implies \int_E |g_n(x)| < \infty \implies \int_E |g_n(x)|^q < \infty$$

Consider the integral

$$\left|\int f_n g\right| = \int |g(x)||g_n(x)|^{q-1} \leq M$$

Where the inequality is by hypothesis. We can then let $n \to \infty$ and use Fatou's lemma to conclude that

$$\|g\|_q \leq M$$

So we have $\|g\|_q \geq M$ be Hölder.

□

Proof of Theorem

First assume that $\mu(X) < \infty$ and $T : L^p \to \mathbb{R}$ is bounded and linear. Define $\nu(E) = T(\mathbb{1}_E), E \in \mathcal{M}$. We have

$$|\nu(E)| \leq C \|\mathbb{1}_E\|_p$$

Recall that $T(\mathbb{1}_E) \leq \|T\| \|\mathbb{1}_E\|_p$. It follows then that

$$(!) \qquad |\nu(E)| \leq C \|\mathbb{1}_E\|_p = C(\mu(E))^{1/p}$$

Since T is linear we know that ν is finitely additive. We can then write

$$\mathbb{1}_{E_1 \cup E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2}, \qquad E_1 \cap E_2 = \emptyset$$

By countable additvity we write

$$E = \bigcup E_n, \qquad \mathbb{1}_E = \sum_1^N \mathbb{1}_{E_n} + \mathbb{1}_{E_N^*}, \qquad E_N^* = \bigcup_{n=N+1}^{\infty} E_n$$

We can then write $\nu(E) = \sum_1^N \nu(E_n) + \nu(E_N^*)$. It remains to show that

$$\nu(E_N^*) \to 0, \qquad N \to \infty$$

But this is true using (!) since $\mu(E_N^*) \to 0$ and thus ν is countable additive. It follows that ν is a signed measure and also $\nu \ll \mu$. By Radon-Nikodym there exists a function $g \in L^1(\nu)$ such that $\nu(E) = \int_E g d\mu$ for all $E \in \mathcal{M}$. We claim that $g \in L^q$. By linearity it follows that $\ell(f) = \int f g d\mu$ holds for simple functions. This implies that $\int f g d\mu = \ell(f)$ for all $f \in L^p$ (because simple functions are dense) since ℓ is continuous. We can apply the lemma (part (ii)) and conclude that $g \in L^q$.

Read the σ-finite case.

13.6 Integral Operators

Let $(X, \mathcal{M}, \mu), (Y, \mathbb{N}, \nu)$ be σ-finite measure spaces. Let K be $\mathcal{M} \otimes \mathbb{N}$ measurable. We can define the operator $Tf(x) = \int K(x,y) f(y) d\nu(y)$. When is T bounded as a functional from $L^p(\nu) \to L^p(\mu)$?

Theorem 57. *Suppose there exists $C > 0$ so that*

$$\int |K(x,y)| d\mu(x) \leq C, \qquad a.e.\ y \in Y$$

$$\int |K(x,y)| d\nu(y) \leq C, \qquad a.e.\ x \in X$$

For $1 \leq p \leq \infty$. Then we have

$$\|Tf\|_{L^p(\mu)} \leq C \|f\|_{L^p(\nu)}$$

Proof. Assume $1 < p < \infty$. We have

$$\int |K(x,y)||f(y)d\nu(y) \overset{*}{\leq} \left(\int |K(x,y)|d\nu(y)\right)^{1/q} \left(\int |K(x,y)||f(y)|^p\right)^{1/p}$$

Where at * we use Hölder's inequality. We continue

$$\begin{aligned}\int |K(x,y)||f(y)d\nu(y) &\overset{*}{\leq} \left(\int |K(x,y)|d\nu(y)\right)^{1/q} \left(\int |K(x,y)||f(y)|^p\right)^{1/p} \\ &\leq C^{1/q} \left(\int |K(x,y)||f(y)|^p\right)^{1/p}\end{aligned}$$

We conclude that

$$\begin{aligned}\int \left[\int K(x,y)f(y)d\nu(y)\right]^p d\mu(x) &\leq C^{p/q} \int\int |K(x,y)||f(y)|^p d\nu(y)d\mu(x) \\ &\overset{**}{=} C^{p/q} \int |f(y)|^p \int |K(x,y)|d\mu(x)d\nu(y) \\ &\leq C^{\frac{p}{q}+1} \|f\|_p^p = C^p \|f\|_p^p\end{aligned}$$

Where at ** we use Tonnelli. This implies that $\|Tf\|_{L^p(\mu)} \leq C \|f\|_{L^p(\nu)}$. □

13.7 Minkowski's Integral Inequality

Recall that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ and by induction $\|\sum f_i\|_p \leq \sum \|f_i\|_p$. The integral inequality is as follows:

Theorem 58 (Minkowski's Integral Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathbb{N}, \nu)$ be σ-finite measure spaces with f measurable with respect to the product σ-algebra.*

(a) Suppose $f \geq 0$, $f \in L^+(X \times Y)$ and $1 \leq p < \infty$. We have

$$\left[\int \left(\int f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int \left[\int f(x,y^p)d\mu(x)\right]^{1/p} d\nu(y)$$

(Note these functions are measurable by Tonelli).

(b) Suppose $1 \leq p \leq \infty$. We know that $x \mapsto f(x,y)$ is in $L^p(\mu)$ and $y \mapsto \|f(\cdot,y)\|_p$ is in $L^1(\nu)$. Then $f(x,\cdot) \in L^1(\nu)$ for almost every x. So $x \mapsto \int f(x,y)d\nu(y)$ is in $L^p(\mu)$ and

$$\left\|\int f(\cdot,y)d\nu(y)\right\|_p \leq \int \|f(\cdot,y)\|_p d\nu(y)$$

Proof. (a) Assume $1 < p < \infty$. Let $h(x) = \int f(x,y)d\nu(y)$. If the right hand side is ∞ we are done. So suppose that $RHS < \infty$. We want to show that $fLx \mapsto \int f(x,y)d\nu(y)$ is in $l^p(\mu)$ and $\|h\|_p \leq \int \left[\int f(x,y^p)d\mu(x)\right]^{1/p} d\nu(y)$.

Recall that

$$\|h\|_p = \sup\left\{\left|\int hg\right| : \|g\|_q = 1\right\}$$

Pick $g \in L^q(\mu)$ with

$$\begin{aligned}\int |hg| &= \int h|g| \\ &= \int \left[\int f(x,y)d\nu(y)\right] |g(x)|d\mu(x) \\ &= \int \left(\int f(x,y)|g(x)|d\mu(x)\right) d\nu(y) \\ &\leq \int \|g\|_q \left(\int f(x,y)^p d\mu(x)\right)^{1/p} d\nu(y)\end{aligned}$$

(b) First apply (a) to $|f|$. This yields

$$\int |f(x,y)d\nu(y) < \infty, \qquad \text{a.e. } x$$

which implies

$$f(x,\cdot) \in L^1(\nu)$$

So the function

$$x \mapsto \int f(x,y)d\nu(y)$$

is measurable in X since □

13.8 Homogeneous Kernels

Theorem 59. *Suppose that $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ with $K(x,y) \geq 0$. Also, assume that $\int K(x,1)x^{-1/p}dx = C < \infty$. Let $1/p + 1/q = 1$. Define the operators*

$$Tf(y) = \int_0^\infty K(x,y)f(y)dy, \qquad Sg(y) = \int_0^\infty K(x,y)g(y)dy$$

Then

$$f \in L^p \implies \|Tf\|_p \leq C\|f\|_p, \qquad g \in L^q \implies \|Sg\|_q \leq C\|g\|_q.$$

Proof. We have $\|Tf\|_p = \left(\int_0^\infty \left|\int_0^\infty K(x,y)f(x)dx\right|^p dy\right)^{1/p}$. Observe that under the change of variable $z = x/y$, letting $f_z(y) = f(zy)$

$$\int_0^\infty K(x,y)|f(x)|dx = \int_0^\infty K(z,1)|f_z(y)|dz$$

Also we have by a simple change of variable

$$\|f_z\|_p = \int_0^\infty |f(y,z)|^p dy = z^{-1}\|f\|_p$$

It follows that

$$\|Tf\|_p = \left(\int_0^\infty \left(\int_0^\infty K(x,1)f_z(y)dz\right)^p dy\right)^{1/p}$$

Using Minkowski with $d\nu(z) = K(z,1)dz$ we have

$$\begin{aligned}
\|Tf\|_p &= \left(\int_0^\infty \left(\int_0^\infty K(x,1)f_z(y)dz\right)^p dy\right)^{1/p} \\
&= \left(\int_0^\infty \left(\int_0^\infty |f_z(y)|d\nu(z)\right)^p dy\right)^{1/p} \\
&\leq \int_0^\infty \left(\int_0^\infty |f_z(y)|dy\right)^p K(z,1)dz \\
&= \int_0^\infty z^{-1/p} \|f\|_p K(z,1)dz = C\|f\|_p
\end{aligned}$$

□

13.8.1 Example

Hardy's inequalities (special cases). Consider the following for $1 < p \leq \infty$ and $1 \leq q < \infty$.

$$Tf(y) = y^{-1}\int_0^y f(x), \qquad Sg(x) = \int_x^\infty y^{-1}f(y)dy$$

We will prove that

$$\|Tf\|_p \leq \frac{p}{p-1}\|f\|_p, \qquad \|S_g\|_q \leq q\|g\|_q$$

Proof. If $Tf(y) = \int_0^\infty K(x,y)f(x)dx$ then $K(x,y) = y^{-1}\cdot \mathbb{1}_{[0,y]}$. Since $\int K(x,y)dy = \infty$ the first theorem doesn't apply. We can apply the previous theorem though since

$$K(\lambda x, \lambda y) = (\lambda y)^{-1}\cdot \mathbb{1}_{[0,\lambda y]}(x) = \frac{y^{-1}\cdot \mathbb{1}_{[0,y]}(x)}{\lambda} = \lambda^{-1}K(x,y)$$

So we have

$$C = \int_0^\infty x^{-1/p}K(x,1)dx = \int_0^1 x^{-1/p}dx$$

□

13.9 Hahn-Banach Theorem

Definition 113. *Suppose V is a real vector space. A map $p : V \to \mathbb{R}$ is **sub-linear** if*

$$p(x+y) \leq p(x) + p(y), \qquad \forall x, y \in V$$

$$p(\lambda x) = \lambda p(x), \qquad \forall x \in V, \forall \lambda \geq 0$$

Theorem 60 (Hahn-Banach)**.** *Let $W \subseteq V$ be a subspace with p a sublinear functional on V. Let $l : W \to \mathbb{R}$ be a linear functional such that $|l(x)| \leq p(x)$ for all $x \in W$. Then there exists $L : V \to \mathbb{R}$ a linear functional such that $|L(x)| \leq p(x)$ for all $x \in V$.*

13.9.1 Separation of Convex Sets

Definition 114. *Let $H_0 \subseteq V$ is a **proper hyperplane** if H_0 is a linear subspace of codimension 1. Equivalently, there exists $l : V \to \mathbb{R}$ linear such that $\ker(l) = H_0$.*

Definition 115. *An **affine hyper-plane** is the space $H_0 + x$ with $x \in V$ or $H = \{v \in V : l(v) = a\}, a \in \mathbb{R}\}$.*

Definition 116. *$K \subseteq V$ is **convex** if for all $v_1, v_2 \in K$ we have $[v_1, v_2] = \{tv_1 + (1-t)v - 2, t \in [0,1]\} \subseteq K$.*

Proposition 51. *A convex set K and any $v_0 \notin K$ can be separated by a hyperplane if and only if there exists $l : V \to \mathbb{R}$ linear such that $l(v_0) < a$ and $l(v) \leq a$ for all $v \in K$.*

Proposition 52. *Assume that $K \neq \emptyset$ and $0 \in K$. We can define the Minkowski gauge function*

$$p(v) = \inf\{r < 0 : \frac{v}{r} \in K\}$$

13.10 Observations

1. $v \in K \iff p(v) < 1$
2. $p(v)$ is sublinear since $p(\lambda v) = \lambda p(v), \quad \lambda > 0$. It follows from convexity that

$$p(v_1 + v_2) \leq p(v_1) + p(v_2)$$

14 Review of 525

14.1 Hardy-Littlewood Maximal Theorem

Let $f \in L^1(\mathbb{R}^d)$, the maximal function $f^*(x) = \sup_{B \ni x} \int_B |f(y)| dm(y)$. The Hardy-Littlewood Theorem states that

Theorem 61 (Hardy-Littlewood)**.** *Let f^* be the maximal function with $f \in L^1(\mathbb{R}^d)$.*

(i) f^* is measurable.

(ii) $f^* < \infty$ a.e.

(iii) f^* is weak type $(1,1)$. This means the set $E_\alpha = \{x : f^*(x) > \alpha\}$ satisfies $m(E_\alpha) \leq \frac{A}{\alpha} \|f\|_1$.

Proof. We use the covering lemma where given a finite collection of balls if there exists a disjoint subcollection such that $\bigcup_{k=1}^n 3B_{i_k} \supset \bigcup_1^N B_j$ then $m(\bigcup B_j) \leq 3^d \sum_{k=1}^n m(B_{i_k})$.

To prove the statement we know that $m(E_\alpha) = \sup\{m(K) : K^{cpct} \subseteq E_\alpha\}$. This means for all $x \in E_\alpha$ there exists $B_x \ni x$ such that $m(b_x) < \frac{1}{\alpha} \int_{B_x} |f|$. We then have a finite subcover $\{B_i\}_1^N$ and apply the lemma to conclude that

$$m(K) \leq m\left(\bigcup_1^N B_i\right) \leq 3^d \sum_{j=1}^n m(B_{i_j}) < \frac{3^d}{\alpha} \int_{\bigcup B_{i_j}} |f| \leq \frac{3^d}{\alpha} \|f\|_1$$

□

Theorem 62 (Lebesgue Differentiation Theorem)**.** *$f \in L^1(\mathbb{R}^d)$ implies that $\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B f(y)dy = f(x), \qquad a.e.$.*

Proof. The idea is to approximate f by continuous functions with compact support. So, for all $\epsilon > 0$ there exists $g \in C_0(\mathbb{R}^d)$ such that $\|f - g\|_1 < \epsilon$. We then have by continuity that

$$\lim_{m(B)\to 0, B\ni x} \frac{1}{m(B)} \int g(y)dy = g(x), \qquad \forall x$$

We then have $E_\alpha = \left\{x : \limsup \left|\frac{1}{m(B)} \int (f(y) - f(x))dy\right| > 2\alpha\right\}$ and we want to show that $m(E_\alpha) = 0$. We can write

$$f(y) - g(y) + g(y) - g(x) + g(x) - f(y)$$

□

14.2 Fundamental Theorem of Calculus

14.2.1 Bounded Variation

Theorem 63 (Theorem 3)**.** *$F \in BV$ implies that there exist $F_1, F_2 \nearrow$ such that $F = F_1 - F_2$.*

Proof. Use positive variation and negative variation. □

14.2.2 Absolute Continuity

We know that $AC \subseteq BV$ and that $f \in L^1(a,b) \implies x \mapsto \int_a^x f(y)dy$ is AC.

Lemma 33. *$F \in AC$ and $F' = 0$ a.e. implies $F \equiv c$ a constant.*

Proof. Suffices to prove that $F(a) = F(b)$ since we could prove that $F(a) = F(x)$ for any $x < b$. Define the set $E = \{x : F'(x) = 0\}$. Obtain a Vitali cover by noting that for every $x \in E$ there exists arbitrarily small $(a_x, b_x) \ni x$ such that $|F(b_x) - F(a_x)| \leq \epsilon(b_x - a_x)$. □

Theorem 64. *If $F \in AC[a,b]$ then $F(x) = \int_a^x F'(y)dy$.*

14.3 Signed Measures

Theorem 65 (Jordan-Decomposition Theorem)**.** *A signed measure $\nu : \mathcal{M} \to (-\infty, \infty]$ can be written as $\nu = \nu_+ - \nu_-$ with ν_+, ν_- positive measures. The total variation of ν, denoted $|\nu| = \nu_+ + \nu_-$.*

Proposition 53. $|\nu| = \sup\{\sum |\nu(E_j)| : E = \bigcup E_j\}$ *is a measure.*

Theorem 66 (Hahn Decomposition Theorem)**.** *ν is a signed measure implies that there exist P, N positive and negative sets such that $P \cup N = X$, $P \cap N = \emptyset$.*

15 Hahn-Banach Theorem

15.1 The Hahn-Banach Theorem

Definition 117. *Let X be a real vector space, $p : X \to \mathbb{R}$ is a* ***sublinear functional*** *if (i) $p(x+y) \leq p(x) + p(y)$ and (ii) $p(\lambda x) = \lambda p(x), \lambda \geq 0$.*

Theorem 67 (Hahn-Banach)**.** *Let X be a real vector space with $p : X \to \mathbb{R}$ sublinear and $M \subseteq X$ as subspace. If $f : M \to \mathbb{R}$ is linear such that $f(z) \leq p(x)$ for all $x \in M$ then there exists $F : X \to \mathbb{R}$ a linear function such that $F|_M = f$ and $F(x) \leq p(x)$ for all $x \in X$.*

Proof. $\boxed{\text{Main Step:}}$ First we will extend f to $M + x\mathbb{R}$ for fixed $x \notin M$. We want $g : M + x\mathbb{R} \to \mathbb{R}$ such that g is linear and $g|_M = f$. We must have $g(y + \lambda x) = f(y) + \lambda\alpha$. So we need to choose an appropriate α to preserve the inequality. Observe that for all $y_1, y_2 \in M$ it holds that

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(y_2 + x).$$

Rearranging gives

$$f(y_1) - p(y_1 - x) \leq p(y_2 + x) - f(y_2), \qquad \forall y_1, y_2 \in M.$$

Therefore $\sup\{f(y) - p(y - x) : y \in M\} \leq \inf\{p(y + x) - f(y) : y \in M\}$. Hence we choose α so

$$\sup\{f(y) - p(y - x) : y \in M\} \leq \alpha \leq \inf\{p(y + x) - f(y) : y \in M\}.$$

Observe that $g(y + \lambda x) = f(y) + \lambda\alpha$. It remains to check that $g(y + \lambda x) \leq p(y + \lambda x)$. We break into two cases

(i) $\boxed{\lambda > 0:}$ Then we have

$$g(y + \lambda x) = \lambda g\left(\frac{y}{\lambda} + x\right) = \lambda\left[f\left(\lambda\frac{y}{\lambda}\right) + \alpha\right] \leq \lambda\left[f(y/\lambda) + p(x + y/\lambda) - f(y/\lambda)\right] = p(y + \lambda x).$$

(ii) $\boxed{\lambda < 0:}$ Observe that

$$g(y - \mu x) = \mu[f(y/\mu) - \alpha] \leq \mu[f(y/\mu) - f(y/\mu) + p(y/\mu - x)] = p(y - \mu x).$$

$\boxed{\text{Zorn's Lemma Finish:}}$ Let $\mathcal{F}$ be the family of subspaces of X, containing M, to which f can be extended linearly with $f \leq p$. Notice that inclusion gives a partial ordering and any linearly ordered family has an upper bound (via taking union). By Zorn's Lemma there exists a maximal element, it has to be X by Step 1.

□

Theorem 68 (Complex Hahn-Banach)**.** *Let X be a $\mathbb{C}$-vector space and $M \subseteq X$. Let $p : X \to [0, \infty)$ be a seminorm ($p(x_1 + x_2) \leq p(x_1) + p(x_2)$ and $p(\lambda x) = |\lambda| p(x)$, $\lambda \in \mathbb{C}$). If $f : M \to \mathbb{C}$ is linear with $|f(x)| \leq p(x)$ then there exists an extension F to X with $|F(x)| \leq p(x)$ for all $x \in X$.*

Proof. Exercise 33.

□

15.2 The Dual Space

Let X be a normed vector space and X^* be the space of bounded linear functionals. We have $\|f\|_{X^*} = \sup\{|f(x)| : \|x\| = 1\}$, and X^* is the dual space to X.

Theorem 69. *Let X be a normed complex vector space. If $M \subseteq X$ is a closed subspace and $x \in X \setminus M$ then there exists $f \in X^*$ such that $f(x) \neq 0$ and $f|_M \equiv 0$. In fact, if $\delta = \operatorname{dist}(x, M)$ we can choose f so that $f(x) = \delta$ and $\|f\| = 1$.*

Proof. The idea is to define f on $M + x\mathbb{C}$ and then use Hahn-Banach. We choose $f(y + \lambda x) = \lambda\delta$. This is a unique linear functional on $M + x\mathbb{C}$ which satisfies

$$|f(y + \lambda x)| \leq |\lambda|\delta \leq \|y + \lambda x\|.$$

So we can use $p(z) = \|z\|$. Hahn-Banach says there exists $f : X \to \mathbb{C}$ linear such that $\|f(z)\| \leq \|z\|$ for all $z \in X$. So we have

□

Corollary 26. *When $M = \{0\}$ we then have for all $x \neq 0$ there exist $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Corollary 27. *$\{f \in X^*\}$ separates points of X. (Recall that $\mathcal{F}$ separates points if for all $x \neq y$ there exist $f \in X^*$ such that $f(x) \neq f(y)$.)*

Definition 118. *For $f \in X^*$ the **norm of** f is given by*

$$\|f\| := \sup\{|f(x)| : \|x\| = 1, x \in X\}.$$

Corollary 28. *For all $x \in X$ we have*

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}.$$

Corollary 29. *$X \hookrightarrow X^{**} := (X^*)^*$. i.e. X embeds in X^{**}.*

Proof. For $x \in X$ let $\hat{x}(f) = f(x)$. Notice that $\hat{x}$ is linear. This is true since $\hat{x}(f+g) = f(x)+g(x) = \hat{x}(f) + \hat{x}(g)$. We can also define the norm of $\hat{x}$ in X^{**}. This is formally given by

$$\|\hat{x}\| = \sup\{|\hat{x}(f)| : \|f\| = 1\} = \sup\{|f(x)| : \|f\| = 1 = \|x\|.$$

This implies that $x \mapsto \hat{x}$ is a linear isometry onto the image.

□

Definition 119. *X is **reflexive** of $X = X^{**}$.*

Proposition 54. *$L^p = L^p(X, \mu)$ is reflexive for $1 < p < \infty$.*

Proof. Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. We know that $(L^p)^* = L^q$. If $g \in L^q$ then $\varphi_g \in (L^p)^*$. This does the trick.

□

Proposition 55. *L^1 is not, in general, reflexive.*

Proof. Let $X = [0, 1]$ with m the Lebesgue measure. We have $C([0, 1]) \subseteq L^\infty([0, 1])$. Let $\varphi_0(f) = f(0), f \in C([0, 1])$. Observe that $\|\varphi_0(f)\| \leq \|f\|_\infty$. By Hahn-Banach theorem there exists $\varphi : L^\infty \to \mathbb{R}$ extending φ_0. We will show that $\varphi \neq \varphi_g$ for any $g \in L^1$. It suffices to show that $\varphi_g \neq \varphi_0$.

To show a contradiction suppose there existed $g \in L^1$ such that $\varphi_g(f) = \int_0^1 fgdm = f(0)$ for all $f \in L^\infty$. This cannot be since by considering the function $\operatorname{sgn}(g)\mathbb{1}_{(1/n,1]}$ will give $g \equiv 0$. □

15.3 Dual Transformations

Let $T : \mathcal{B}_1 \to \mathcal{B}_2$ be linear and bounded. Then there exists a map $T^* : \mathcal{B}_2^* \to \mathcal{B}_1^*$ with $T^*(\ell_2) = \ell_2 \circ T$ and $T^*(\ell_2)(x) = \ell_2(T(x))$.

$$\begin{tikzcd} \mathcal{B}_1 \arrow[r, "T"] \arrow[dr] & \mathcal{B}_2 \arrow[d, "2"', "\ell"] \\ & \mathbb{C} \end{tikzcd}$$

Proposition 56. $\|T^*\| = \|T\|$.

Proof. Observe that $\|T^*\| = \sup\{\|T^*(\ell_2)\| : \|l_2\| = 1\}$. We then have

$$\sup\{\|T^*(\ell_2)(x)\| : \|x\| = 1\} \leq \|\ell_2\| \, \|T(x)\| \leq \|\ell_2\| \, \|T\| \, \|x\| = \|.\| \, T$$

Hence $\|T^*\| \leq \|T\|$. Next we check that $\|T^*\| \geq \|T\| - \epsilon$. Fix $\epsilon > 0$. We have

$$\|T^*\| = \sup\{\|T^*(\ell_2)\| : \|\ell_2\| = 1\}.$$

We also have that $\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}$. Hence there exists x such that $\|x\| = 1$ and $\|T(x)\| \geq \|T\| - \epsilon$ (by definition of the sup). Recall that a corollary of Hahn-Banach tells us that $\|y\| = \sup\{\|\ell_2(y)\| \, L \, \|\ell_2\| = 1\}$ is obtained. So for all y there exists $\ell_2 \in \mathbb{B}_2^*$ such that $\ell_2(y) = \|y\|$ and $\|\ell_2\| = 1$. It follows that

$$\|T^*(\ell_2)\| \geq \|T^*(\ell_2)(x)\| = \|\ell_2(T(x))\| = \|\ell_2(y)\| = \|y\| = \|T(x)\| \geq \|T\| - \epsilon.$$

□

There is a simple, but useful (especially on the homework) lemma.

Lemma 34 (Riesz Lemma)**.** *Suppose that X is a normed space and $Y \subseteq X$ is a non-dense linear subspace. Then for any $\epsilon > 0$ there exists vector $x \in X$ such that $\|x\| = 1$ and* $\operatorname{dist}(x, M) > 1 - \epsilon$.

Proof. Choose $x_1 \in X \setminus \overline{Y}$ and let $R = \inf_{y \in Y} |y - x_1| > 0$. For $\alpha \in (0, 1)$ let $y_1 \in Y$ be such that $|y_1 - x_1| < R/\alpha$. Let $\mathbf{b} = \frac{x_1 - y_1}{|y_1 - x_1|}$ so $|\mathbf{b}| = 1$. Also $\mathbf{b} \in X \setminus Y$ since if $\mathbf{b} \in Y$ then $x_1 - y_1 \in Y$ and thus $(x_1 - y_1) + y_1 = x_1 \in Y$ a contradiction. Now, for any $y \in Y$ we have

$$\begin{aligned} \|\mathbf{b} - y\| &= \left\| \frac{x_1 - y_1}{\|x_1 - y_1\|} - y \right\| \\ &= \left\| \frac{x_1}{\|x_1 - y_1\|} - \frac{y_1}{\|x_1 - y_1\|} - y \right\| \\ &= \frac{1}{\|x_1 - y_1\|} \|x_1 - (y_1 + \|x_1 - y_1\| \cdot y)\| \\ &> (R/\alpha)^{-1} \cdot \inf_{y \in Y} |x_1 - y| \\ &= \frac{\alpha}{R} R \\ &= \alpha \end{aligned}$$

□

Corollary 30. *If* $\dim(X) = \infty$ *then the unit ball is not precompact. Hence* X *is not locally compact.*

Proof. Pick $x_1 \in X$ such that $|x_1| = 1$ and let S_1 be the subspace generated by x_1. The Riesz Lemma guarantees that there exists an element $x_2 \in X$ such that $|x_2| = 1$ and $d(x_2, S_1) \geq 1/2$. Let S_2 be the subspace generated by x_1 and x_2. Since X is infinite dimensional we can continue to apply the Riesz Lemma to proceed inductively and create a sequence $\{x_n\}$ such that each x_n satisfies $|x_n| = 1$ and $d(x_n, S_{n-1}) \geq 1/2$. Clearly $\{x_n\} \subseteq \overline{\mathbb{D}}$, however it has no convergent subsequence since $|x_n - x_m| \geq 1/2$ for all $m \neq n$.

□

15.4 Applications of Hahn-Banach

15.4.1 Problem of Measure

Theorem 70 (Banach)**.** *There exists a function* $\hat{m} : 2^{\mathbb{R}^d} \to [0, \infty]$ *such that*

(i) $\hat{m}$ *is finitely additive.*

(ii) $\hat{m}|_{\mathscr{L}} = m$ *(where* $\mathscr{L}$ *denote Lebesgue measurable sets).*

(iii) $\hat{m}$ *is translation invariant.*

Proof. See SSIV, §5.4.

□

Remark: The Banach-Tarski paradox show you cannot ensure $\hat{m}$ is also rotation-invariant in $\mathbb{R}^d$, $d \geq 3$.

16 Consequences of the Baire Category Theorem

16.1 The Baire Category Theorem

Theorem 71. *Suppose X is a complete metric space, the following are equivalent.*

(a) U_n open and dense in X implies that $\bigcap_1^\infty U_n$ is dense in X.

(b) X is not a countable union of nowhere dense sets.

Proof. $\boxed{(a \implies b)}$ If F_n is nowhere dense this means by definition that $(\overline{F}_n)^c$ is open and dense. This implies that $\bigcap_n \left(\overline{F}_n\right)^c$ is dense and thus non-empty. We then have $\left(\bigcap \overline{F}_n^c\right)^c = \bigcup F_n \neq X$.

$\boxed{(b \implies a)}$ We want to show that for all nonempty open W we have $W \cap \bigcap_1^\infty U_n \neq \emptyset$. Since U_1 is dense and open we have $W \cap U_1$ is a nonempty open set. So there exists $\overline{B}_1 = \emptyset B(x_1, r_1)$ with $0 < r < 1$ and $\overline{B}_1 \subseteq W \cap U_1$. Similarly, we can find $\overline{B}_1 \subseteq W \cap U_2$. Continuing inductively we keep going and let $\overline{B}_n = \overline{B(x_n, r_n)} \subseteq B_{n-1} \cap U_n \neq \emptyset$. We then have

$$\left(\bigcap \overline{F}_n^c\right)^c = \bigcup F_n \neq X, \qquad 0 < r_n < 2^{-n}.$$

We claim that $\{x_n\}$ is Cauchy since $d(x_n, x_{n-1}) \leq 2^{-n}$. So there exists $x = \lim x_n \in \overline{B}_k \subseteq U_k$. This implies that $x \in \bigcap_1^\infty U_k$. □

Definition 120. *Let X be a topological space. We say that $A \subseteq X$ is **first category** or **meager** if $A = \bigcup F_n$ with F_n nowhere dense. We say that A is **second category** if A is not first category. We say that A is **residual** or **generic** if A^c is first category.*

Proposition 57. *If X is homeomorphic to a complete metric space then X is second category.*

16.1.1 Remark

For $A \subseteq \mathbb{R}^d$, being generic and having large lebesgue measure are orthogonal properties (i.e. they don't correlate between one another).

16.1.2 Example

- There exists $E \subseteq [0,1]$ such that E is generic and $m(E) = 0$. Consider $Q \cap (0,1) = \{q_i\}$. We have $U_n = \bigcup_j \mathbb{B}(q_j, e^{-j-n})$ is open and dense. Notice that $m(U_n) \leq 2^{-n+1}$. This implies that $\bigcap_n U_n$ is residual and measure zero.
- There exists $F \subseteq [0,1]$ such that $m(F) = 1$ and F is first category. Letting $C_n = \mathcal{C}_{\frac{n-1}{n}}$ be a fat cantor set of measure $\frac{n-1}{n}$ we let $F = \bigcup C_n$.

Theorem 72 (Uniform Boundedness Principle)**.** *(i) Let X, Y be normed vector spaces. Let $\mathcal{A} \in \mathscr{L}(X,Y)$ be collection of bounded linear operators $T : X \to Y$. Suppose $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ for all x in a non-meager set. It follows that $\sup_{T \in \mathcal{A}} \|T\| < \infty$.*

(ii) If X, Y are Banach spaces then $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ for all $x \in X$ implies that $\sup_{\mathcal{A}} \|T\| < \infty$.

Proof. (i) Let $E_n = \{x \in X : \sup_{T\in\mathcal{A}} \|Tx\| \leq n\}$ and $\bigcup E_n$ is non-meager and so (by definition) not a countable union of nowhere dense sets. Therefore there exist sn such that $\overline{E}_n = E_n \supset \overline{\mathbb{B}(x_0, r)}$ for some $r > 0$. We claim that $\overline{B(0,r)} \subseteq E_{2n}$. This is easy to see since we know there is $x_0 - x \in \overline{\mathbb{B}(x_0, r)} \subseteq E_n$.

$$\|T(x_0)\| \leq n$$

$$\|T(x_0 - x)\| = \|Tx - Tx_0\| \leq n \qquad \forall T \in \mathcal{A}$$

Hence $\|Tx\| \leq \|Tx_0\| + \|Tx - Tx_0\| \leq 2n$. So $\|Tx\| \leq 2n$ fora ll $x \in \mathbb{B}(0, r)$. This implies that

$$\|T\| \leq \frac{2n}{r}.$$

Since

$$\|T\| := \sup\{\|Tx\| : \|x\| = 1\} = \sup\{\frac{\|Tx\|}{r} : \|x\| = r\}.$$

□

16.1.3 Application

Let X be a Banach space and X^* be the dual. If $x_n \to x$ weakly this means by definition that $\ell(x_n) \to \ell(x)$ for all $\ell \in X^*$.

Corollary 31. *$x_n \to x$ weakly implies that* $\sup_n \|x_n\| < \infty$.

Proof. We have $\|x\| = \sup\{|\ell(x)| : \|\ell\| = 1, \ell \in X^*\}$. Notice that $\ell(x_n) \to \ell(x)$ implies that $\ell(x_n)$ is bounded. Looking at $x_n \in (X^*)^*$, by the uniform boundedness principle we have $\|x_n\| < \infty$.

□

16.2 The Open Mapping Theorem

Theorem 73 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces. Suppose that $T : X \to Y$ is linear, bounded and surjective then T is open.*

Proof. We proceed in four steps.

$\boxed{\text{Step 1:}}$ From linearity is enough to show that there is $r > 0$ such that the image of the unit ball $T(\mathbb{D}) \supseteq \mathbb{B}(0, r)$. This is true since by translation we will have $T(\mathbb{B}(x, 1)) \supseteq \mathbb{B}(T(x), r)$. So if we scale then we have an open neighborhood of Tx.

$\boxed{\text{Step 2:}}$ Since T is surjective we can write

$$T\left(\bigcup_1^\infty \mathbb{B}_n\right) = T(X) = Y.$$

This means that $\bigcup T(\mathbb{B}_n) = Y$, by Baire Category theorem there exists n such that $T(\mathbb{B}_n)$ is not nowhere dense. This implies that $T(\mathbb{B}_1)$ is not nowhere dense since all of the $\mathbb{B}_n$ are homeomorphic ($T(\mathbb{B}_n) = nT(\mathbb{D})$. This means that $\overline{T(\mathbb{B}_1)}$ contains a non-trivial ball.

$\boxed{\text{Step 3:}}$ Now we need a ball centered at $0 \in Y$ that belongs to $T(\mathbb{D})$. We know that there exists $r > 0$ such that $\mathbb{B}(y_0, 4r) \subseteq \overline{T(\mathbb{D})}$. Observe that $y_0 \in \overline{T(\mathbb{D})}$ implies that $(y_0, 2r) \cap T(\mathbb{D}) \neq \emptyset$. Pick $y_1 \in \mathbb{B}(y_0, 2r) \cap \mathbb{D}$. Observe that

$$\|y_1 - y_0\| < 2r, \qquad y_1 = T(x_1) \qquad \text{for some } x_1 \in \mathbb{D}.$$

We now have

$$\mathbb{B}(y_1, 2r) \subseteq \mathbb{B}(y_0, 4r) \subseteq \overline{T(\mathbb{D})}.$$

We translate to the origin

$$\mathbb{B}(0, 2r) = -y_1\mathbb{B}(y_1, 2r) = -T(x_1) + \mathbb{B}(y, 2r),$$

$$T(x_1) \in T(\mathbb{D}), \quad -T(x_1) \in -T(\mathbb{D}) = T(\mathbb{D}).$$

Observe that

$$\mathbb{B}(0, 2r) \subseteq T(\mathbb{D}) + \overline{T(\mathbb{D})} \subseteq \overline{T(\mathbb{D}) + T(\mathbb{D})} \subseteq \overline{\mathbb{B}(0, 2)}.$$

Hence $\mathbb{B}_r \subseteq \overline{T(\mathbb{D})}$.

$\boxed{\text{Step 4:}}$ We claim that $\mathbb{B}_{r/2} \subseteq T(\mathbb{D})$. We know that $\overline{T(\mathbb{B}_a)} \supset ra$. If we choose y such that $\|y\| < \frac{r}{2}$ this implies that $\mathbb{B}(0, r/2) \subseteq \overline{T(\mathbb{B}(0, 1/2))}$. This means that there exists $x_1 \in \mathbb{B}_{y_2}$ suh that $\|y - T(x_1)\| < r/4$. Note that $y - T(x_1) \in \mathbb{B}_{r/4} \subseteq \overline{T(\mathbb{B}_{1/4})}$. so there exists $x_2 \in \mathbb{B}_{1/4}$ such that

$$\|y - T(x_1) - T(x_2)\| < r/8.$$

Continue inductively and we get $x_n \in \mathbb{B}_{2^{-n}}$ and

$$\left\| y - \sum_1^T (x_j) \right\| < r/2^{n+1} \qquad (*)$$

Clearly the partial sums are Cauchy. By completeness this converges and so there exists $x = \lim \sum_1^n x_j$ with $\|x\| < 1$ (since $\|x\| \leq \sum \|x\|_j < \|x_1\| + \sum_2^\infty 2^{-n} < 1/2 + 1/2 = 1$). We claim that $y = T(x)$. This is true since by continuity

$$T\left(\sum_1^n x_j\right) = \sum T(x_j) \longrightarrow T(x).$$

And we know from (*) that the above also converges to y. □

Corollary 32. *If X, Y are Banach spaces and $T \in \mathscr{L}(X, Y)$ with T bijective then $T^{-1} \in \mathscr{L}(Y, X)$ (i.e. T is an isomorphism).*

Proof. We need only prove that T^{-1} is continuous. But this follows immediately from the fact that T is open.

□

Definition 121. *Suppose that X is a vector space. Recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be* ***equivalent*** *if there exist $m, M > 0$ such that $m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_2$.*

Corollary 33. *Suppose X has two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and that $\|x\|_2 \leq M\|x\|_1$ for all $x \in X$. Also assume that X is complete with respect to both norms. It follows then that the norms are equivalent.*

Proof. Let I denote the identity map. We have $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2) : x \mapsto x$. This is clearly linear and bijective. Our hypothesis tells us that I is bounded and thus by the previous corollary I^{-1} is bounded. This implies that $\|x\|_2 \leq m\|x\|_1$. □

16.3 The Closed Graph Theorem

Definition 122. *Let $T : X \to Y$ be normed vector spaces. Let $\Gamma(T) = \{(x, T(x)) \in X \times Y\}$. Observe that $X \times Y$ is a normed vector space with $\|(x, y)\| = \|x\| + \|y\|$. We say that T is* ***closed*** *if $\Gamma(T)$ is a closed set.*

Note that if T is continuous then T is closed.

Theorem 74 (Closed Graph Theorem (Banach))**.** *For X, Y Banach spaces. If $T : X \to Y$ is linear and closed then T is continuous ($T \in \mathscr{L}(X, Y)$).*

Proof. Notice that $\Gamma(T)$ is a closed subspace of a Banach space (since $X \times Y$ is also a Banach space). Hence $\Gamma_1(T)$ with the induced norm is also Banach. It follows that the projection maps π_X, π_Y are linear maps. We then have $\pi_X : \gamma(T) \to X$ is a continuous, bijection, therefore π_X^{-1} is continuous (by the corollary to the open mapping theorem). Also, we know that $T = \pi_Y \circ \pi_X^{-1}$, whence T is continuous. □

16.3.1 Why is this useful?

T is continuous means that $x_n \to x$ implies that $T(x_n) \to T(x)$. If T is closed then $x_n \to x$ implies that $T(x_n) \to y$ implies that $y = T(x)$.

Proposition 58. *Let X, Y be normed vector spaces. Let $T_n \in \mathscr{L}(X, Y)$ with $\sup_n \|T_n\| < \infty$. Let $T \in \mathscr{L}(X, Y)$ with $T_n x \to Tx$ for all $x \in E$ with E dense in X. Then*

$$T_n x \to Tx, \qquad \forall x \in X.$$

Proof. Without loss of generality suppose that $T = 0$ (take $T_n - T$) and suppose that $T_n x \to 0$ for all $x \in E$. Pick $x \in X$ and let $\epsilon > 0$ There exists $x' \in E$ such that $\|x - x'\| < \epsilon$. So there exists N such that for all $n \geq N$ we have$\|T_n x' - Tx'\| = \|T_n x'\| < \epsilon$. By the triangle inequality we have

$$\|T_n x\| \leq \left\|T_n x'\right\| + \left\|T_n(x - x')\right\| < \epsilon + \|T_n\| \|x - x_n\| < \epsilon(M + 1).$$

□

17 Modes of Convergence

17.1 Weak and Weak* Convergence

Definition 123. *Let X be a Banach space with X^* the dual. For $\{x_n\} \subseteq X$ we say that $x_n \to x$* ***weakly*** *if for all $\ell \in X^*$ we have $\ell(x_n) \to \ell(x)$. This also lets us define the* ***weak topology*** *which is the smallest topology such that $\ell : X \to \mathbb{C}$ are continuous. A basis for this topology is finite intersections of $\{y : |\ell(y-x)| < \epsilon\}$.*

Definition 124. *For $\ell_n \in X^*$ we say that ℓ_n converges* ***weak**** *to ℓ if for all $x \in X$ we have $\ell_n(x) \to \ell(x)$. The* ***weak*-topology*** *is the smallest topology for which $\ell \mapsto \ell(x)$ is continuous for all $x \in X$.*

17.1.1 Remark

- If X is a Banach space then so is X^*. The two topologies coincide when X is reflexive.
- By the uniform boundedness principle, if $x_n \to x$ weakly then $\sup \|x_n\| < \infty$. And if $\ell_n \to \ell$ weak* then $\sup \|\ell_n\| < \infty$.

17.1.2 The case of L^p

Proposition 59. *Let $1 \le p < \infty$ and let q be such that $1/p + 1/q = 1$. We have $f_n \to f$ weakly if and only if for all $g \in L^q$ with $\|g\|_q \le 1$ we have*

$$\int f_n g \to \int fg.$$

Proposition 60. *Suppose $L^q(X, \mu)$ is separable, then the unit ball in L^p is weakly sequentially compact. (Note: this is not true in general for L^1.)*

Proof. We know that $\|f_n\|_p \le 1$ and we want to show that there exists $f_{n_k} \to f$ weakly.

$\boxed{\text{Step 1:}}$ Choose $\{g_k\} \subseteq \mathbb{D} \subseteq L^q$ a dense countable set in the unit ball in L^q. We will show that there exists f_{n_i} such that $\int f_{n_i} g_k$ converges as $i \to \infty$ for every k. To prove this we make use of a diagonal type argument. Start with g_1 and look at

$$\left| \int f_n g_1 \right| \le \|f_n\|_p \|g_1\|_q \le 1.$$

There exists a subsequence n_j such that $\int f_{n_j} g_1 \to \alpha_1$. Then take g_2 and look at a subsequence of $\{n_j\}$ such that $\int f_{n_j^{(1)}} \to \alpha_2$. Continue and then take the diagonal.

$\boxed{\text{Step 2:}}$ By the useful proposition we have that $\int f_{n_i} g \to \alpha(g)$ for all $g \in L^q$.

$\boxed{\text{Step 3:}}$ Observe that $g \mapsto \alpha(g)$ is a bounded linear functional on L^q. So we have

$$|\alpha(g)| \le \sup_n |f_n g| \le \sup \|f\|_p \|g\|_q .$$

And this implies that $\alpha(g) = \int fg$ for some $f \in L^p$ so $f_n \to f weakly$. □

17.1.3 Remark

This is not necessarily true in L^1 consider the sequence $f_n = n\mathbb{1}_{[0,1/n]}$. We claim that there is no weakly convergent subsequence. To show a contradiction suppose that $\int f_{n_i} g \to \int fg$ for all $g \in L^\infty$. Then $f = 0$ almost everywhere. But if we take $\int f_{n_i} 1 = 1$ for all n, so this is impossible.

17.1.4 Recall Past Homework

- Exercise 6.20 in Folland we proved that for $1 < p < \infty$ if $f_n \to f$ pointwise a.e. and $\sup \|f_n\|_p < \infty$ then $f_n \to f$ weakly.
- Exercise 6.22 we proved that $f_n \to f$ weakly does not imply that $f_n \to f$ a.e.

17.1.5 More Remarks

(i) If X is an infinite dimensional normed vector space then the unit ball is not compact.

Theorem 75 (Alaoglu)**.** *If X is a normed vector space then the unit ball in X^* is compact in the weak* topology.*

(ii) If X is a separable normed vector space then the weak* topology is second countable implies metrizable. This cannot be defined by a translation invariant metric.

17.2 Fréchet Spaces

Definition 125. *X is a **topological vector space** if addition and scalar multiplication are continuous with respect to the given topology.*

Definition 126. *Recall that a **seminorm** is a function $p : X \to [0, \infty)$ if*

(i) $p(\alpha x) = |\alpha| p(x), \qquad \forall x, \alpha.$

(ii) $p(x + y) \leq p(x) + p(y), \qquad \forall x, y.$

Let $\{p_\alpha\}_{\alpha \in A}$ be a family of seminorms. We define the set $U_{x,\alpha,\epsilon} := \{y \in X : p_\alpha(y - x) < \epsilon\}$. Let $\mathcal{T}$ be the topology generated by $U_{x,\alpha,\epsilon}$ with $x \in X, \alpha \in A, \epsilon > 0$.

Definition 127. *X is Fréchet if X is complete Hausdorff topological vector space with the topology defined by countably many seminorms.*

17.2.1 Example

Consider $C^\infty([0, 1])$ is Fréchet with $p_k = \sup_{x \in [0,1]} |f^{(k)}(x)|$.

17.3 Radon Measures

Let X be compact and let $C(X)$ denote the space of continuous functions on X. Our goal is to prove the following theorem.

Theorem 76 (Riesz)**.** *For all $\ell \in C(X)^*$ there exists a unique μ (finite signed Borel measure) such that*

$$\ell(f) = \int f d\mu, \qquad \forall f \in C(X).$$

The first major step is to introduce the following definition.

Definition 128. *A linear functional $\ell : C(X) \to \mathbb{R}$ is called **positive** if $\ell(f) \geq 0$ if $f \geq 0$.*

Lemma 35. *If ℓ is positive linear then ℓ is bounded and $\|l\| = \ell(\mathbb{1})$ (Here $\mathbb{1}$ denotes the identity map).*

Proof. We can say that $f(x) \leq \|f\|_u \implies \ell(f) \leq \ell(\mathbb{1})\,\|f\| \implies \|l\| \leq \ell(\mathbb{1})$. Therefore

$$\|l\| = \sup\{|\ell(g) : \|g\| = 1\} \geq \ell(\mathbb{1}).$$

□

Theorem 77. *Let X be a compact metric space and $\ell : C(X) \to \mathbb{R}$ be a positive linear functional. Then there exists a unique finite (positive) Borel measure μ such that*

$$(*) \qquad \ell(f) = \int f d\mu, \qquad \forall f \in C(X).$$

Proof. $\boxed{\text{Step 1:}}$ First we define on open sets. If $\mathcal{O}$ is an open set we let

$$\rho(\mathcal{O}) := \sup\{\ell(f) : 0 \leq f \leq 1, \operatorname{supp} f \subseteq \mathcal{O}\}.$$

Also define an exterior measure for all $E \subseteq X$

$$\mu_*(E) := \inf\{\rho(\mathcal{O}) : E \subseteq \mathcal{O}, \mathcal{O} \text{ open}\}.$$

We prove that μ_* is a measure.

(i) $E_1 \subseteq E_2$ clearly implies that $\mu_*(E_1) \leq \mu_*(E_2)$.

(ii) $E = \mathcal{O}$ implies that $\mu_*(\mathcal{O}) = \rho(\mathcal{O})$.

(iii) The proof of sub-additivity takes some work.

(I) First we prove this holds on open sets. Let $\mathcal{O} = \bigcup_1^\infty \mathcal{O}_k$ with the $\mathcal{O}_k$ open. Pick $f \in C(X)$ with $0 \leq f \leq 1$ and $K = \sup(f) \subseteq \mathcal{O}$. We have K is compact and since $K \subseteq \bigcup \mathcal{O}_k$ we can refine to a finite cover $\bigcup_1^N \mathcal{O}_k$. Let $\{\eta_k\}_1^N$ be a partition of unity subordinate to $\bigcup_1^N \mathcal{O}_k$. Recall that $\operatorname{supp} \eta_k \subseteq \mathcal{O}_k$ with $\sum_1^N \eta_k = 1$ on K. We then have $f = \sum_1^N f\eta_k$ this implies that

$$\ell(F) = \sum_1^N \ell(f\eta_k) \leq \sum_1^N \rho(\mathcal{O}_k).$$

It follows that $0 \leq f\eta_k \leq 1$ and $\operatorname{supp}(f\eta_k) \subseteq \mathcal{O}_k$. Letting $N \to \infty$ we have

$$\ell(f) \leq \sum_1^\infty \rho(\mathcal{O}_k)$$

then the supremum over all f gives $\rho(\mathcal{O}) \leq \sum_!^\infty \rho(\mathcal{O}_k)$. This is true if and only if

$$\mu_*(\mathcal{O}) \leq \sum \mu_*(\mathcal{O}_k).$$

(II) Now we prove sub-additivity on all sets. Let $E_k \subseteq X$ and $\epsilon > 0$. We know that for all k there exists $\mathcal{O}_k \supset E_k$ such that

$$\mu_*(\mathcal{O}_k) \leq \mu_*(E_k) + \epsilon/2^k.$$

Letting $\mathcal{O} = \bigcup \mathcal{O}_k$ we have $\mathcal{O}$ covers $\bigcup E_k$. This yields

$$\mu_*\left(\bigcup E_k\right) \leq \mu_*(\mathcal{O}) \leq \sum \mu_*(\mathcal{O}_k) \leq \sum \mu_*(E_K) + \epsilon.$$

(iv) Now we prove that μ_* is a metric outer measure. That is, we wish to prove that

$$d(E_1, E_2) > 0 \implies \mu_*(E_1 \cup E_2) = \mu_*(E_1) + \mu_*(E_2).$$

Also note that μ_* is additive on disjoint open sets since

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \implies \rho(\mathcal{O}_1 + \mathcal{O}_2) = \rho(\mathcal{O}_1) + \rho(\mathcal{O}_2).$$

It follows that $\rho(\mathcal{O}) = \sup\{\ell(f) : \operatorname{supp} f \subseteq \mathcal{O}, \mathcal{O} \leq f \leq 1\}$, and so write $f = f_1 + f_2$.

Continuing along we have

$$(E_1, E_2) > 0 \implies \exists \mathcal{O}_1, \mathcal{O}_2 \text{ disjoint}, \quad E_i \subseteq \mathcal{O}_i.$$

For al $\mathcal{O} \supset E_1 \cup E_2$ we have

$$\mathcal{O} \supset (\mathcal{O} \cap \mathcal{O}_1) \cup (\mathcal{O} \cap \mathcal{O}_2).$$

Also,

$$\mu_*(\mathcal{O}) \geq \mu_*(\ O \cap \mathcal{O}_1) + \mu_*(\mathcal{O} \cap \mathcal{O}_2),$$

and this implies that $\mu_*(E_1 \cup E_2) \geq \mu_*(E_1) + \mu_*(E_2)$.

So μ_* is indeed a measure on the open sets.

$\boxed{\text{Step 2:}}$ Applying the Caratheodory extension theorem we conclude that there exists a Borel measure, μ, on the Borel sets such that $\mu = \mu_*$ on the open sets. Observe that μ satisfies $\ell(f) = \int f d\mu$ and since we can write $f = f^+ - f^-$ it suffices to consider $f \geq 0$. Rescaling lets us simplify further to just consider functions $0 \leq f \leq 1$.

Fix such an f and fix $N \geq 1$. Define $\mathcal{O}_0 = X$ for $n \geq 1$ we define

$$\mathcal{O}_n := \{x : f(x) > \frac{n-1}{N}\}.$$

Notice that $\mathcal{O}_n$ is a decreasing family of open sets. We also define a sequence of functions

$$f_n(x) := \begin{cases} \frac{1}{N}, & x \in \mathcal{O}_{n+1} \\ f(x) - \frac{n-1}{N}, & x \in \mathcal{O}_n \setminus \mathcal{O}_{n+1} \\ 0, & x \in \mathcal{O}_n^c \end{cases}.$$

Note that $\sum_1^n f_n = f$ and $Nf_n \equiv 1$ on $\mathcal{O}_{n+1}$. We also have $\operatorname{supp}(Nf_n) \subseteq \overline{\mathcal{O}}_n \subseteq \mathcal{O}_{n-1}$ and so $0 \leq Nf_n \leq 1$.

We now have

$$\mu(\mathcal{O}_{n+1}) \leq \ell(Nf_n) \leq \mu(\mathcal{O}_{n-1}).$$

This means we can average and write

$$(1/N)\sum_1^N \mu(\mathcal{O}_{n+1}) \leq (1/N)\sum_1^N \ell(Nf_n) \leq (1/N)\sum_1^N \mu(\mathcal{O}_n).$$

Also we have

$$\mu(\mathcal{O}_{n+1}) \leq \int Nf_n d\mu \leq \mu(\mathcal{O}_{n-1}).$$

This means that

$$(1/N)\sum_1^N \mu(\mathcal{O}_{n+1}) \leq \int f_n d\mu \leq (1/N)\sum_1^N \mu(\mathcal{O}_{n-1}).$$

Therefore as $N \to \infty$

$$\left|\ell(f) - \int f d\mu\right| \leq (1/N)\sum_1^N (\mu(\mathcal{O}_{n-1}) - \mu(\mathcal{O}_{n+1}) = \frac{2\mu(X)}{N} \to 0.$$

$\boxed{\text{Step 3:}}$ We finish by proving uniqueness. Suppose ν is another finite Borel measure satisfying the theorem. It is enough to prove that the measures agree on open sets. Let $\mathcal{O}$ be open. Recall that finite Borel measures are inner-regular. Meaning, for any $\epsilon > 0$ we can find K a closed set such that

$$m(\mathcal{O} \setminus K) < \epsilon.$$

There exists $f \in C(X)$ such that $0 \leq f \leq 1$, $\operatorname{supp} f \subseteq \mathcal{O}$ and $f \equiv 1$ on K. We now have

$$\mu(\mathcal{O}) < \mu(K) + \epsilon = \int_K f d\mu + \epsilon \leq \int f d\mu + \epsilon = \ell(f) + \epsilon = \int f d\nu + \epsilon \leq \int_{\mathcal{O}} d\nu + \epsilon = \nu(\mathcal{O}) + \epsilon.$$

Since ϵ is arbitrary we have $\mu(\mathcal{O}) \leq \nu(\mathcal{O})$, by symmetry it holds that $\mu(\mathcal{O}) = \nu(\mathcal{O})$.

□

17.4 Riesz Representation Theorem

Proposition 61. *Let X be a compact metric space and $\ell : C(X) \to \mathbb{R}$ be a bounded linear functional. Then there exists ℓ^+, ℓ^- positive linear such that $\ell = \ell^+ - \ell^-$. Moreover,*

$$\|\ell\| = \ell^+(1) - \ell^-(1) = \left\|\ell^+\right\| + \left\|\ell^-\right\|.$$

Proof. Let $f \in C(X)$ and $f \geq 0$. Let $\ell^+ := \sup\{\ell(\varphi) : 0 \leq \varphi \leq f\}$. We have to check several things. First note that

- $|\ell(\varphi)| \leq \|\ell\| \, \|\varphi\| \leq \|\ell\| \, \|f\|$.
- $0 \leq \ell^+(f) \leq \|\ell\| \, \|f\|$ (since we can let $\varphi \equiv 0$).
- $\ell(f) \leq \ell^+(f)$ since we can take $\varphi = f$.

$\boxed{\text{Step 1:}}$ We check that ℓ^+ is linear. Let $\alpha \geq 0$ and $f \geq 0$. It is not hard to see that $\ell^+(\alpha f) = \alpha \ell^+(f)$. Let $f, g \geq 0$, we have for any $0 \leq \varphi \leq f$ and $0 \leq \varphi + \psi \leq f + g$. This implies that

$$\ell^+(f+g) \geq \ell^+(f) + \ell^+(g).$$

Let $0 \leq \varphi \leq f + g$ and let $\varphi_1 = \min\{\varphi, f\}, \varphi_2 = \varphi - \varphi_1$. Observe that

$$0 \leq \varphi_1 \leq f, \qquad 0 \leq \varphi_2 \leq f.$$

If $\varphi_1(x) = f(x) \leq \varphi(x)$ then

$$\varphi_2(x) = \varphi(x) - \varphi_1(x) \leq \varphi(x) - f(x) \leq g(x).$$

Also, $\ell(\varphi) = \ell(\varphi_1) + \ell(\varphi_2) \leq \ell^+(f) + \ell^+(g)$. Taking the supremum over φ gives

$$\ell^+(f+g) \leq \ell^+(f) + \ell^+(g).$$

$\boxed{\text{Step 2:}}$ Next we extend ℓ^+ to be a positive linear functional on $C(X)$. For $f \in C(X)$ we write $f = f^+ - f^-$ and let $\ell^+(f)\ell^+(f) - \ell^+(f)$. This is independent on the decomposition and it is easy to check that $|^+$ is linear on all of $C(X)$. Also note that $\|\ell^+\| \leq \|l\|$. To see this notice that for $f = f^+ - f^-$ we have from the fact that $\ell(f) \leq \ell^+(f)$ that

$$\ell^+(f^+ - f^-)| = |\ell^+(f^+) - \ell^+(f^-)| \leq \max\{\ell^+(f^+), \ell^+(f^-)\} \leq \max\{\|\ell\| \left\|f^+\right\|, \|l\| \left\|f^-\right\|\} = \|l\| \|f\|.$$

$\boxed{\text{Step 3:}}$ We define $\ell^- = \ell^+ - \ell$.

$\boxed{\text{Step 4:}}$ At this point we have proven that $\ell = \ell^+ - \ell^-$ and ℓ^+, ℓ^- are positive linear. We finish by proving the equality statement at the end of the theorem. We then note that

$$\|\ell\| = \left\|\ell^+ - \ell^-\right\| \leq \left\|\ell^+\right\| + \left\|\ell^-\right\| = \ell^+(1) + \ell^-(1).$$

For the reverse let $0 \leq \varphi \leq 1$, then

$$|2\varphi - 1| \leq 1 \implies \|\ell\| \geq \ell(2\varphi - 1) \implies \ell(2\varphi - 1)| \leq \|\ell\| \|2\varphi - 1\| \leq \|l\|.$$

And this implies that

$$\ell(2\varphi - 1) = 2\ell(\varphi) - \ell(1) \leq \|\ell\|,$$

which yields

$$2\ell(\varphi) \leq \|l\| + \ell(1).$$

Taking the supremum over all φ and noting that $\ell^+(1) := \sup\{\ell(\varphi) : 0 \leq \varphi \leq 1\}$ we obtain

$$2\ell^+(1) \leq \|\ell\| + \ell^+(1) - \ell^-(1) \implies \ell^+(1) + \ell^-(1) \leq \|\varphi\| \implies \|\ell\| = \ell^+(1) + \ell^-(1).$$

□

Theorem 78 (Riesz Representation Theorem)**.** *Let X be a compact metric space. For every bounded linear functional $\ell : C(X) \to \mathbb{R}$ ther exists a unique finite Borel (signed) measure μ on X such that*

$$\ell(f) = \int f d\mu, \qquad \forall f \in C(X),$$

and

$$\|\ell\| = \|\mu\| := |\mu|(X).$$

Proof. Given ℓ write $\ell = \ell^+ - \ell^-$. By the previous theorem on positive linear functionals there exist μ^+, μ^- positive finite Borel measures satisfying

$$\ell^+(f) = \int f d\mu^+, \quad \ell^-(f) = \int f d\mu^-, \qquad \forall f \in C(X).$$

To prove the equality at the end of theorem we let $\mu = \mu^+ - \mu^-$ and note that by the previous proposition and the triangle inequality we have

$$\|\ell\| = \ell^+(1) + \ell^-(1) = \mu^+(X) + \mu^-(X) \geq |\mu|(X).$$

Also we have from last quarter that

$$\ell(f)| = \left|\int f d\mu\right| \leq \int |f| d|\mu| \leq \|f\| \, |\mu|(X) \implies \|\ell\| \leq \|\mu\| \, .$$

To prove uniqueness we suppose that

$$\int f d\mu = \int f d\nu, \qquad \forall f.$$

Let $\eta = \mu - \nu$. We then have $\int f d\eta = 0$ for all f. We can write

$$\eta = \eta^+ - \eta^-.$$

Also, for all f we have $\int f d\eta^+ = \int f d\eta^-$ which implies that $\eta^+ = \eta^-$ by uniqueness for positive functions.

□

17.5 Extensions of Riesz Representation

Definition 129.

$$C_c(X) = \{f \in C(X) : \operatorname{supp} f \textit{ is compact}\}$$
$$C_0(X) = \{f \in C(X) : \forall \epsilon > 0, \exists K_\epsilon \textit{ compact, such that } |f(x)| \leq \epsilon \textit{ on } K_\epsilon^c\}.$$
$$\overline{C_c(X)} C_0(X).$$

Definition 130. *$\mu \geq 0$ is a Radon measure if μ is*

(i) Borel.

(ii) Finite on compact sets.

(iii) Outer regular on Borel sets.

(iv) Inner regular on open sets.

Theorem 79 (see Folland)**.** *Let X be LCH. For all ℓ positive linear functionals on $C_c(X)$ there exists μ a positive Radon measure on X such that*

$$\ell(f) = \int f d\mu, \qquad \forall f \in C_c(X).$$

Theorem 80 (From Autumn Quarter SSIII)**.** *If X is a metric space and μ is a Borel measure which is finite on balls then it is regular (inner regular on closed sets). If also X is σ-compact then μ is Radon.*

Definition 131. *Let X be LCH. We define $M(X)$ to be the space of finite complex Radon measures.*

Theorem 81. *Let X be LCH then $C_0(X)^* \simeq M(X)$ are naturally isomorphic. More precisely for μ Radon, let $I_\mu : f \mapsto \int f d\mu$. It hold that $\|I_\mu\| = \|\mu\|$ and for all $: C_0(X) \to \mathbb{C}$ bounded and linear there exists $\mu \in M(X)$ such that $\ell = I_\mu$.*

Proof. This can be proven in full generality using one-point compactification. We will give the proof for $X = \mathbb{R}^n$ using Riesz representation theorem for compact sets. Take $\ell : C_0(X) \to \mathbb{C}$ a bounded linear functional. We have ℓ induces a bounded linear functional on [PROOF ERROR!] □

Proposition 62. *If μ is a regular Borel measure on X then $C_c(X)$ is dense in $L^p(\mu)$, $1 \leq p < \infty$.*

Proof. We know that simple functions are dense in L^p. Hence it is enough to approximate $\mathbb{1}_E$ where E is Borel with finite measure. By regularity we know for every $\epsilon > 0$ there exists $K \subseteq E$ compact and $\mathcal{O} \supset E$ open such that $m(\mathcal{O} \setminus K) < \epsilon$. By Urysohn's lemma there exists f which is continuous and $\mathbb{1}_K \leq f \leq \mathbb{1}_{\mathcal{O}}$. It follows that

$$\|f - \mathbb{1}_E\|_p^p \leq \int_{\mathcal{O}\setminus K} 1^p d\mu < \epsilon.$$

□

17.6 The Vague Topology

Definition 132. *Let X be LCH, the **vague topology** (also called **weak*-topology**) on $M(X)$ is the topology generated by*

$$\mu \mapsto \int f d\mu, \qquad \forall f \in C_0(X).$$

We write $\mu_n \rightharpoonup \mu$ (or $\mu_n \to \mu$ vaguely) if and only if for all $f \in C_0(X)$ it holds that $\int f d\mu_n \to \int f d\mu$.

17.6.1 Recall

If $C_0(X)$ is separable then for all $\{\mu_n\} \subseteq M(X)$ with $\sup \|\mu_n\| < C$ there exists a subsequence such that $\mu_{n_k} \rightharpoonup \mu$.

Note: Weak convergence arguments for L^p usually fail for L^1 because $L^1 \neq (L^\infty)^*$, but consider $L^1 \subseteq M(X)$ then use vague convergence.

Proposition 63. *Suppose $\mu, \mu_n \in M(\mathbb{R})$. Consider their cumulative distribution function (CDF) given by*

$$F_n(x) := \mu_n((-\infty, x]), \qquad F(x) := \mu((-\infty, x]).$$

(a) If $\sup \|\mu_n\| < \infty$ and $F_n(x) \to F(x)$ for all x where F is continuous then $\mu_n \rightharpoonup \mu$.

(b) Assume that $\mu_n \geq 0, \mu_n \to \mu$ vaguely. Then $\sup \|\mu_n\| < \infty$ and $F_n(x) \to F(x)$ for all x where F is continuous.

Proof. Sketch of (a). We know that F is continuous except maybe on a countable set. Hence $F_n \to F$ a.e. (Lebesgue) and $\|F_n\|_u \leq \|\mu_n\|$. We want to show that $\int f d\mu_n \to \int f d\mu$ for every $f \in C_0$. We will prove this for a dense set of $C_0(\mathbb{R})$. We take $C_c^1(\mathbb{R})$ which is dense in $C_0(\mathbb{R})$. If $f \in C_c^1(\mathbb{R})$ then by integration by parts we have from the fact that $dF_n = d\mu_n$

$$\int f d\mu_n = -\int f'(x) F_n(x) dx \overset{*}{\to} -\int f'(x) F(x) dx = \int f d\mu.$$

Where at $(*)$ we use dominated convergence.

□

17.6.2 Example

In the (b) it is necessary to assume that the measures are positive. Consider $\mu_n = \delta_{1/n} - \delta_{-1/n}$. Then we have $\sup \|\mu_n\| = 2$ and for every $f \in C_0(\mathbb{R})$ it holds that

$$\int f d\mu = f(1/n) - f(-1/n) \to 0.$$

Therefore $\mu_n \to 0$ vaguely.

18 Compact Operators

18.1 Introduction

Definition 133. *Let X, Y be Banach spaces. A linear operator $T : X \to Y$ is* ***compact*** *if given any bounded sequence $\{u_k\} \subseteq X$ (i.e. $\sup_k \|u_k\| < \infty$) then there exists a subsequence u_{n_k} such that $T(u_{n_k})$ converges in Y.*

Proposition 64. *If T is compact than T is bounded.*

Proof. Notice that if $\mathbb{D}$ is compact then $\overline{T\mathbb{D}}$ is compact. Let $\{\mathbb{B}(0,m)\}_1^\infty$ be an open cover for $\overline{T\mathbb{D}}$. There exists $M \in \mathbb{N}$ such that $\overline{TB} \subseteq \mathbb{B}(0,M)$. Hence for all $x \in X$ with $\|x\| \leq 1$ we have

$$|Tx| \leq M \implies \|T\| \leq M.$$

□

18.1.1 Example

(i) Let $\mathcal{H}$ be a Hilbert space. We know tht $\mathbb{D} = \{f \in \mathcal{H} : \|f\| = 1\}$ is not sequentially compact. This means that $\mathbb{1} : \mathcal{H} \to \mathcal{H}$ is *not* a compact operator.

(ii) Let $T : \mathcal{H} \to \mathcal{H}$, with $\mathcal{H}$ a Hilbert space and T bounded with finite rank. If $T\mathcal{H}$ is finite dimensional we have T is compact. So, if $\{u_k\} \subseteq \mathcal{H}$ is bounded since T is bounded we know that $\{Tu_k\}$ is bounded in $\mathbb{R}^d$ which implies that there exists a convergent subsequence $\{u_{n_k}\}$ and thus $\{Tu_{n_k}\}$ converges by continuity.

Proposition 65. *Let $T : \mathcal{H} \to \mathcal{H}$ with T bounded and linear then*

1. *If $S : \mathcal{H} \to \mathcal{H}$ is compact then ST and TS are compact.*
2. *If $\{T_n\}$ is a family of compact operators with $T_n : \mathcal{H} \to \mathcal{H}$ and $\|T_n - T\| \to 0$ then T is compact.*
3. *Conversely, if T is compact there exists a sequence $\{T_n\}$ of finite rank operators such that*
$$\|T_n - T\| \to 0, \qquad n \to \infty.$$
4. *T is compact if and only if T^* is compact.*

Proof. 1. If $\{f_k\} \subseteq \mathcal{H}$ is bounded it follows that $|Tf_k| \leq \|T\| \|f_k\|$ and thus $\{Tf_k\}$ is bounded. Since S is compact, there exists a subsequence such that $\{STf_{n_k}\}$ converges. For the other direction, since $\{f_k\}$ is bounded there exists a subsequence so that $\{Sf_{n_k}\}$ converges. By continuity we must then have $\{TSf_{n_k}\}$ converges.

2. Let $\{f_k\}$ be bounded by M. Since T_1 is compact there exists a subsequence $\{f_{n_k^{(1)}}\} \subseteq \{f_k\}$ such that $\{T_1 f_{n_k^{(1)}}\}$ converges. Let $\{f_{n_k^{(2)}}\} \subseteq \{f_{n_k^{(1)}}\}$ such that $\{T_1 f_{n_k^{(2)}}\}$ converges. Repeat this so that for each j we have $\{T_1 f_{n_k^{(j)}}\}$ converges. Let $f_{n_k} = f_{n_k^{(k)}}$. We claim that $\{Tf_{n_k}\}$ is Cauchy. We can use the triangle inequality to write

$$\begin{aligned}\left\|Tf_{n_k} - Tf_{n_j}\right\| &\leq \|Tf_{n_k} - Tmf_{n_k}\| + \left\|T_m f_{n_k} - T_m f_{n_j}\right\| + \left\|Tf_{n_k} - T_m f_{n_j}\right\| \\ &\leq \|T - T_m\| \|f_{n_k}\| + \|T - T_m\| \left\|f_{n_j}\right\|\end{aligned}$$

Notice that

$$\left\|Tf_{n_k} - Tf_{n_j}\right\| \leq 2\left\|T - T_m\right\| M + \left\|T_m f_{n_k} - T_m f_{n_j}\right\|.$$

Let $\epsilon > 0$. There exists m_0 such that for all $m \geq m_0$ it holds that

$$\|T - T_m\| \leq \frac{\epsilon}{4M}.$$

Fix $m \geq m_0$. For this m there exist s$k_0 \in \mathbb{N}$ such that for $k, j \geq k_0$ it holds that

$$\left\|T_m f_{n_k} - T_m f_{n_j}\right\| \leq \epsilon/2.$$

This lets us bound

$$\left\|Tf_{n_k} - Tf_{n_j}\right\| \leq 2\left\|T - T_m\right\| M + \left\|T_m f_{n_k} - T_m f_{n_j}\right\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

3. Let T be compact and let $\{e_i\}$ be an orthonormal basis. Consider the span $\Gamma_n = \text{span } \{e_1, \ldots, e_n\}$. We know that the projection map $P_n : \mathcal{H} \to \Gamma_n$ and there is a complimentary projection $Q_n : \mathcal{H} \to \text{span } \{e_j\}_{j>n}$. We can write $\mathbb{1} = P_n + Q_n$. Each map $T_n := P_n T$ is finite rank. We claim that $\|T_n - T\| \to 0$ as $n \to \infty$. We proceed by contradiction. Let $f \in \mathcal{H}$. We can write

$$\|Q_{n+1}Tf\|^2 = \sum -j \geq n + 1(Tf, e_j)^2 \leq \sum_{j \geq n}(Tf, e_j)^2 \leq \|Q_n Tf\|^2.$$

In particular we have $\lim_{n\to\infty} \|Q_n Tf\| = c_0 < \infty$. Also observe that

$$\|T - T_n\| = \|T - P_n T\| = \|Q_n T\|.$$

So it suffices to prove that $\|Q_n T\| \to 0$. Our calculation tells us that

$$\|Q_{n+1}Tf\| \leq \|Q_n Tf\| \leq \|Q_n T\| \|f\|.$$

Hence

$$\|Q_{n+1}T\| \leq \|Q_n T\| \implies \lim_{n\to\infty} \|Q_n T\| = c < \infty.$$

To show a contradiction, assume that $c > 0$. There exists $\{f_n\}$ with $\|f_n\| = 1$. We know from the definition of the norm and the supremum that $\|Q_n T f_n\| \geq c/2$. Since T is compact there exists $Tf_{n_k} \to g \in \mathcal{H}$. We can write

$$Q_{n_k} g = Q - n_k T f_{n_k} + Q_{n_k}(g - Tf_{n_k}).$$

It follows from the fact that $\|Q_{n_k}\| \leq 1$ that

$$|Q_{n_k}(g - Tf_{n_k}| \leq \|Q_{n_k}\| \|g - Tf_{n_k}\| \leq \|g - Tf_{n_k}\| \to 0.$$

Thus there exist $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$\|Q_{n_k} g\| \geq c/4.$$

This gives the contradiction

$$\sum_{j \geq n_k} (g, e_{n_j})^2 \geq c^2/16.$$

4. It is a fact of the dual that $(Tf, g) = (f, T^*g)$ and that $\|T\| = \|T^*\|$. Because T is compact we have $P_nT \to T$. Hence

$$\|P_nT - T\| = \|(P_nT - T^*)\| = \|(P_nT)^* - T^*\| = \|T^*P_n^* - T^*\|.$$

It is a general fact about projections that

$$(P_nf, g) = (P_nf, P_ng) = (f, P_ng),$$

and thus $P_n = P_n^*$. Applying this to the above line we have

$$\|P_nT - T\| = \|T^*P_n - T^*\| \to 0.$$

Since P_n is finite rank and P_n is compact we know from (1) that T^*P_n is compact. Property (2) tells us that T^* is compact.

□

Proposition 66. *If $T : \mathcal{H} \to \mathcal{H}$ is diagonalizable with respect to the basis $\{\varphi_k\}$ of eigenfunctions. In other words*

$$T\varphi_k = \lambda_k\varphi_k, \qquad \lambda_k \in \mathbb{R}.$$

Then T is compact if and only if $|\lambda_k| \to 0$.

Proof. Exercise in SSIV. □

18.1.2 Remark

For $f \in L^2(\mathbb{R}^d)$ we can consider operators of the form $Tf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$ with $k(x,y) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. These operators come up when one is trying to solve Dirichlet's problem on the disc.

18.2 Hilbert Schmidt Operators

Definition 134. *Let $\mathcal{H} = L^2(X)$ with $(X, \mathcal{M}, \mu)$ a σ-finite measure space. The map $T : \mathcal{H} \to \mathcal{H}$ give by*

$$Tf(x) = \int K(x,y)f(y)dy, \qquad f \in L^2(X).$$

*We call $K(x,y)$ the **kernel**. We say that T is **Hilbert-Schmidt** (HS) if $K \in L^2(X \times X)$.*

Proposition 67. *Suppose that T is HS.*

(i) $f \in L^2(X)$ implies that for a.e. $x \in X$ the map $y \mapsto K(x,y)f(y)$ is integrable.

(ii) $T : \mathcal{H} \to \mathcal{H}$ is bounded with $\|T\| \le \|K\|_{L^2(X\times X)}$.

(iii) T^* is also HS with kernel $\overline{K(y,x)}$.

Proof. (i) First we know that

$$\int \left(\int |K(x,y)||f(y)|d\mu(y) \right) d\mu(x) \le \infty.$$

It follows from Tonelli that almost everywhere we have $\int |K(x,y)|^2 d\mu(y) < \infty$.

We then have from this and Hölder that

$$\int |K(x,y)||f(y)|d\mu(y) \le \left(\int |K(x,y)|^2 |f(y)|d\mu(y) \right)^{1/2} \|f\|_2 < \infty.$$

(ii) Let $f \in L^2(X)$. We can write $\|Tf\|_2^2 = \int |T(f(x)|^2 d\mu(x)$. So we have

$$|T(f(x)|^2 \leq \int |K(x,y)|^2 d\mu(Y) \, \|f\|_2^2 \, .$$

Thus,

$$\|Tf\|_2^2 = \int |T(f(x)|^2 d\mu(x) \leq \|K\|_2^2 \, \|f\|_2^2 \, .$$

(iii) See Folland.

□

Proposition 68. *Hilbert-Schmidt operators on $\mathbb{R}^d$ are compact.*

Proof. We will prove on next weeks homework that for $\{\varphi_k\}$ an orthonormal basis of $L^2(\mathbb{R}^d)$ then $\{\varphi_k(x)\varphi_l(y)\}_{k,l=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. So let us write for any $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$K(x,y) = \sum_{k,l} a_{kl}\varphi_k(x)\varphi_l(y), \qquad \sum_{k,l}^{\infty} |a_{kl}|^2 < \infty.$$

Define $K_n(x,y) := \sum_{k,l=1}^{n} a_{k,l}\varphi_k(x)\varphi_l(y)$. Also define the operator

$$T_n f(x,y) := \int K_n(x,y) f(y) dy.$$

We claim that T_n has finite rank. This follows from the fact that

$$\int \varphi_k(x)\varphi_l(y) f(y) dy = \varphi_k(x) \int \varphi_y(y) f(y) dy = c_y \varphi_k(x).$$

Therefore rank$T_n \leq n$. Next we claim that $T_n \to T$. To see this observe that by the last proposition

$$\|T - T_n\|_2 \leq \|K - K_n\|_2 = \sum_{\max k,l > n} |a_{kl}|^2 \to 0.$$

Therefore it is compact. □

18.3 Spectral Theory

Definition 135. *A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is **symmetric** (or **self-adjoint**) if $T^* = T$.*

Our goal is to prove the following theorem:

Theorem 82 (Spectral Theorem for Self-Adjoint Compact Operators)**.** *If T is symmetric and compact then T is diagonalizable. That is there exists and orthonormal basis $\{e_n\}$ of eigenvectors (vectors which satisfy $Te_n = \lambda_n e_n$ with $\lambda_n \in \mathbb{R}$ and $\lambda_n \to 0$).*

18.3.1 Remark

The theorem fails if either of the conditions (i) symmetry or (ii) compactness are omitted.

(i) There exists a compact, non-symmetric operator without eigenvalues. Consider the operator T such that $Te_k = \lambda_k e_{k+1}$.

(ii) The Voleterra operator on $L^2([0,1])$ given by $Tf(x) = \int_0^x f(y)dy$ is HS (and thus compact). Notice that $K(x,y) = \mathbb{1}_{\{y \le x\}}$. However it is not symmetric because $K(y,x) \neq K(x,y)$.

(iii) The map $f(x) \mapsto xf(x)$ on $L^2([0,1])$ is symmetric and non-compact. It can be proven that this operator has no eigenvalues.

Lemma 36. *Let T be a bounded linear symmetric operator. Then*

(i) If λ is an eigenvalue of T then $\lambda \in \mathbb{R}$.

(ii) If $\lambda_1 \neq \lambda_2$ are eigenvalues then the corresponding eigenvectors f_1, f_2 are orthogonal.

Proof. (i) There exists $f \neq 0$ such that$Tf = \lambda f$. We have

$$\overline{\lambda}\,\|f\|^2 = (f, \lambda f) = (f, Tf) = (f, T^* f) = (Tf, f) = (\lambda f, f) = \lambda\,\|f\|^2 \implies \lambda = \overline{\lambda}.$$

(ii) We have $Tf_1 = \lambda_1 f_1$ and $Tf_2 = \lambda_2 f_2$. Therefore

$$(Tf_1, f_2) = \lambda_1 f_1, f_2) = \lambda_1 (f_1, f_2).$$

We also have the above is equal to

$$(f_1, T^* f_2) = (f_2, Tf_2) = (f_1, \lambda_2 f_2) = \lambda_2 (f_1, f_2),$$

since $\lambda_1 \neq \lambda_2$ it follows that $(f_1, f_2) = 0$.

□

Definition 136. *Given an operator T with eigenvalue λ we let $N(T - \lambda I)$ denote the* ***dimension of the eigenspace at*** *λ.*

Lemma 37. *[Lemma 4]*
Suppose that T is compact and $\lambda \neq 0$ it follows that

(i) $\dim \mathrm{Null}(T - \lambda I) < \infty$.

(ii) The eigenvalues of T for a sequence $\{\lambda_k\}$ satisfies $\lambda_k \to 0$. And for all $\mu > 0$ it holds that

$$\dim(\mathrm{Span}\{f_k : Tf_k = \lambda_k f_k, |\lambda_k| \ge \mu\}) < \infty.$$

Proof. (i) Let $V_\lambda := \mathrm{Null}(T - \lambda I)$. Suppose that $\dim(V_\lambda) = \infty$. Because we are working with an infinite dimensional Hilbert space we can form a sequence $\{\varphi_n\}_1^\infty$ an orthonormal basis. As T is compact we know that $\{T\varphi_n\}$ contains a convergent subsequence. But since the φ_n are eigenvectors we know that $\{T\varphi_n\} = \{\lambda\varphi_n\}$, but this contradicts the fact that the φ_n are orthogonal since the distance between any two elements is constant and thus cannot go to zero. More precisely,

$$\|\lambda\varphi_{n_k} - \lambda\varphi_{n_l}\|^2 = |\lambda|^2(\|\varphi_{n_k}\|^2 + \|\varphi_{n_l}\|^2) = 2|\lambda|^2 \not\to 0.$$

(ii) Suppose that T is symmetric. If not there exists $\{\varphi_n\}_1^\infty$ such that $T\varphi_n = \lambda_n\varphi_n$ with the λ_n are distinct. Without loss of generality, passing to a subsequence suppose that the $T\varphi_n$ converge. Thus the $\lambda\varphi_n$ converge, but as in (i) we have

$$\|\lambda_n\varphi_n - \lambda_k\varphi_k\|^2 \overset{*}{=} |\lambda_n|^2 \|\varphi_n\|^2 + |\lambda_k|^2 \|\varphi_k\|^2 > 2\mu^2 > 0.$$

Where the equality at (*) follows from the first lemma since $(\varphi_n, \varphi_k) = 0$. □

Lemma 38. *If $T = T*$ then $\|T\| = \sup\{|(Tf, f)|, \|f\| = 1\}$.*

Proof. Recall that

$$\|T\| = \sup\{\|Tf\| : \|f\| = 1\} = \sup\{|(Tf, g)| : \|f\| = 1, \|g\| = 1\}.$$

We use the polarization identity to write

$$4(Tf, g) = (T(f+g), f+g) - (T(f-g), f-g) + i(T(f+ig), f+ig) - i(T(f+ig), f-ig).$$

Denote $M = \sup\{|(T, f, f)|L \|f\| = 1$. We then have

$$|(Th, h)| \leq M \|h\|^2, \quad \forall h \in \mathcal{H}.$$

Also observe that since $\|f\| = 1$

$$|(Tf, f)| \leq \|Tf\| \|f\| \leq \|T\| \|f\|^2 = \|T\|.$$

Hence $M \leq \|T\|$. For the reverse inequality we use the fact that $(Th, h) \in \mathbb{R}$ for all h and the polarization identity to write

$$\begin{aligned}
4\operatorname{Re}(Tf, g) &= \operatorname{Re}[(T(f+g), f+g) - (T(f-g), f-g) + i(T(f+ig), f+ig) - i(T(f+ig), f-ig)] \\
&= (T(f+g), f+g) - (T(f-g), f-g) \\
&\leq M \|f+g\|^2 + M \|f-g\|^2 \\
&= M(2\|f\|^2 + 2\|g\|^2 +) \\
&= 2M(\|f\|^2 + \|g\|^2) \\
&= 4M
\end{aligned}$$

Therefore $\operatorname{Re}(Tf, g) \leq M$. Replace g by αg with $|\alpha| = 1$ so that $(Tf, \alpha g) \geq 0$. Hence, $|(Tf, g)| \leq M$ and the lemma is proven.

□

Lemma 39. *[Lemma 5] Given a symmetric compact operator T. If follows that T has at least one nonzero eigenvalue. In fact either $\lambda = \|T\|$ or $\lambda = -\|T\|$ is an eigenvalue.*

Proof. Either $\|T\| = \sup(Tf, f)$ or $-\|T\| = \inf(Tf, f)$ over $\|f\| = 1$. Since both cases are similar we need only consider the first possibility. Suppose that $\|T\| = \sup(Tf, f)$. Then there exists $f_n, \|f_n\| = 1$ such that $(Tf_n, f_n) \to \|T\| = \lambda$. Without loss of generality we can (by compactness of T) assume that $Tf_n \to g$. Therefore

$$\begin{aligned}
\|Tf_n - \lambda f_n\|^2 &= (Tf_n - \lambda f_n, Tf_n - \lambda f_n) \\
&= \|Tf_n\|^2 + \lambda^2 \|f_n\|^2 - 2\lambda(Tf_n, f_n) \\
&\leq \left\|T^2\right\| \|f_n\|^2 \\
&\leq 2\lambda^2 - 2\lambda(Tf_n, f_n) \\
&\to 0.
\end{aligned}$$

We can conclude that $Tf_n - \lambda f_n \to 0$ as $Tf_n \to g$ therefore $\lambda f_n \to g$, $f_n \to \lambda^{-1} g$. This means that

$$1 = \|f_n\| \to \frac{1}{\lambda} \|g\| .$$

Which implies that $\|g\| = \lambda \neq 0$. Since $g \neq 0$ we use continuity of T to conclude that in addition to the fact that $Tf_n \to g$ we have

$$Tf_n \to \frac{1}{\lambda} Tg.$$

Hence, $g = \frac{1}{\lambda} Tg$ and $Tg = \lambda g$. □

Lemma 40. *Let T be symmetric (T need not be compact). If V is an invariant subspace for T, then $V^\perp$ is also invariant.*

Proof. We need to show that $T(V^\perp) \subseteq V^\perp$. To have $x \in V^\perp$ means that $(x, y) = 0$ for all $y \in V$. Notice that

$$Tx \in V^\perp \iff (Tx, y) = 0, \ \forall y \in V \iff (x, Ty) = 0, \ \forall y \in V.$$

So let

$$V = \text{Span}\{ \text{ of all eigenvectors}\}.$$

Notice that $T(V) \subseteq V$ and so $T(V^\perp) \subseteq V^\perp$ by Lemma 5. Since $T|_{V^\perp}$ is also a compact symmetric operator we have $V^\perp$ is closed. Thus by Lemma 4 it follows that $T|_{V^\perp} = 0$. □

Theorem 83 (Spectral Theorem for Self-Adjoint Compact Operators)**.** *If T is symmetric and compact then T is diagonalizable. That is there exists and orthonormal basis $\{e_n\}$ of eigenvectors (vectors which satisfy $Te_n = \lambda_n e_n$ with $\lambda_n \in \mathbb{R}$ and $\lambda_n \to 0$).*

Proof. The theorem follows from the just proven lemmas.

□

Theorem 84 (Fredholm Alternative)**.** *Suppose that T is compact and $\lambda \neq 0$. Either*

(a) $T_u - \lambda u = f$ has a unique solution for all $f \in \mathcal{H}$, or

(b) $T_u - \lambda u = 0$ has solutions $u \neq 0$.

Moreover, if (b) holds then $Tu - \lambda u = f$ has a solution if and only if $f \in \text{Null}\{(T^* - \overline{\lambda} I)^\perp\}$.

Theorem 85. *Let $T : \mathcal{H} \to \mathcal{H}$ be compact and $\lambda \neq 0$. Then*

(i) $\dim \text{Null}(T - \lambda I) < \infty$.

(ii) $\text{Range}(T - \lambda I)$ *is closed.*

(iii) $\text{Range}(T - \lambda I) = \text{Null}(T^* - \overline{\lambda} I)^\perp$.

(iv) $\text{Null}(T - \lambda I) = \{0\} \iff \text{Range}(T - \lambda I) = \mathcal{H} \iff$ *Fredholm Alternative.*

(v) $\dim \text{Null}(T - \lambda I) = \dim \text{Null}(T^* - \overline{\lambda} I)$.

Lemma 41. *[Lemma 8] Suppose that T is compact and $u_k \rightharpoonup u$ then $Tu_k \to Tu$ (in norm).*

Proof. Since $u_k \rightharpoonup u$ we know that $\sup \|u_k\| < \infty$. Therefore Tu_k has a convergent subsequence $Tu_{k_j} \to w$. Since $u_k \rightharpoonup u$ we claim that $Tu_k \rightharpoonup Tu$. To see this notice that

$$Tu_k \rightharpoonup Tu \iff (Tu_k, g) \to (Tu, g).$$

We have

$$(Tu_k, g) - (Tu, g) = (T(u_k - u), g) = (u_k - u, T^* g) \to 0.$$

Since this applies to any subsequence has a convergent subsequence which converges to Tu we conclude that $Tu_k \to Tu$. □

Lemma 42. *[Lemma 9] There exists $\gamma > 0$ such that $\|Tu - \lambda u\| \geq \gamma \|u\|$ for all $u \in \mathrm{Null}(T - \lambda I)^\perp$.*

Proof. Suppose the contrary. Then there exists a sequence $\{u_k\}$ with $\|u_k\| = 1$ where we have $u_k \in \mathrm{Null}(T - \lambda I)^\perp$ so that $\|Tu - \lambda u\| < 1/k$. Since $Tu_k - \lambda u_k \to 0$ we conclude that there exists a weakly convergent subsequence $\{u_{k_j}\}$. We then have $u_{k_j} \rightharpoonup u$. By Lemma 8 we know that $Tu_{j_j} \to Tu$. Therefore

$$Tu_{k_j} - \lambda u_{k_j} \to 0 \implies \lambda u_k \to T_u.$$

This means that

$$[\lambda u_{k+j} \rightharpoonup Tu \text{ and } \lambda u_{k_j} \rightharpoonup \lambda u] \implies Tu = \lambda u.$$

Where $u \in \mathrm{Null}(T - \lambda I)$, but $u_{k_j} \to u$ (since $\lambda u_{k_j} \to Tu = \lambda u$). This implies $\|u\| = 1$. Therefore

$$0 = (u_{k_j}, u) \to (u, u) = 1,$$

a contradiction.

□

Proof of Theorem

Proof. (i) We have already proven this.

(ii) Let $\{v_k\} \subseteq \mathrm{Range}(T - \lambda I)$ be such that $v_k \to v$. We want to show that $v \in \mathrm{Range}(T - \lambda I)$. Observe that

$$v_k = (T - \lambda I)u_k \to v.$$

We can choose $u_k \in \mathrm{Null}(T - \lambda I)^\perp$. By Lemma 9 we know that

$$\|v_k\| \geq \gamma \|u_k\|.$$

Thus,

$$\|v_k - v_l\| = \|(T - \lambda I)(u_k - u_l)\| \geq \gamma \|u_k - u_l\| \implies \{u_k\} \text{ is Cauchy.}$$

By completeness it follows that $u_k \to u$. This yields

$$(T - \lambda I)u_k \to (T - \lambda I)u \implies v_k \to (T - \lambda I)u = v.$$

(iii) We just proved that $\mathrm{Range}(T - \lambda I)$ is closed. Using the general fact that for any linear operator A we have $\overline{\mathrm{Range}\, A} = \mathrm{Null}(A^*)^\perp$. This fact follows easily by proving containment both ways using the definitions. First we prove that $\mathrm{Range}(A) = \mathrm{Null}(A^*)^\perp$. This is true since $(Ax, y) = (x, A^* y) = 0$ for all $y \in \mathrm{Null}(A^*)$. Since orthogonal complements are closed this further implies that $\overline{\mathrm{Range}\, A} \subseteq \mathrm{Null}(A^*)^\perp$. Next we use if $V_1 \subseteq V_2$ then $V_1^\perp \supset V_2^\perp$. It is therefore enough to show that $\overline{\mathrm{Range}(A)}^\perp = \mathrm{Range}(A)^\perp \subseteq (\mathrm{Null}(A^*)^\perp)^\perp = \mathrm{Null}(A^*)$. If we take $y \in \mathrm{Range}(A)^\perp$ this means that $(Ax, y) = 0$ for all $x \in H$. We then have

$$(Ax, y) = 0 \ \ \forall x \in \mathcal{H} \iff (x, A^* y) = 0, \ \ \forall x \in \mathcal{H} \iff A^* y = 0 \iff y \in \mathrm{Null}(A^*).$$

(iv) Assume that $\text{Null}(T - \lambda I) = \{0\}$. Next, suppose that $\mathcal{H}_1 = \text{Range}(T - \lambda I) \neq H$. We then have $H_1 \subsetneq \mathcal{H}$. We can write

$$\mathcal{H}_1 = (T - \lambda I)\mathcal{H} \implies \mathcal{H}_2 = (T - \lambda I)\mathcal{H}_1 \subsetneq \mathcal{H}_1.$$

Where the proper containment follows from the fact that $T - \lambda I$ is injective. We can continue and write

$$\mathcal{H}_k = (T - \lambda I)H_{k-1} \subsetneq \mathcal{H}_{k-1}.$$

We can choose a sequence $u_k \in \mathcal{H}_k$ with $\|u_k\| = 1$ and $(u_k \in \mathcal{H}_{k+1}^{\perp}$. Let $k > l$ and define the sequence

$$Tu_k - Tu_l = -(\lambda u_k - \lambda u_l) + (\lambda u_l - Tu_l) + (\lambda u_k - \lambda u_l).$$

We have $\lambda u_k - Tu_k = -(T - \lambda I)u_k \in \mathcal{H}_{k+1} \subseteq \mathcal{H}_{l+1}$ and also $(\lambda u_l - Tu_l) \in \mathcal{H}_{l+1}$. Lastly since $u_l \in \mathcal{H}_{l+1}^{\perp}$ so we have

$$P(Tu_k - Tu_l) = -\lambda e_l.$$

Therefore we obtain the contradiction

$$\|Tu_k - Tu_l\| \geq \|\lambda u_k l\| = |\lambda| > 0.$$

□

Definition 137. *Let $T \in \mathscr{L}(\mathcal{H}, \mathcal{H})$. The **spectrum** of T is $\sigma(T) := \{\lambda \in \mathbb{C} : (T_\lambda I)^{-1} \notin \mathscr{L}(\mathcal{H}, \mathcal{H})\}$. We also define the **point spectrum***

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \text{Null}(T - \lambda I) \neq \{0\}\}.$$

(Notice that $\sigma_p(T) \subseteq \sigma(T)$).

Corollary 34. *If T is compact then $\sigma(T) = \{0\} \cup \sigma_p(T)$.*

Proof. We know that $0 \in \sigma(T)$ (i.e. T^{-1} is not bounded which is true since $TT^{-1} = I$ and if T were continuous we would have I is compact, a contradiction). Suppose that $\lambda \neq 0$ with $\lambda \notin \sigma_o(T)$ then $(T - \lambda I^{-1}) \in \mathscr{L}(\mathcal{H}, \mathcal{H})$. By (iv) of the last theorem we know that

$$\text{Null}(T - \lambda I) = \{0\} \implies \text{Range}(T - \lambda I_{=}\mathcal{H} \overset{*}{\implies} (T - \lambda I)^{-1} \in \mathscr{L}(\mathcal{H}, \mathcal{H}).$$

Where the implication at * follows from the corollary to the open mapping theorem. There exists $\gamma > 0$ such that

$$\|(T - \lambda I)u\| \geq \gamma \|u\|,$$

so for all $u \in \text{Null}(T - \lambda I)^{\perp} = \mathcal{H}$ which implies that $\left\|(T - \lambda I)^{-1}\right\| \leq \gamma^{-1}$. □

19 Elements of Fourier Analysis

19.1 Introduction

We will work mostly with $\mathbb{R}^n$. The main topics will be convultions and the Fourier transform. This falls in sections 8.1-8.3 of Folland.

Definition 138. *We will let*

$$C^k(U) := \{f : U \to \mathbb{C} \mid \text{ there exist continuous partials of order less than or equal to } k\},$$
$$C\infty(U) := \bigcap_{k=1}^{\infty} C^k(U)$$
$$C_c^\infty(U) := C\infty(U) \cap C_c(U).$$

$$x \cdot y = \sum_1^n x_i y_i, \qquad |x| = \sqrt{x \cdot x}, \partial_j = \frac{\partial}{\partial x_j}.$$

19.1.1 Remark

$C_c^\infty(\mathbb{R}^n)$ is nontrivial. For example it can be verified that the function

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}.$$

Definition 139. *We will use the following* ***multiindex notation****. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \geq 0$ integers. We denote*

$$|\alpha| = \sum_1^n \alpha_j, \qquad \alpha! = \prod \alpha_j!.$$

If $x \in \mathbb{R}^n$ then $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$. Also

$$\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}.$$

19.1.2 Example

For example we can write Taylor's formula as

$$f(x) = \sum_{|\alpha| \leq k} (\partial^\alpha f)(x_0) \frac{(x - x_0)^\alpha}{\alpha!} + R_k(x), \qquad \lim_{x \to x_0} \frac{R_k(x)|}{x - x_0|^k} = 0.$$

A future exercise will be to prove that

$$\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g).$$

19.2 Schwartz Space

Definition 140. *The* ***Schwartz space*** *is the set*

$$\mathcal{S} := \{f \in C^\infty : \|f\|_{(N,\alpha)} < \infty, \quad \forall N, \quad \forall \alpha\}.$$

Where $\|f\|_{(N,\alpha)} := \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$, (this defines a semi-norm) and α is a multi-index.

19.2.1 Remark

Note that $C_c^\infty \subseteq \S$, also letting $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ we have $x^\alpha e^{-|x|^2} \in \mathcal{S}$ while $x^\alpha e^{-|x|} \notin \mathcal{S}$. This is true since $|x|$ is not differentiable at 0 while $|x|^2$ is.

Proposition 69. *$\mathcal{S}$ is Frechet, with topology defined by the semi-norms $\|\cdot\|_{(N,\alpha}$ with $N \in \mathbb{N}$ and α a multi-index.*

Proof. Suppose that $\{f_k\}$ is a Cauchy sequence. This means that for all N, α we have $\|f_i - f_k\|_{(N,\alpha)} \to 0$. And this implies that for all α it holds that $\{\partial^\alpha f_k\}$ is Cauchy in $\|\cdot\|_u$. By completeness we then have $\partial^\alpha f_k \to g_\alpha \in C$ in $\|\cdot\|_u$ for all α. Let $\alpha = 0 = (0, \ldots, 0)$. We know that $f_k \to g_0$ in the uniform norm, we claim that $\partial^\alpha f_k \to \partial^\alpha g_0$ in $\|\cdot\|_u$. To see this let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ and let $\partial_j g = \partial^{e_j} g$. We then have

$$f_k(x + te_j) - f_k(x) = \int_0^t \partial_j f_k(x + se_j)ds \implies g_0(x + te_j) - f_0(x) = \int_0^t g_{e_j}(x + se_j)ds.$$

By the fundamental theorem of calculus we then have $g_{e_j} = \partial_j g$. □

19.3 Translates

Definition 141. *We define the **translate** of a function to be $\tau_y f(x) = f(x - y)$.*

19.3.1 Remark

We have in L^p it holds that $\|\tau_y f\|_p = \|f\|$ for all $0 < p \leq \infty$. Also, $\|\tau_y f\| = \|f\|_u$ and so τ_y is a bounded linear operator on L^p. ($\|\tau\|_y = 1$.)

Definition 142. *A function f is **uniformly continuous** if $\|\tau_y f - f\|_u \to 0$ as $y \to 0$.*

Lemma 43. *$f \in C_c(\mathbb{R}^n)$ implies that f is uniformly continuous.*

Proof. Straightforward. □

Proposition 70 (Proposition 17)**.** *Let $1 \leq p < \infty$ and let $f \in L^p$. This implies that $\tau_y f \to f$ in L^p as $y \to 0$.*

Proof. Since $\|\tau_y\|_p = 1$ it is enough to prove that $\tau_y f \to f$ on a dense set. We know that $C_c(\mathbb{R}^n)$ is dense in L^p. Fix $g \in C_c$ and assume that $|y| \leq 1$. Then $\tau_y g$ are supported on $K = \operatorname{supp} g + [-1, 1]^n$. It follows that

$$\int |\tau_y g - g|^p \leq \|\tau_y g - g\|_u^p \, m(K) \to 0.$$

□

19.4 Convolutions

Definition 143. *Suppose that f, g are measurable. We define the **convolution** to be*

$$f * g := \int f(x - y)g(y)dy.$$

Note that $f(x - y)$ is Borel measurable.

Proposition 71. *So long as the integrals exist the following properties hold:*

(a) $f * g = g * f$.

(b) $(f * g) * h = f * (g * h)$.

(c) $f, g \in L^1 \implies f * g \in L^1$.

(d) $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$.

(e) $\operatorname{supp}(f * g) \subseteq \operatorname{supp} f + \operatorname{supp} g$.

Proof. (a),(b),(c), and (d) are straightforward. We prove (e). Let $A = \operatorname{supp} f + \operatorname{supp} g$. Suppose $x \notin A$. We then have $f * g(x) = \int f(x-y)g(y)dy \equiv 0$ since $f(x-y)g(y) = 0$ for all y. □

Corollary 35. *(a), (b), (c) together imply that L^1 with the convolution product is an associative algebra without a unit.*

Theorem 86 (Titchmarsh's Theorem)**.** *Let $f, g \in L^1([0,\infty))$, then* $\inf \operatorname{supp}(f * g) = \inf(\operatorname{supp} f) + \inf(\operatorname{supp} g)$.

Theorem 87 (Young's Inequality)**.** *Let $f \in L^1$ and $g \in L^p$, then $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.*

Proof. Recall that Theorem 6.18 of Folland concerns the operator that $Tg(x) = \int K(x,y)g(y)dy$. If we let $K(x,y) = f(x-y)$ then $Tg(x) = f * g(x)$ and thus

$$\int |f(x-y)|dx = \|f\|_1, \qquad \forall y$$

$$\int |f(x-y)dy = \|f\|_1, \qquad \forall x.$$

Apply the theorem and we're done. □

Proposition 72. *Let $f \in L^p$ and $g \in L^q$ with $1/p + 1/q = 1$. It holds that*

(i) *$f * g$ is bounded with $\|f * g\|_u \leq \|f\|_p \|g\|_q$.*

(ii) *$f * g$ is uniformly continuous.*

(iii) *$1 < p < \infty$ implies that $f * g \in C_0(\mathbb{R}^n)$.*

Proof. (i) follows from Hölder. To see this notice that

$$\left| \int f(x-y)g(y)dy \right| \leq \|f\|_p \|g\|_q.$$

(ii) requires a little more work. Recall that $\tau_y(f(x) = f(x-y)$. We can write

$$\|\tau_y(f * g) - f * g\|_u = \|\tau_y f - f) * g\|_u \leq \|\tau_y f - f\|_p \|g\|_q.$$

If $1 < p < \infty$ then we can apply Proposition 17 and conclude that the above goes to zero as $y \to 0$. If $p = \infty$ then we have $q = 1$, so we can interchange the roles of f and g to obtain the result.

To prove (iii) we fix $1 < p < \infty$ and let f_n, g_n be compactly supported functions with

$$\|f_n - f\|_p \to 0, \qquad \|g_n - g\|_q \to 0.$$

We know that $\operatorname{supp}(f_n * g_n) \subseteq \operatorname{supp} f_n + \operatorname{supp} g_n$ therefore $f_n * g_n$ is compactly supported.We can write

$$\|f_n * g_n - f * g\|_u \leq \|f_n - f\|_p \|g_n\|_q + \|f\|_p \|g_n - g\|_q \to 0.$$

Thus, $f * g \in \overline{C}_c = C_0$. □

Proposition 73. *Let $f \in L^1, g \in C^k$ with $\partial^\alpha g$ bounded for all $|\alpha| \leq k$, then $f * g \in C^k$ and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ with $|\alpha| \leq k$.*

Proposition 74. *This follows immediately from Theorem 2.27 in Folland.*

Proposition 75. *Let $\mathcal{S}$ denote the Schwartz space. If $f, g \in \mathcal{S}$ then $f * g \in \mathcal{S}$.*

Proof. The previous proposition implies that $f * g \in C^\infty$. We need to prove that the convolution decays rapidly. To prove this we utilize the inequality

$$1 + |x| \leq 1 + |x - y| + |y| \leq (1 + |x - y|)(1 + |y|).$$

We now have

$$\begin{aligned}(1 + |x|)^N |\partial^\alpha (f * g)(x)| &\leq \int (1 + |x - y|^N |\partial^\alpha f(x - y)|(1 + |y|)^N |g(y)| dy \\ &\leq \|f\|_{(N,\alpha)} \|g\|_{N+n+1,0} \int (1 + |y|)^{-n-1} dy < \infty.\end{aligned}$$

□

19.5 Approximate Identities

Let φ be a function in$\mathbb{R}^n$ with $\varphi \in L^1$. For $t > 0$ we let $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$. Observe that

$$\int \varphi_t(x)dx = \int \varphi(t^{-1}x)t^{-n}dx = \int \varphi(y)dy = \int \varphi.$$

Thus, the integral is invariant under t. As $t \to 0$ we know that φ_t approaches the δ-function.

Theorem 88. *Let $\varphi \in L^1$, and $\int \varphi(x)dx = a$ (usually we will take $a = 1$). Then*

(i) $f \in L^p$ $(1 \leq p < \infty)$ implies that $f * \varphi_t \to af$ in L^p as $t \to 0$.

(ii) If f is bounded and uniformly continuous then $f * \varphi_t \to af$ uniformly as $t \to 0$.

(iii) If $f \in L^\infty$ and f is continuous on some open set U then $f * \varphi_t \to af$ uniformly on compact subsets of U.

Proof. (i) Letting $y = tz$ we have

$$\begin{aligned}f * \varphi_t(x) - af(x) &= \int [f(x - y) - f(x)]\varphi_t(y)dy \\ &= \int [f(x - tz) - f(x)]\varphi(z)dz \\ &= \int [\tau_{tz} f(x) - f(x)]\varphi(z)dz\end{aligned}$$

Minkowski's integral inequality yields

$$\begin{aligned}\|f*\varphi_t - af\|_p &= \left(\int \left|\int [\tau_{tz}f(x) - f(x)]\varphi(z)dz\right|^p dx\right)^{1/p} \\ &\le \int \left(\int |\tau_{tz}f(x) - f(x)|^p dx\right)^{1/p} |\varphi(z)|dz \\ &= \int \|\tau_{tz}f - f\|_p |\varphi(z)|dz \to 0\end{aligned}$$

By the dominated convergence theorem since $\|\tau_{tz}f - f\|_p \le 2\|f\|_p$ and $\varphi \in L^1$.

(ii) We can write $f*\varphi_t(x) - af(x) = \int \tau_{tz}f(x) - f(x))\varphi(z)dz$. So we can make the estimate

$$\|f(\varphi_t(x) - af\|_u \le \|\tau_{tz}f - f\|_u \int |\varphi| \to 0.$$

□

Proposition 76. *C^∞ is dense in L^p and in C_0 ($1 \le p < \infty$).*

Proof. There exists $\varphi \in C_c^\infty$ with $\int \varphi = 1$. Also, for all $g \in C_c$ we have $g*\varphi_t \to g$ in the L^p-norm. This implies that $g*\varphi_t \in C^\infty$ and $\operatorname{supp}(g*\varphi_t)$ is compact. □

19.6 C^∞ Urysohn

Theorem 89 (Urysohn)**.** *For all $K \subseteq \mathbb{R}^n$ compact and for all open $U \supset K$ there exists $f \in C_c^\infty$ with $0 \le f \le 1$ such that $f \equiv 1$ on K and* $\operatorname{supp} f \subseteq U$.

Proof. Let $\delta = \rho(K, U^c) > 0$. We have $V = \{x : \rho(x,K) < \delta/3\}$. Choose $\varphi \in C_c^\infty$ such that $\int \varphi = 1$ and $\operatorname{supp}\varphi \subseteq \mathbb{D}(0,\delta/3)$. We then let $f := \mathbb{1}_V * \varphi \in C_c^\infty$. Observe that

$$\operatorname{supp} f \subseteq \operatorname{supp}\mathbb{1}_V + \operatorname{supp}\varphi \subseteq \{x : \rho(x,K) \le \frac{2\delta}{3}\} \subseteq U.$$

Therefore $f(x) = \int_{|y|<\delta/3} \mathbb{1}_V(x-y)\varphi(y)dy \le \int \varphi(y) = 1$. This gives $x \in K$ implies that $\mathbb{D}(x,\delta/3) \subseteq V$ which implies $f(x) = 1$.

□

19.7 Convergence

Theorem 90. *Suppose that $|\varphi(x)| \le C(1+|x|)^{-n-\epsilon}$ and with $1 \le p \le \infty$ we have $\int \varphi = a$. Then $f \in L^p$ implies that $f*\varphi_t(x) \to af(x)$ for all $x \in L_f$, where*

$$L_f := \left\{x : \lim_{m(B)\to 0} \frac{1}{m(B)} \int_{y\in B} |f(x-y) - f(x)|dy \to 0, \quad x \in B\right\}.$$

Proof. Fix $x \in L_f$. Then for all $\delta > 0$ there exists $\eta > 0$ such that for all $r \le \eta$ we have

$$\int_{|y|<r|} |f(x-y) - f(x)|dy \le \delta r^n. \qquad (!)$$

We can make the bound

$$|f * \varphi_t(x) - af(x)| = \left| \int (f(x-y) - f(x))\varphi_t(y)dy \right| \leq \int |f(x-y) - f(x)||\varphi_t(y)|dy = I_1 + I_2.$$

Where I_1 is the integral over $|y| < \eta$ and I_2 is the integral over $|y| \geq \eta$. We will show that $I_1 \leq A\delta$ (independent of t) and $I_2 \to 0$ as $t \to 0$. First we consider I_1. We would like to integrate over a ball of radius smaller than t, so we need to shrink η as necessary by breaking up over annuli.

$$I_1 = \sum_{k=1}^{N} \int_{2^{-k}\eta < |y| < 2^{1-k}\eta} |f(x-y) - f(x)||\varphi_t(y)|dy, \qquad 2^N \leq \frac{\eta}{t} < 2^{N+1}.$$

On the kth annulus we have $2^{-k}\eta \leq |y| \leq 2^{1-k}\eta$ and so

$$|\varphi_t(y)| = t^{-n}|\varphi(\frac{y}{t})| \leq Ct^{-n}\left|\frac{t}{y}\right|^{-n-\epsilon} = Ct^{\epsilon}|y|^{-n-\epsilon}.$$

And on the inside ball we have $|y| < 2^{-N}\eta$ and we use $|\varphi_t(y)| \leq Ct^{-n}$. Using (!) we obtain

$$\begin{aligned} I_1 &\leq Ct^{-n}\left(\int_{|y|<2^{-N}\eta} |f(x-y) - f(x)|dy\right) \\ &\qquad + \sum_{k=1}^{N} Ct^{\epsilon}(2^{-k}\eta)^{-n-\epsilon} \int_{2^{-k}\eta \leq |y| < 2^{1-k}\eta} |f(x-y) - f(x)|dy \\ &\leq \delta(2^{-N}\eta)^n + \int_{|y|<2^{1-k}\eta} |f(x-y) - f(x)|dy \\ &\leq \delta(2^{-N}\eta)^n + \delta(2^{1-k}\eta)^n. \end{aligned}$$

Hence,

$$I_1 \leq C\delta\left[\sum_{k=1}^{N} 2^n \left(\frac{\eta}{t}\right)^{-\epsilon} 2^{k\epsilon} + \left(\frac{2^{-k}\eta}{t}\right)^n\right].$$

But we can write $C\delta \sum_{k=1}^{N} 2^n \left(\frac{\eta}{t}\right)^{-\epsilon} 2^{k\epsilon} \leq \frac{2^{(N+1)\epsilon}}{2^{\epsilon}-1}$.

It remains to prove that $I_2 \to 0$ as $t \to 0$. See Folland Theorem 8.15 for remainder of proof. □

19.8 The Fourier Transform

We will mostly work with $\mathbb{R}^n$, but will occasionally use the torus $\mathbb{T}^n$. Both are abelian groups. The building blocks of harmonic analysis are called *characters*.

Definition 144. *A **character** is a continuous homomorphism into $\mathbb{T}$. That is $\varphi(x+y) = \varphi(x)\varphi(y)$.*

Theorem 91. *If φ is measurable on $\mathbb{R}^n$ (or $\mathbb{T}^n$) such that $\varphi(x+y) = \varphi(x)\varphi(y)$ with $|\varphi| = 1$ then there exists $\xi \in \mathbb{R}^n$ (or $\xi \in \mathbb{Z}^n$) such that $\varphi(x) = e^{e\pi i\xi x}$.*

Proof. There exists $a \in \mathbb{R}$ such that $\int_0^a \varphi(t)dt \neq 0$. Denote $\frac{1}{\int_0^a \varphi(t)dt} =: A$. Observe that

$$\varphi(x) = A\int_0^a \varphi(x)\varphi(t)dt = A\int_0^a \varphi(x+t)dt = A\int_x^{x+a} \varphi(t)dt.$$

Therefore $\varphi \in C$, repeating this argument inductively gives $\varphi \in C^\infty$. In particular, $\varphi \in C^1$. We then have

$$\begin{aligned}\varphi'(x) &= A[\varphi(x+a) - \varphi(x)] \\ &= A[\varphi(a)\varphi(x) - \varphi(x)] \\ &= A(\varphi(a) - 1)\varphi(x) = B\varphi(x)\end{aligned}$$

We then have $\varphi' = B\varphi$ and $\varphi(0) = 1$. Therefore $\varphi = e^{Bx}$. We take $B = 2\pi i$ and we are done.

On $\mathbb{T}$ we know that φ has to be $\mathbb{Z}$-periodic and since $e^{e\pi i\xi} = 1$ we must have $\xi \in \mathbb{Z}$. In $\mathbb{R}^n$ we want $\psi_j(t) = \varphi(te_j) = e^{2\pi i\xi_j t}$.

In $\mathbb{R}^n$ we have $\psi_j(t) = \varphi(te_j) = e^{2\pi i\xi_j t}$ and so

$$\varphi(x) = \varphi\left(\sum_1^n x_j e_j\right) = \prod_1^n \psi_j(x_j) = e^{2\pi i\xi\cdot x}.$$

□

Theorem 92 (8.20)**.** *The set $\{e_\kappa = e^{2\pi i\kappa\cdot x}, \kappa \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$.*

Proof. To prove it is orthonormal is standard. Completeness follows from Stone-Weierstrass since $\mathrm{Span}\{E_\kappa\}$ is an algebra.

□

Definition 145. *For all $f \in L^2(\mathbb{T}^n$ we can define the **Fourier Transform***

$$\hat{f} := (f, E_\kappa) = \int_{\mathbb{T}^n} f(x)e^{-2\pi i\kappa x}dx.$$

19.8.1 Remark

Notice that $f = \sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa)E_\kappa$ and that $\hat{f}(\kappa) = \int_{\mathbb{T}^n} f(x)e^{-2\pi i\kappa\cdot x}dx$. From Parseval's identity we get $f \mapsto \hat{f}$ is a unitary map from $L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$. The formal analogue in $\mathbb{R}^n$ is

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi, \qquad \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}dx.$$

Definition 146. *Let $f \in L^1(\mathbb{R}^n)$ we define $\mathcal{F}f(\xi) := \hat{f}(\xi)\int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}dx$. So, $\hat{f}$ is continuous by 2.27. Also $\mathcal{F}$ is a linear map with $\left\|\hat{f}\right\|_u \le \|f\|_1$ and $\mathcal{F} : L^1(\mathbb{R}^n) \to BC(\mathbb{R}^n)$.*

Theorem 93 (Folland 8.22)**.** *The following properties of Fourier transforms hold:*

(a) $\widehat{(\tau_y f)}(\xi) = e^{-2\pi i\xi\cdot y}\hat{f}(\xi)$, and $\tau(\hat{f}) = \hat{h}$ with $h(x) = e^{2\pi i\eta\cdot x}$.

(b) Suppose that T is an invertible linear transformation of $\mathbb{R}^n$ and let $S = (T^*)^{-1} = (T^{-1})^*$. We have $\widehat{(f\circ T)} = |\det T|^{-1}\hat{f}\circ S$.

(i) $T = (T^*)^{-1}$ and $T = S$ then $\widehat{(f\circ T)} = \hat{f}\circ T$.

(ii) $Tx = t^{-1}x$, $|\det T| = t^{-n}$ and $Sx = tx$ then

$$\widehat{(f\circ T)}(\xi) = t^n\hat{f}(t\xi).$$

(c) $\widehat{(f * g)} = \hat{f}\hat{g}$.

(d) If $x^\alpha f \in L^1$ *then for all* $|\alpha| \leq k$ *then* $\hat{f} \in C^k$, $\partial^\alpha \hat{f} = (\widehat{-2\pi i x})^\alpha f$.

(e) (Smoothness of f *translates into decay of* $\hat{f}$ *at* ∞*) Let* $f \in C^k$*, assume that* $\partial^\alpha f \in L^1$ *for all* $|\alpha| \leq k$ *and that* $\partial^\alpha f \in C_0$ *for all* $|\alpha| \leq k-1$*. It follows that*

$$\widehat{(\partial^\alpha f)}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi).$$

(f) (Riemann-Lebesgue Lemma) $\mathcal{F}(L^1(\mathbb{R}^n)) \subseteq C_0(\mathbb{R}^n)$*. (Here* $\mathcal{F}$ *denotes a Fourier transform).*

Proof. (a) Exercise.

(b) We make the change of variables $y = Tx$ and thus $x = T^{-1}y$ and $dy = |\det T| dx$.

$$\begin{aligned}
\widehat{(f \circ T)}(\xi) &= \int_{\mathbb{R}^n} f(Tx) e^{-2\pi i \xi \cdot x} dx \\
&= |\det T|^{-1} \int f(y) e^{-2\pi i S\xi \cdot y} dy \\
&= |\det T|^{-1} \hat{f} \circ S
\end{aligned}$$

(c) Follows from Fubini.

(d) We will use induction on $|\alpha|$ and Theorem 2.26 (Folland). Observe that

$$\partial^\alpha \hat{f}(\xi) = \partial_\xi^\alpha \int f(x) e^{-2\pi i \xi \cdot x} dx = \int f(x)(-2\pi i x)^{-\alpha} e^{-2\pi i \xi \cdot x} dx.$$

(e) Suppose first that $|\alpha| = 1$. This implies that $f' \in L_1$ and $f \in C_0$. We then have

$$\begin{aligned}
\hat{f'}(\xi) &= \int f'(x) e^{-2\pi i \xi \cdot x} dx \\
&\overset{*}{=} f(x) e^{-2\pi i \xi \cdot x} \Big|_{-\infty}^{+\infty} - \int f(x)(-2\pi i \xi) e^{-2\pi i \xi \cdot x} dx \\
&= (-2\pi \xi) \hat{f}(\xi).
\end{aligned}$$

Where at * we use integration by parts. The inductive case is similar.

(f) Let $f \in C^1 \cup C_c$. Because $C^1 \cap C_c$ is dense in L^1, Property (e) implies that $2\pi i \xi_j \hat{f}(\xi) = \widehat{(\partial_j f)}(\xi) \in BC(\mathbb{R}^n)$ and thus $f \in C_0(\mathbb{R}^n)$. Since we have proven this on a dense set it is easy to extend it to arbitrary $f \in L^1$ by considering a sequence $f_n \to f$ and using the fact that $|hat f_n \to \hat{f}$ uniformly and using the fact that C_0 is closed in $\|\cdot\|_u$.

□

Corollary 36 (Folland 8.23)**.** $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ *continuously.*

Proof. Let $f \in \mathcal{S}$. This implies that $x^\alpha \partial^\beta f \in l^1 \cap C_0$. Hence $\hat{f} \in C^\infty$ by (d) (where we use $\beta = 0$). Let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We can write $(cx)^\alpha = c^{|\alpha|} x^\alpha$. Observe that

$$\begin{aligned}
\widehat{(x^\alpha \partial^\beta f)} &\overset{(d)}{=} (-2\pi i)^{-|alpha|} \partial^\alpha \widehat{(\partial^\beta f)}. \\
&\overset{(e)}{=} (-1)^{|\alpha|} (2\pi i)^{|\beta| - |\alpha|} \partial^\alpha (\xi^\beta \hat{f})
\end{aligned}$$

This then implies tha t$\hat{f} \in \mathcal{S}$ by Proposition 8.3. □

Proposition 77 (Folland 8.3)**.** *Let $f \in C^\infty$ the following are equivalent*

(i) $f \in \mathcal{S}$.

(ii) $x^\beta \partial^\alpha f \in B$ *for all* α, β.

(iii) $\partial^\alpha (x^\beta f) \in B$ *for all* α, β.

Proof. (i) $\implies$ (ii) is trivial since $|x^\beta| \leq (1+|x|)^N$ for $|\beta| \leq N$.

To see $(ii) \implies (i)$ we note that $\sum_1^n |x_j|^N \geq \delta > 0$ on $|x| = 1$ and so $\sum -1^n |x_j|^N \geq \delta |x|^N$ for all $x \in \mathbb{R}^n$. We then have

$$(1+|x|)^N \leq 2^N(1+|x|^N) \leq 2^N \left[1 + \delta^{-1} \sum_1^N |x_j^N|\right] \leq 2^N \delta^{-1} \sum_{|\beta| \leq N} |x^\beta|.$$

Lastly, we know that $(iii) \iff (ii)$ from last weeks homework. □

19.9 Main Results in Fourier Analysis

Proposition 78 (Folland 8.24)**.** *Let $a > 0$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = e^{-\pi a|x|^2}$ then $\hat{f}(\xi) = a^{-n/2} e^{-\pi|\xi|^2/a}$.*

Proof. We proceed by induction. First we consider the case $n = 1$.

$\boxed{n=1}$ We will show that $e^{\pi\xi^2/z}\hat{f}(\xi) = C$ with $C = a^{-1/2}$ a constant. It suffices to prove that $\frac{d}{d\xi}\left(e^{\pi\xi^2/a}\hat{f}(\xi)\right) \equiv 0$. We can write

$$\begin{aligned}
\frac{d}{d\xi}\left(e^{\pi\xi^2/a}\hat{f}(\xi)\right) &= e^{\pi\xi^2/a}\frac{\pi\xi}{a}\hat{f}(\xi) + e^{\pi\xi^2/a}(\hat{f}(\xi))' \\
&= e^{\pi\xi^2/a}\left[\frac{\pi\xi}{a}\hat{f}(\xi) + (\hat{f}(\xi))'\right] \\
&= e^{\pi\xi^2/a}\left[\frac{\pi\xi}{a}\hat{f}(\xi) + (\hat{f}(\xi))'\right]
\end{aligned}$$

If we make the substitutions

$$\widehat{f'(\xi)} = 2\pi i\xi\hat{f}(\xi); \qquad f'(\xi) = (-w\pi i a\xi)f(\xi).$$

We see that the above is zero.

$\boxed{n=n}$ In $\mathbb{R}^n$ we us Fubini's Theorem to write

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\pi a|x|^2 - 2\pi i\xi\cdot x}dx = \prod_1^n \int_{\mathbb{R}} e^{-\pi a x_j^2 - 2\pi i\xi_j x_j}dx = \prod_1^n (a^{-1/2}e^{-\pi\xi_j^2/a}.$$

□

Definition 147. *We define the* ***Inverse Fourier Transform*** $\check{f} := \hat{f}(-x) = \int f(\xi)e^{2\pi i\xi\cdot x}d\xi$.

Lemma 44. *If $f, g \in L^1$ then $\int \hat{f}g = \int f\hat{g}$.*

Theorem 94 (Folland 8.26)**.** *Suppose $f \in L^1$ and $\hat{f} \in L^1$ then $f = (\hat{f})\check{}$ a.e.*

Consider $g(x) = e^{-\pi|x|^2}$ we know that $\int g = 1$ and $\hat{g}(\xi) = g(\xi)$. Also define $g_t(x) = t^{-n}g(x/t)$. Recall that $f * g_t \to f$ in L^1. We define $\varphi(\xi) := e^{2\pi i\xi\cdot x - \pi t^2|\xi|^2}$. It follows that

$$\hat{\varphi}(y) = \int_{\mathbb{R}^n} e^{2\pi i\cdot x - 2\pi i t^2|\xi|^2 - 2\pi i\xi\cdot y} d\xi = \widehat{e^{-\pi t^2|\xi|^2}}(y-x) = t^{-n}e^{-\frac{\pi|y-x|^2}{t^2}} = g_t(x-y).$$

It follows that

$$\begin{aligned} f * g_t(x) &= \int g_t(x-y)f(y)dy \\ &= \int f\hat{\varphi} \\ &= \int \hat{f}\varphi \\ &= \int e^{-\pi t^2|\xi|^2}e^{2\pi i\xi\cdot x}\hat{f}(\xi)d\xi \\ &\to (\hat{f})\check{}(x) \end{aligned}$$

by the dominated convergence theorem.

Corollary 37. *If $f \in L^1$ and $\hat{f} \equiv 0$ then $f = 0$ a.e.*

Corollary 38. *$\mathcal{F}(\mathcal{S}) = \mathcal{S}$.*

Theorem 95 (Follad 8.29)**.** *(Plancherel) If $f \in L^1 \cap L^2$ then $\hat{f} \in L^2$ and $\mathcal{F}|_{L^1\cap L^2}$ extends uniquely to a unitary isomorphism on L^2.*

Proof. Start with the class of functions $X = \{f \in L^1 : \hat{f} \in L^1\}$. Certainly $\mathcal{S} \subseteq X$ and also notice that if $f \in L^1$ and $\hat{f} \in L^1$ then $f \in L^\infty$ by 8.26. Therefore $f \in L^2$. Hence $\S \subseteq X \subseteq L^2$ and X is dense in L^2. We want to show that $\mathcal{F}|_X$ preserves inner product. Let $f, g \in X$ with $h = \overline{\hat{g}}$ (the bar represents complex conjugation). We then have

$$\hat{h}(\xi) = \int e^{-2\pi i\xi\cdot x}\overline{\hat{g}(x)}dx = \int \overline{e^{2\pi i\xi\cdot x}\hat{g}(x)}dx \overset{a.e.}{=} \overline{g(\xi)}.$$

□

Where to conclude the a.e. equality we used 8.26. Note also that

$$\int f\ell g = \int f\hat{h} = \int \hat{f}h = \int \hat{f}\overline{\hat{g}} \implies (f,g) = (\hat{f},\hat{g}).$$

Take $g = f$ this implies that $\left\|\hat{f}\right\|_2 = \|f\|$. So we have $\mathcal{F}(X) = X$ by 8.26. Therefore $\mathcal{F}$ extends to be unitary on L^2. It remains to prove that $\mathcal{F}|_{L^1\cap L^2}$ agrees with the just defined extension. To see this we use the Gaussian kernel and consider $g(x) = e^{-\pi|x|^2}$ where $f * g_t \in L^1$. We then have

$$(\widehat{f * g_t}) = e^{-\pi t^2|\xi|^2}\hat{f} \in L^1 \implies f * g_t \in X.$$

Since we assumed $f \in L^1 \cap L^2$ we have $f * g_t \to f$ in L^1 and in L^2. Also we know that

$$\widehat{f * g_t} \to \hat{f}, \qquad \text{uniformly.}$$

Hence $\widehat{f * g_t}$ converges to the extension of $\mathcal{F}$ and therefore they agree.

19.10 Poisson Summation Forumula

Let $f \in L^1(\mathbb{R}^n$. We seek a periodic function that relates to f. There are two ways to proceed. We can consider

$$\sum_{k\in\mathbb{Z}^n} f(z-k), \qquad \text{(if it converges)}.$$

Or we can take the Fourier series

$$\sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{-2\pi i\xi\cdot x}, \qquad \text{(if it converges)}.$$

Theorem 96 (Folland 8.31)**.** *Suppose $f \in L^1(\mathbb{R}^n)$ then $\sum_{k\in\mathbb{Z}^n}\tau_k f$ converges almost everywhere pointwise and also converges in $L^1(\mathbb{T}^n)$ to the function Pf, which satisfies $\|Pf\|_{L^1(\mathbb{T}^n} \leq \|f\|_1$. Moreover, $\widehat{Pf}(\kappa) = \hat{f}(\kappa)$.*

Proof. Write $Q = [-\frac{1}{2}, \frac{1}{2})^n$ and $\mathbb{R}^n = \bigsqcup_{k\in\mathbb{Z}^n}(Q+k)$. We then have

$$\int_{\mathbb{R}^n} |f(x)|dx = \sum_{k\in\mathbb{Z}^n}\int_{Q+k} |f(x)\ dx = \sum_{k\in\mathbb{Z}^n}\int_Q |f(x-k)|dx \overset{*}{=} \int_Q \left(\sum_{k\in\mathbb{Z}^n} |f(x-k)|\right) dx.$$

Where at * we use Tonelli's Theorem. Applying Theorem 2.25 (a corollary to the dominated convergence theorem) yields

$$\sum_j \int |f_j|dx < \infty \implies \sum_j f_j, \text{ converges a.e. and in } L^1.$$

We now have $\sum_{k\in\mathbb{Z}^n}\tau_k f \to Pf$ a.e. and in $L^1(Q) \simeq L^1(\mathbb{T}^n)$ and also $\|Pf\|_1 \leq \|f\|_1$. Next observe that

$$\begin{aligned}
\widehat{Pf}(\kappa) &= \int_Q \sum_{k\in\mathbb{Z}^n} f(x-k)e^{-2\pi i\kappa\cdot x}dx \\
&= \sum_{k\in\mathbb{Z}^n}\int_Q f(x-k)e^{-2\pi i\kappa\cdot x}dx \\
&= \sum_{k\in\mathbb{Z}^n}\int_{Q+\kappa} f(x-k)e^{-2\pi i\kappa\cdot x}dx \\
&= \hat{f}(\kappa)
\end{aligned}$$

□

Theorem 97 (8.32)**.** *Suppose that $f \in C(\mathbb{R}^n)$ with $|f(x)| \leq C(1+|x|)^{-n-\epsilon}$ and $|\hat{f}(x)| \leq C(1+|x|)^{-n-\epsilon}$. Then*

$$\sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{\kappa\in\mathbb{Z}^n}\hat{f}(\kappa)e^{2\pi i\kappa\cdot x}.$$

In particular $x=0$ then $\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{\kappa\in\mathbb{Z}^n}\hat{f}(\kappa)$.

Proof. We know the series converge absolutely and uniformly on Q since $\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-\epsilon} < \infty$. So

$$Pf = \sum_{k\in\mathbb{Z}^n}\tau_k f \in C(\mathbb{T}^n) \subseteq L^2(\mathbb{T}^n).$$

And therefore

$$\sum_{\kappa\in\mathbb{Z}^n} \widehat{Pf}(\kappa)e^{2\pi i\kappa\cdot x} \to Pf, \qquad \text{in } L^2.$$

We finish by applying 8.31 and concluding we have equality almost everywhere. □

19.10.1 Remark

If $f, g \in L^1$ and $\hat{f} = \hat{g}$ then $f = g$.

19.11 Recovering a Function from a Transform

We seek to answer the question: If $f \in L^1$ and $\hat{f} \in C_0$ can we recover f?

The idea is to replace

$$(\hat{f})\check{}(x) = \int \hat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi$$

with $\int \hat{f}(\xi)\varphi(t\xi)e^{2\pi i\xi\cdot x}d\xi$ where $\varphi \to 0$ rapidly at ∞. We can use $\varphi(\xi) = e^{-\pi|\xi|^2}$.

Theorem 98 (8.35)**.** *(L^1 Part) Suppose $\varphi \in L^1 \cap C_0$ and $\varphi(0) = 1$, $\varphi = \check{\varphi} \in L^1$. Let $f \in L^1$, for $t > 0$, define $f^t(x) = \int \hat{f}(\xi)\varphi(t\xi)e^{2\pi i\xi\cdot x}d\xi$. The following statements hold*

(a) $f \in L^p$ with $1 \le p < \infty$ we have $f^t \in L^p$ and $\left\|f^t - f\right\|_p \to 0$ as $t \to 0$.

(b) f is bounded and uniformly continuous implies f^t is also bounded and uniformly convergent and $f^t \to f$ uniformly.

(c) If $|\varphi(x)| \le C(1 + |x|)^{-n-\epsilon}$ implies $f^t(x) \to f(x)$ for every $x \in L_f$ and therefore almost everywhere and for every x at which f is continuous.

Proof. It suffices to verify that $f^t = f * \varphi_t$ with $\varphi_t = t^{-n}\varphi(x/t)$ after which we will use Theorems 8.14 and 8.15. We know that

$$\widehat{\varphi_t}(\xi) = \hat{\varphi}(t\xi) = \widehat{(\check{\varphi})}(t\xi) = \varphi(t\xi).$$

Also we have

$$\hat{\varphi}(0) = \int \varphi(x) = \varphi(0) = 1.$$

Since $f * \varphi \in L^1$ and $\hat{f}\varphi \in L^1$ we know that

$$\widehat{f * \varphi_t}(\xi) = \hat{f}(\xi)(\hat{\varphi}_t)(\xi) = \hat{f}(\xi)\varphi(t\xi) \in L^1.$$

Taking the Fourier inversion we obtain the desired formula.

□

19.12 Applications to Partial Differential Equations

19.12.1 Notation

Definition 148. *A **differential operator** is a map $Lf(x) := \sum_{|\alpha|\le m} a_\alpha \partial^\alpha f(x)$ with $a_\alpha \in C^\infty$.*

For $f \in \mathcal{S}$ we have

$$\widehat{Lf}(\xi) = \sum_{|\alpha| \leq m} a_\alpha (2\pi i \xi)^\alpha \hat{f}(\xi), \qquad b_\alpha = (2\pi i)^{|\alpha|} a_\alpha.$$

Let $D^\alpha = (2\pi i)^{-|\alpha|} \partial^\alpha$ then $L = \sum_{|\alpha| \leq m} b_\alpha D^\alpha$ and $\widehat{Lf} = \left(\sum_{|\alpha| \leq m} b_\alpha \xi^\alpha \right) \hat{f}$.

Definition 149. *Let p be any polynomial in n complex variables. Let $P(\zeta) = \sum_{|a| \leq m} b_\alpha \xi^\alpha$. Define the constant-corefficient operator $P(D) := \sum_{|\alpha| \leq m} b_\alpha D^\alpha$.*

We have $\widehat{P(D)f} = P \cdot \hat{f}$. We would like to sole $P(D)u = f$ on $\mathbb{R}^n$. Then $\hat{u} = p^{-1}\hat{f}$ and so $u = (P^{-1}\hat{f})\check{}$. Formally if $P^{-1} = \hat{\varphi}$ then $u = f * \varphi$. *Note: Even when $u = (P^{-1}\hat{f})\check{}$ is well defined it is not a unique solution.*

Definition 150. *The **Laplace operator** $\Delta = \sum_1^n \frac{\partial^2}{\partial x_j^2} = -4\pi^2 \sum_1^n D_j^2 = P(D)$.*

Theorem 99. *If L is a differential operator and $L(f \circ T) = (Lf) \circ T$ for all translations and rotations T, then there exists P a 1 variable polynomial such that $L = P(\Delta)$.*

Proof. See Folland. □

19.13 The Dirichlet Problem for Δ

Let $\Omega \subseteq \mathbb{R}^n$ and f be defined on $\partial\Omega$. We would like to find u defined on $\overline{\Omega} : \Delta u = 0$ in Ω and $u|_{\partial\Omega} = f$.

Consider the case of $\Omega = UHP \subseteq \mathbb{R}^{n+1}$. We then have $\Delta = \sum_1^n \frac{\partial^2}{\partial x_j^2}$ and $\Delta + \partial_t^2$ is Laplace on $\mathbb{R}^{n+1}$.

Theorem 100. *Let $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then $u(x,t) = (f * P_t)(x)$ satisfies $(\Delta + \partial^2 t)u = 0$ on $\mathbb{R}^n \times (0, \infty)$. and $\lim_{t \to 0} u(x,t) = f(x)$ for a.e. x and for al lx where f is continuous, and if $p < \infty$ then it converges in L^p.*

19.14 Fourier Analysis of Measures

Definition 151. *We let $M(\mathbb{R}^n) = \{$complex Borel measures on $\mathbb{R}^n\}$. Notice that $L^1(\mathbb{R}^n) \subseteq M(\mathbb{R}^n)$. We define the **convolution of measures** ν, μ to be (for $E \in \mathcal{M} \otimes \mathbb{N}$ Borel)*

$$\mu * \nu(E) = \int \int \mathbb{1}_E(x+y) d\mu(x) d\nu(y).$$

Recall that

$$d(\mu \times \nu)(x,y) = \frac{d\mu}{d|\mu|}(x) \frac{d\nu}{d|\nu|}(y) d(|\mu| \times |\nu|)(x,y).$$

Proposition 79. *The following properties of $\mu * \nu$ hold:*

(a) $\mu * \nu = \nu * \mu$ is associative.

(b) for all h bounded and Borel we have

$$\int h d(\mu * \nu) = \int \int h(x+y) d\mu(x) d\nu(y).$$

(c) $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$.

(d) If $d\mu = f dm$ *and* $d\nu = g dm$*, then* $d(\mu * \nu) = (f * g) dm$.

Proof. (a) Straightforward.

(b) Know for $h = \mathbb{1}_E$.

(c) Let $h = \frac{d(\mu*\nu)}{d(|\mu*\nu|)}$. Observe that $|h| = 1$ a.e.

$$\begin{aligned}
\|\mu * \nu\| &= \int d|\mu * \nu| \\
&= \int \frac{1}{h} d(\mu * \nu) \\
&\left| \leq \int \int \frac{1}{h}(x+y) d\mu(x) d\mu(y) \right| \\
&\leq \int \left| \frac{1}{h} d(\mu * \nu) \leq \int \int \frac{1}{h}(x+y) \right| d\mu(x) d\mu(y) \\
&\leq \int \int d|\mu(x) d|\nu(y) \\
&= \|\mu\| \|\nu\|.
\end{aligned}$$

(d) Straightforward.

□

Definition 152. *The* ***Fourier Transform*** *is given by*

$$\hat{\mu} = \int e^{-2\pi i \xi \cdot x} d\mu(x).$$

This is bounded and continuous with $\|\hat{\mu}\|_u \leq \|\mu\|$.

Definition 153. *A measure* μ *on* $\mathbb{R}$ *is called a* ***Raijmann measure*** *if* $\hat{\mu} \in C_0(\mathbb{R})$.

19.14.1 Example

A measure δ_0 a point mass at zero is of this type since $\hat{\delta}_0 \equiv 1$.

Proposition 80. *Let* $\mu_n, \mu \in M(\mathbb{R}^n)$ *with* $\|\mu_n\| \leq C < \infty$*. If* $\hat{\mu}_k \to \hat{\mu}$ *pointwise then* $\mu_k \to \mu$ *vaguely (or weak*).*

Proof. Recall that $\mu_k \to \mu$ vaguely means that for all $f \in C_0(\mathbb{R}^n)$ we have $\int f d\mu_k \to \int f d\mu$. The norms are uniformly bounded implies that it is enough to check convergence on a dense subset of $f \in C_0(\mathbb{R}^n)$. Since $\mathcal{S}$ is dense in C_0 we let $f \in \mathcal{S}$ and note that $\check{f} \in \mathcal{S}$. By Fourier inversion we

have

$$\begin{aligned}
\int f d\ mu_k &= \int \left(\int \check{f}(y) e^{-2\pi i y \cdot x} dy \right) d\mu_k(x) \\
&= \int \check{f}(y) \left(e^{-2\pi i} d\mu_k(x) \right) dy \\
&= \int \check{f}(y) \hat{\mu}_k(y) dy \\
&\xrightarrow{DCT} \int \check{f}(y) \hat{\mu}(y) dy \\
&= \int f d\mu.
\end{aligned}$$

□

20 Distributions

20.1 Introduction

We will start with continuous linear functionals on $C\infty(U)$ where $U \subseteq \mathbb{R}^n$ is open. Later we will consider tempered distributions which are continuous linear functionals on the Schwarz space $\mathcal{S}$.

Definition 154. *We let $C_c^\infty(U) := \bigcup_{K \subseteq U} C_c^\infty(K)$, with K compact where $f \in C_c^\infty(K)$ if $f \in C^\infty$ and* $\operatorname{supp} f \subseteq K$.

Definition 155. *We say that $\varphi_j \to \varphi$ in $C_c^\infty(U)$ if there exists $K \subseteq U$ such that $\varphi_j \in C_c^\infty(K)$ and $\partial^\alpha \varphi_j \to \partial^\alpha \varphi$ uniformly on K for all α.*

Definition 156. *Let $U \subseteq \mathbb{R}^n$ be open with $\mathcal{D}'(U) := \{F : C_c^\infty(U) \to \mathbb{C},$ which are linear and continuous$\}$, where continuity means that $\varphi_j \to \varphi$ in $C_c^\infty(U)$ implies that $F(\varphi_j) \to F(\varphi)$.*

20.1.1 Example

(a) $f \in L^1_{loc}(U)$ (recall this means that $\int_K |f| < \infty$ for all $K \subseteq U$ compact) defines a distribution on U. This is true since $\varphi \mapsto \int f\varphi$ is linear and continuous with $\operatorname{supp}(\varphi) = K \subseteq u$. and $\varphi \in \mathcal{D}(U)$.

(b) Any μ, a Radon-measure on U, defines a distribution on U. We have $\varphi \mapsto \int \varphi d\mu$.

(c) $x_0 \in U$ and α a multi-index gives $\varphi \mapsto \partial^\alpha \varphi(x_0)$ is a distribution (this only comes from a measure if $\alpha = 0$.)

20.1.2 Notation

We identify L'_{loc} with a subspace of $\mathcal{D}'(U)$. We use the following conventions

$$(F, \varphi) = F(\varphi), \qquad \varphi \in \mathcal{D}(U) = C_c^\infty(U), F \in \mathcal{D}'(U).$$

$$\tilde{\varphi}(x) = \varphi(-x)$$

$$(\delta, \varphi) = \varphi(0).$$

Proposition 81. *Let $f \in L^1(\mathbb{R})$ with $\int f = a$ and $f_t(x) = t^{-n} f(t^{-1}x)$, then $f_t \to a\delta$ in $\mathcal{D}'$ as $t \to 0$.*

Proof. by 8.14 take $\varphi \in C_c^\infty$, we have

$$(f_t, \varphi) = \int f_t \varphi = t_t * \tilde{\varphi}(0) \to a\tilde{\varphi}(0 = a\varphi(0) = a(\delta, \varphi).$$

□

Definition 157. *We say that $F = G$ on V an open set if $(F, \varphi) = (G, \varphi)$ for all $\varphi \in C_c^\infty(V)$.*

Proposition 82. *Suppose $\{V_\alpha\}$ is a collection of open sets. Let $V = \bigcup_\alpha V_\alpha$. If $F, G \in \mathcal{D}'(U)$ agree on each V_α then they agree on V.*

Proof. Let $\varphi \in C_c^\infty(V)$. By compactness there exist finitely many $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $\operatorname{supp}\varphi \subseteq \bigcup_1^m V_{\alpha_j}$. Using C_c^∞ a partition of unity we can find $\psi_1, \ldots, \psi_m \in C_c^\infty$ with $\operatorname{supp}(\psi_j) \subseteq V_\alpha$ such that $\sum_1^m \psi_j = 1$ on $\operatorname{supp}(\varphi)$. We then have

$$(F, \varphi) = (F, \sum_1^m \varphi\psi_j) = (F\varphi \sum_1^m \psi_j) = \sum_1^m (F, \varphi\psi j) = \sum_1^m (G, \varphi\psi_j) = (G, \varphi).$$

□

Definition 158. *Suppose that $F \in \mathcal{D}'$ with $\operatorname{supp} F = V^c$ where V is the union of all open U such that $F = 0$ on U.*

20.2 Operations

(i) (Differentiation) If $F \in \mathcal{D}'(U)$ and α is a multi-index then we define $(\partial^\alpha F, \varphi) = (-1)^{|\alpha|}(F, \partial^\alpha \varphi)$ if $F = f \in C^{|\alpha|}(U)$ then

$$\int (\partial^\alpha f) \cdot \varphi = (-1)^{|\alpha|} \int f \cdot (\partial^\alpha \varphi).$$

(ii) (Multiplication by Smooth Functions) Let $\psi \in C^\infty(U)$ and $F \in \mathcal{D}'(U)$. Then we have

$$(\psi F, \varphi) := (F, \psi\varphi).$$

(iii) (Translation) If $F \in \mathcal{D}'(U)$ then $\tau_y F \in \mathcal{D}'(U + y)$ and it acts by

$$(\tau_y F, \varphi) = (F, \tau_{-y}\varphi).$$

(iv) (Composition with Linear Maps) Suppose $S : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible. We want to define $F \circ S \in \mathcal{D}'(S^{-1}(U))$, the way to do this is

$$(F \circ S, \varphi) = \frac{1}{\det S}(F, \varphi \circ S^{-1}).$$

20.3 Convolutions

There are two different methods for extending the notion of a convolution.

20.3.1 Method 1

Let $\psi \in C_c^\infty$ and U be open, $F \in \mathcal{D}'(U)$. Start with $f \in L^1_{loc}(U)$. We can define

$$f * \psi(x) = \int f(x-y)\psi(y)dy = \int f(y)\psi(x-y)dy = \int f\tau_x\tilde{\psi}(y)dy,$$

which is defined on $V := \{x : x - y \in U \text{ for all } y \in \operatorname{supp}\psi\} = \{x : x + \operatorname{supp}\tilde{\psi} \subseteq U\}$. Next, for $F \in \mathcal{D}'(U)$ we have

$$F * \psi(x) := (F, \tau_x\tilde{\psi}),$$

which is defined on the same set V. Note that $F * \psi$ is continuous on V. If $x \to x_0$ in V then $\tau_x\tilde{\psi} \to \tau_{x_0}\tilde{\psi}$ in C_c^∞ and this implies $(F, \tau_x\tilde{\psi}) \to (F, \tau_{x_0}\tilde{\psi})$ which is the definition of continuity.

20.3.2 Example

When we consider the point-mass function δ we obtain

$$\delta * \psi(x) = (\delta, \tau_x \tilde{\psi}) = \tau_x \tilde{\psi}(0) = \tilde{\psi}(-x) = \psi(x).$$

20.3.3 Method 2

Let $\psi, \tilde{\psi}$ and V be as above. Let $f \in L^1_{loc}(U)$ and $\varphi \in C_c^\infty(V)$. We have

$$\int (f * \psi)\varphi = \int \int f(y)\psi(x-y)\varphi(x)dydx = \int f \cdot (\varphi * \tilde{\psi}).$$

Hence it makes sense to give the definition for $F \in \mathcal{D}'(U)$

$$(F * \psi, \varphi) = (F, \varphi * \tilde{\psi}).$$

20.3.4 Example

Again with the δ function we have

$$\delta * \psi = (\delta * \psi, \varphi) = (\delta, \varphi * \tilde{\psi}(0) = \int \varphi(x)\psi(x) = (\psi, \varphi).$$

Proposition 83. *If $U \subseteq \mathbb{R}^n$ is open and $\psi \in C_c^\infty$ and V as above. For $F \in \mathcal{D}'(U)$, $x \in V$, let $F * \psi(x) = (F, \tau_x \tilde{\psi})$. The following hold:*

(a) $F * \psi \in C^\infty(V)$.

(b) $\partial^\alpha(F * \psi) = (\partial^\alpha F) * \psi = F * (\partial^\alpha \psi)$.

(c) $\forall \varphi \in C_c^\infty(V)$ *we have* $\int (F * \psi)\varphi = (F, \varphi * \tilde{\psi})$.

Proof. First we prove (b) Let e_j be a unit vector. We have

$$\partial_j(F*\psi)(x) := \lim_{t\to 0} \frac{(F, \tau_{x+te_j}\tilde{\psi}) - (F, \tau_x \tilde{\psi})}{t} = \lim_{t\to 0} \left(F, \frac{\tau_{x+te_j}, \tilde{\psi} - \tau_x \tilde{\psi}}{t} \right) \to \left(F, \tau_x \partial_j \tilde{\psi} \right) = F*(\partial_j \psi).$$

So, since $(\partial^\alpha f) * \psi = (\partial^\alpha F, \tau_x \tilde{\psi}) = (-1)^{|\alpha|}(F, \partial^\alpha \tau_x \tilde{\psi})$ we conclude (b). Notice that (b) $\implies$ (a).

Lastly, we prove (c). For any $\varphi \in C_c^\infty(V)$ we claim that

$$\int (F * \psi)\varphi = (F, \varphi * \tilde{\psi}).$$

To see this notice that

$$\varphi * \tilde{\psi}(x) = \int \varphi(y)\psi(y-x)dy = \int \varphi(y)\tau_y \tilde{\psi}(x)dy.$$

Taking a Riemann sum, $S^m = 2^{-nm} \sum_j \varphi(y_j^m)\tau_{y_j^m}\tilde{\psi}$. Observe that $S^m \to \varphi * \tilde{\psi}$ uniformly as $m \to \infty$. So we have

$$(F, \varphi * \tilde{\psi}) = \lim_{m\to\infty}(F, S^m) = \lim_{m\to\infty} 2^{-nm} \sum_j \varphi(y_j^m)(F, \tau_{y_j^m}\tilde{\psi}) = \int \varphi(y)(F * \psi)(y)dy.$$

□

Corollary 39 (Folland 9.5)**.** *For all open $U \subseteq \mathbb{R}^n$ we have $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$ (in the weak* topology).*

Proof. We will use Lemma 9.4 which states that for $\varphi, \psi \in C_c^\infty$ with $\int \psi = 1$ and the function $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. It follows that

(a) For all open $U \subseteq \operatorname{supp}\varphi$ we have $\operatorname{supp}(\varphi * \psi_t) \subseteq U$ for t small. Therefore $\operatorname{supp}(\psi_t) = t \operatorname{supp}\psi$.

(b) it holds that $\varphi * \psi_t \to \varphi$ as $t \to 0$ in C_c^∞ (by 8.14)

That said, we prove the proposition.

$\boxed{\text{Step 1}}$ First we approximate F by distributions of compact support in U and take $V_j \nearrow U$ (open and precompact sets) by C^∞-Urysohn. So there exists $\zeta_j \in C_c^\infty(u)$ with $\zeta_j \equiv 1$ on $\overline{V}_j$. We then have $\zeta_k F \to F$ as $k \to \infty$ in $\mathcal{D}'(U)$. Hence for all $\varphi \in \mathcal{C}(U)$ there exists j such that $\operatorname{supp}\varphi \subseteq V_j$. We then have

$$(F, \varphi) = (F, \zeta_j\varphi) = (\zeta_j, F\varphi).$$

Since $\operatorname{supp}\zeta_j$ is compact we know that $\zeta_j F$ is a distribution on $\mathbb{R}^n$. We can take ψ, ψ_t as in Lemma 9.4. So we would like to show that

$$(\zeta_j F) * \psi_t \to \zeta_j F, \qquad \text{in } \mathcal{D}'(U).$$

Let $\varphi \in C_c^\infty$ by Proposition 9.3 we have

$$((\zeta_j F) * \psi_t, \varphi) = (\zeta_j F, \varphi * \tilde{\psi}_t) \to (\zeta_j F, \varphi),$$

by 9.4. The last step is to check that $\operatorname{supp}((\zeta_j F) * \psi_t) \subseteq U$ for t small and j large. Note that $\operatorname{supp}(\zeta_j) \subseteq V_k$ for some k. If $\operatorname{supp}\varphi \cap \overline{V}_k = \emptyset$ then for small t we have $\operatorname{supp}(\varphi * \tilde{\psi}_t) \cap \overline{V}_k = \emptyset$. This implies that $((\zeta_j F) * \psi_t, \varphi) = (F, \zeta_j(\varphi * \tilde{\psi}_t)) = 0$. □

20.4 Compactly Supported Distributions

Recall that the **support of a distribution** F is given by $\operatorname{supp}(F) = (\bigcup_V \{\text{open } V \subseteq F : F|_V \equiv 0\})^c$.

Definition 159. *Let $U \subseteq \mathbb{R}^n$ be open. Define the space $\mathcal{E}'(U) := \{$all distributions of compact support on $U\}$.*

We know that $\mathcal{D}'(U) = C_c^\infty(U)^*$. We will show that $\mathcal{E}'(U) = C^\infty(U)^*$.

Proposition 84. *$C^\infty(U)$ is a Fréchet space.*

Proof. We have $V_m \nearrow U$ (with the V_m precompact) and $\bigcup_m V_m = U$. Given m, α we can write

$$\|f\|_{[m,\alpha]} = \sup_{x \in \overline{V}_m} |\partial^\alpha f(x)|,$$

is a seminorm. □

Lemma 45. *If X is a topological vector space with the topology defined by the seminorms $\{p_n\}$ then a linear map $F : X \to \mathbb{C}$ is continuous if and only if there exists an $N \in \mathbb{N}$ and $A > 0$ such that $|F(x)| \leq A \sum_1^N p_j(x)$.*

Proof. ($\Leftarrow$) is easy.

($\Rightarrow$) Suppose not. Let $x_N \in X$ be such that $|F(x)| \leq C_N \sum_1^N p_j(x)$ with $C_N \nearrow \infty$. Take $y_N := \frac{x_N}{C_N \sum_1^N p_N(x_N)}$, then

$$|F(y_N)| = \left| \frac{F(x_N)}{c_N \sum_1^N p_n(x_N)} \right| \geq 1.$$

BY $y_N \to 0$ in X because for all k we have $p_k(y_N) \to 0$ as $N \to \infty$. Hence if we take $N \geq k$ it follows that

$$p_k(y_N) = \frac{p_k(x_N)}{c_N \sum_1^N p_n(x_N)} \leq \frac{1}{c_N} \to 0.$$

□

Proposition 85 (9.8)**.** *We have $\mathcal{E}'(U) = (C^\infty(U))^*$, more precisely,*

(i) Every $F \in \mathcal{E}'(U)$ extends continuously to $G \in (C^\infty(U))^*$.

(ii) Every $G \in (C^\infty(U))^*$ satisfies $G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$.

Proof. (i) Let $F \in \mathcal{E}'(U)$. Choose $\psi \in C_c^\infty(U)$ with $\psi = 1$ on $\operatorname{supp} F$ (exist by C^∞ Urysohn). Let $(G, \varphi) = (F, \psi\varphi)$ for $\varphi \in C^\infty(U)$. This defines a linear functional on $C^\infty(U)$. Also, this is an extension since if $\varphi \in C_c^\infty(U)$ then $(F, \varphi) = (G, \varphi) = (F, \varphi\psi)$. The difficult property to check is that this is continuous. We know that F is continuous on $C_c^\infty(\operatorname{supp}\psi)$. Because $C_c^\infty(U)$ is dense in $C^\infty(U)$ we know that there exist N, C such that for all $\varphi \in C^\infty(U)$ it holds that

$$|(G, \varphi)| = |(F, \psi\varphi)| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha(\psi\varphi)\|_u \overset{*}{\leq} C' \sum_{|\alpha| \leq N, x \in \operatorname{supp}\psi} \sup |\partial^\alpha \varphi(x)| \leq C; \sum_{|\alpha| \leq N} \|\varphi\|_{[m,\alpha]}.$$

Where the inequality at * follows from the product rule. So we conclude that G is continuous. Moreover G is unique because it is continuous and defined on a dense set.

(ii) Let $G \in (C^\infty(U))^*$ the lemma implies that there exist C, m, N such that

$$|(G, \varphi)| \leq C \sum |\alpha \leq N \|\varphi\|_{[m,\alpha]},$$

Therefore $\|\varphi\|_{[m,\alpha]} \leq \|\partial^\alpha \varphi\|_u$ and thus G is continuous on $C_c^\infty(K)$ for all compact $K \subseteq U$. Letting $F = G|_{C_c^\infty(U)} \in \mathcal{D}'(u)$ it remains to prove that $\operatorname{supp} F \subseteq \overline{V}_m$. If $\operatorname{supp}\varphi \cap \overline{V}_m = \emptyset$ then $(G, \varphi) = 0$. □

Definition 160. *We define $\mathcal{S}'$ to be the set of continuous linear functionals on $\mathcal{S}$. Any $F \in \mathcal{S}$ is called a* ***tempered distribution****. Notice that $\mathcal{D}' \supset \mathcal{S}' \supset \mathcal{E}'$ and $C_c^\infty \subseteq \mathcal{S} \subseteq C^\infty$.*

Recall that $\mathcal{S}$ is a Fréchet space with topology defined by

$$\|\varphi\|_{[N,\alpha]} := \operatorname{supp}_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \varphi(x)|.$$

Proposition 86. *C_c^∞ is dense in $\mathcal{S}$. In fact if $\psi \in C_c^\infty$ with $\psi(0) = 1$, let $\psi^\epsilon(x) = \psi(\epsilon x)$, then $\psi^\epsilon \varphi \to \varphi$ in $\mathcal{S}$ as $\epsilon \to 0$ for all $\varphi \in \mathcal{S}$.*

Proof. Homework.

□

20.4.1 Example

1. Let $f \in L^1_{loc}(\mathbb{R}^n)$ with $\int (1+|x|)^{-N}|f(x)|dx < \infty$ and the corresponding distribution F_f. We have F_f is tempered since

$$\left|\int f\varphi\right| \leq C\,\|\varphi\|_{(0,N)} \qquad \text{and} \qquad |(F_f,\varphi)| = \left|\int f\varphi\right|.$$

2. The function $f(x) = e^{ax}$ is not tempered for $a > 0$. This is true because for $\psi \in C_c^\infty$ with $\int \psi = 1$ we can take $\varphi_j(x) = e^{-ax}\psi(x-j)$. It follows that $\varphi_j \to 0$ in $\mathcal{S}$ as $j \to \infty$, but $\int f\varphi_j = \int \psi(x-j)dx = 1$. So it is not continuous on $\mathcal{S}$ (since if it were continuous we would have $\varphi_j \to 0$ in $\mathcal{S}$ implies $F(\varphi_j) = to0$).

3. However, $e^x\cos(e^x)$ is a tempered distribution. To see this we employ integration by parts to write

$$\left|\int f\varphi\right| = \left|-\int \varphi'(x)\sin(e^x)dx\right| \leq C\,\|\varphi\|_{(2,1)}\,.$$

20.4.2 Operations

Definition 161. *$\psi \in C^\infty$ is **slowly increasing** if for all α there exists c_α, N_α such that $|\partial^\alpha\psi(x)| \leq c_\alpha(1+|x|)^{N_\alpha}$.*

20.4.3 Example

(i) Polynomials are slowly increasing.

(ii) $(1_|x|^2)^s$ is slowly increasing for all $s \in \mathbb{R}$.

- (Multiplication) Recall that for general $F \in \mathcal{D}'(U)$ we define $(F\psi,\varphi) = (F,\psi\varphi)$ If $F \in \mathcal{S}'$ to define $F\psi$ we need ψ to be C^∞ and *slowly* increasing.

- (Convolutions) Recall that (I) we first saw convolutions for $F \in \mathcal{D}'(U)$ and $\psi \in C_c^\infty$, then $F * \psi \in C^\infty(V)$. Next we (II) defined the convolution for $F \in \mathcal{E}'$, $\psi \in C^\infty$ and we proved that $F * \psi \in C^\infty$. We can also (III) define convolutions of distributions with distributions. This is defined for $F \in \mathcal{D}'(U)$ and $G \in \mathcal{E}'(U)$ then for test functions φ we take

$$(F * G, \varphi) := (F, \tilde{G} * \varphi).$$

  Lastly, for $F \in \mathcal{S}'$ and $\psi \in \mathcal{S}$ we can define $F * \psi(x) = (F, \tau_x\tilde{\psi})$.

Proposition 87 (9.10)**.** *Let $F \in \mathcal{S}', \psi \in \mathcal{S}$. This implies that $F * \psi \in C^\infty$ and is slowly increasing. And, for all $\varphi \in \mathcal{S}$ it holds that $\int (F*\psi)\varphi = (F, \varphi * \tilde{\psi})$.*

Proof. See Folland. □

20.5 Fourier Transform

Definition 162. *We define the **Fourier Transform** to be*

$$(\hat{F}, \psi) := (F, \hat{\varphi}),$$

where $F \in \mathcal{S}'$ and $\varphi \in S$.

Proposition 88 (9.11)**.** *Let $F \in \mathcal{E}'$, then $\hat{F}$ is slowly increasing and C^∞. Also, $\hat{F}(\xi) = (F, E_{-\xi})$ where $E_\xi = e^{2\pi i \xi \cdot x}$.*

Proof. Begin by defining the function $g(\xi) = (F, E_{-\xi})$. Then, as in 9.3, we know that $g \in C^\infty$ and we can write

$$\partial^\alpha g(\xi) = (F, \partial_\xi^\alpha E_{-\xi}) = (-2\pi i)^{|\alpha|}(F, x^\alpha E_{-\xi}).$$

By 9.9 and Lemma 5.15 we know that there exist m, n, c such that

$$|\partial^\alpha g(\xi)| \leq c \sum_{|\beta| \leq N} \sup_{|x| \leq m} |\partial_x^\beta (x^\alpha E_{-\xi}(x))| \leq c'(1+m)^{|\alpha|}(1+|\xi|)^N.$$

Hence g is slowly increasing. To show that $g = \hat{F}$ we need to show that

$$(g, \varphi) = (\hat{F}, \varphi), \qquad \forall \varphi \in \mathcal{S}.$$

Since C_c^∞ is dense in $\mathcal{S}$ it is enough to check for $\varphi \in C_c^\infty$. Then using Riemann sums we can write

$$\begin{aligned}
\int g\varphi &= \lim \sum g(\xi_j)\varphi(\xi_j)\Delta\xi_j \\
&= \lim \sum (F, E_{-\xi_j})\varphi(\xi_j)\Delta\xi_j \\
&= \lim (F, \sum \varphi(\xi_j) E_{-\xi_j} \Delta\xi_j) \\
&= (F, \hat{\varphi})
\end{aligned}$$

□

20.6 Sobolev Spaces

20.6.1 Informal Introduction

Let $k \geq 0$. Define the set $H_k := \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2 \forall |\alpha| \leq k\}$. This is a Hilbert space with inner product

$$(f, g) = \sum_{|\alpha| \leq k} \int (\partial^\alpha f)\overline{(\partial^\alpha g)}.$$

Note that by 8.22.e and Plancherel we know that $f \in H_k \iff \xi^\alpha \hat{f} \in L^2$ for all $|\alpha| \leq k$.

It turns out to be more convenient to use an equivalent norm.

20.6.2 Remark

There exist constants $C_1, C_2 > 0$ such that

$$C_1(1+|\xi|^2)^k \leq \sum_{|\alpha| \leq k} |\xi^\alpha|^2 \leq C_2(1+|\xi|^2)^k.$$

For a proof see 8.3. This means that we can work with $C(1+|\xi|^2)^k$ instead of the more cumbersome summation $\sum_{|\alpha| \leq k} |\xi|^2$.

Proposition 89. $f \in H_k \iff (1+|\xi|^2)^{\frac{k}{2}} \hat{f} \in L^2$.

20.6.3 Formal Definition

We know that for all $s \in \mathbb{R}$ we have $\xi \mapsto (1+|\xi|^2)^{\xi/2}$ is C^∞ and slowly increasing. Define the operator

$$\Lambda_s f := [(1+|\xi|^2)\hat{f}]\check{}.$$

Definition 163. *For $s \in \mathbb{R}$ we define the **Sobolev Space** to be*

$$H_s := \{f \in \mathcal{S}' : \Lambda_s f \in L^2\}.$$

H_s is equipped with an inner product given by

$$(f,g)_s = \int (\Lambda_s f)\overline{(\Lambda_s g)} \overset{*}{=} \int \hat{f}(\xi)(1+|\xi|^2)^s \overline{\hat{g}(\xi)} d\xi.$$

*Where at * we use Plancherel.*

Proposition 90. *H_s has the following properties:*

(i) H_s is a Hilbert Space and $\mathcal{F} : H_s \to L^2(\mathbb{R}^n, \mu_s)$ is an isomorphism where $d\mu_s(\xi) = (1+|\xi|^2)^s d\xi$ and $\mathcal{F}(f) = \hat{f}$ (this gives $\|f\|_s = \|\Lambda_s f\|_2$).

(ii) $\mathcal{S}$ is dense in H_S for al $ls \in \mathbb{R}$.

(iii) If $t < s$ then $H_s \subseteq H_t$, but H_s is dense in H_t and $\|\cdot\|_t \leq \|\cdot\|_2$.

(iv) Λ_t is a unitary operator from $H_s \to H_{s-t}$ for all $s, t \in \mathbb{R}$.

(v) $H_0 \in L^2$ and $\|\cdot\|_0 = \|\cdot\|_2$ (and so $H_s \subseteq L^2$ for all $s \geq 0$).

(vi) $\partial^\alpha : H_s \to H_{s-|\alpha|}$ is bounded for all s, α.

Proof. (i) Straightforward.

(ii) By (i) this is equivalent to $\mathcal{S}$ being dense in $L^2(\mathbb{R}^n, \mu_s)$ which we know to be true.

(iii) Straightforward.

(iv) Straightforward.

(v) Straightforward.

(vi) True since $|\xi^\alpha| \leq (1+|\xi|^2)^{|\alpha|/2}$.

□

20.6.4 Example

$\delta \in H_s \iff s \leq -\frac{n}{2}$. To see this we note that $\hat{\delta}(\xi) \equiv 1$ and so $\|\delta\|_2 = \int (1+|\xi|^2)^{s/2} < \infty$ iff $s < -\frac{n}{2}$.

Proposition 91. *$H_{-s} \simeq (H_s)^*$.*

Proof. We are skipping this due to a time shortage. □

20.6.5 Sobolev Embedding Theorem

Theorem 101 (Sobelev Embedding Theorem)**.** *Suppose that $s > k + \frac{n}{2}$ with $k \geq 0$ an integer. The following hold regarding H_s:*

(a) $f \in H_s$ implies that $(\partial^\alpha f)\hat{} \in L^1$ have $\|(\partial^\alpha f)\check{}\|_1 \leq C \|f\|_s$. for all $|\alpha| \leq k$ and C depends only on $k - s$.

(b) $H_s \subseteq C_0^k$ (in the sense up to measure zero).

Proof. First we prove that $(a) \implies (b)$. Notice that for $|\alpha| \leq k$ it holds that

$$(\partial^\alpha f)\hat{} = g \in L^1 \implies \partial^\alpha f = \check{g}.$$

And since $\check{g} \in C_0$ (by Riemann-Lebesgue Lemma) we have (b).

Next we prove (a). We can write

$$\begin{aligned}
(2\pi)^{-|\alpha|} \int |(\partial^\alpha f)\hat{}(\xi) d\xi &= \int |\xi^\alpha \hat{f}(\xi)| d\xi \\
&\leq \int (1 + |\xi|^2)^{k/2} |\hat{f}(\xi)| d\xi \\
&\overset{*}{\leq} \left(\int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \left(\int (1 + |\xi|^2)^{k-s} d\xi \right)^{1/2} \\
&\overset{**}{=} \|f\|_2 \cdot C.
\end{aligned}$$

Where at * we use Cauchy-Schwarz and at ** we use the hypothesis that $s > k + \frac{n}{2}$ to conclude that the right side integral is a constant C.

□

Corollary 40. *If $f \in H_s$ for all $s \in \mathbb{R}$ then $f \in C^\infty$.*

20.7 The Elliptic Regularity Theorem

Our goal for the next lecture is to prove the elliptic regularity theorem. Let's state it without explaining anything.

Theorem 102 (Elliptic Regularity Theorem)**.** *Let L be an elliptic, constant coefficient differential operator of order m. Let Ω be an open subset of $\mathbb{R}^n$. Let $u \in \mathcal{D}'(U)$.*

(a) If $Lu \in H_s^{loc}(\Omega)$ for some $s \in \mathbb{R}$ then $u \in H_{s+m}^{loc}$.

(b) And if $Lu \in C^\infty$ then $u \in C^\infty(\Omega)$.

Let $P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$ be a polynomial. Define the differential operator $L = P(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$. We say that L is **order** m if there exists α with $|\alpha| = m$ and $c_\alpha \neq 0$. Also we say that $P(\xi)$ is the **symbol** of $P(D)$. One more term is $P_m(\xi) = \sum_{|\alpha| = m} c_\alpha \xi^\alpha$ which we call the **principle symbol**.

Definition 164. *We say that L is **elliptic** if $P_m(\xi) \neq 0$ for all $\xi \neq 0$.*

20.7.1 Example

Δ is elliptic but $\partial_t - \Delta$ and $\partial_t^2 - \Delta$ are not.

Lemma 46. *$P(D)$ is elliptic of order m if and only if there exists $R, C > 0$ such that $|P(\xi)| \leq C|\xi|^m$ for $|\xi| \geq R$.*

Proof. See Folland. □

Proof of Elliptic Regularity Theorem ($\Omega = \mathbb{R}^d$ case)

Proof. First we consider the case $\Omega = \mathbb{R}^d$. We know that

$$P(D)u \in H_s \iff (1+|\xi|^2)^{s/2}\widehat{P(D)}u \in L^2 \iff (1+|\xi|^2)^{s/2}|P(\xi)|\hat{u} \in L^2,$$

and also that

$$u \in H_s \iff (1+|\xi|^2)^{s/2}\hat{u} \in L^2.$$

Let $|\xi| \geq R \geq 1$. This implies that $(1+|\xi|^2)^{m/2} \leq 2^m|\xi|^m \leq C^{-1}2^m|P(\xi)|$ and so

$$|\xi| \leq R \implies (1+|\xi|^2)^{m/2} \leq (1+R^2)^{m/2}.$$

We would like to show that $u \in H_{s+m}$ which we know is equivalent to proving $(1+\xi)^{\frac{s+m}{2}}|\hat{u}| \in L^2$. Observe that

$$(1+|\xi|^2)^{m/2}(1+|\xi|^2)^{s/2}|\hat{u}| \leq C'(1+|\xi|^2)^{s/2}(|P\hat{u}| + |\hat{u}|) \in L^2.$$

□

The more general setting takes too much work to prove. We just offer a sketch. First some more terminology.

Definition 165. *We define the space*

$$H_s^{loc}(\Omega) := \{f \in \mathcal{D}'(\Omega) : \text{ for all precompact } V \text{ such that } \overline{V} \subseteq \Omega \text{ there exists } g \in H_s \text{ such that } g = f \text{ on } V\}.$$

Lemma 47 (9.23)**.** *$f \in H_s^{loc}$ if and only if $\varphi f \in H_s$ for all $\varphi \in C_c^\infty(\Omega)$.*

Proof. See Folland. This should be intuitively obvious from the definition. □

Theorem 103 (9.26)**.** *Assume $u \in \mathcal{D}'(\Omega)$ and $Lu \in H_s^{loc}(\Omega)$ with L elliptic of order m. Then $u \in H_{s+m}^{loc}(\Omega)$.*

Proof. Given $\varphi \in C_c^\infty(\Omega)$ we want to show that $\varphi u \in H_{s+m}$. Choose φ such that

$$\operatorname{supp}\varphi \subseteq V \subseteq \overline{V} \subseteq \Omega.$$

There exists $\psi \in C_c^\infty(\Omega)$ such that $\psi \equiv 1$ on $\overline{V}$ and so $\psi u \in \mathcal{E}'$. We next claim that $\psi u \in H_\sigma$ for some $\sigma \in \mathbb{R}$. This follows from 9.11 which states that

$$F \in \mathcal{E}' \implies \hat{F}(\xi) \text{ is slowly increasing.}$$

We then have $|\hat{F}(\xi)| \leq C(1+|\xi|)^N$ for some N. Hence

$$(1+|\xi|^2)^{\sigma/2}|\hat{F}(\xi)| \in L^2.$$

Without loss of generality suppose that $s + m - \sigma = k \in \mathbb{N}$. Set $\psi_0 = \psi_1$ and $\psi_k = \varphi$ with $\psi_1, \ldots, \psi_k \in C_c^\infty$ such that $\psi \equiv 1$ on $\overline{\operatorname{supp}\varphi} + \epsilon$ and so that $\operatorname{supp}(\psi_j) \subseteq \{\psi_{j-1} \equiv 1\}$. We will show by induction that

$$\psi_j u \in H_{\sigma+j}, \qquad j = 0, \ldots, k.$$

The trick is to use commutators. We define the operator for $\rho \in C_c^\infty$

$$[L, \rho]f := L(\rho f) - \rho L f.$$

This is a differential operator of order $m-1$ whose coefficients are linear combinations of derivatives of ρ, and which vanish on a set where $\rho \equiv A$ with A a constant.

We then have $f \in H_t \implies \partial^\alpha f \in H_{t-(m-1)}$ for $|\alpha| \leq m-1$. Proposition 9.20 tells us that for $\gamma \in C_0$ it holds that

$$\int (1 + |\xi|^2)^{a/2} |\hat{\gamma}(\xi)| d\xi < \infty.$$

Therefore $M_\gamma f = \gamma f$ is bounded on H_s for $|s| \leq a$.

Now we can perform the inductive step. Suppose that $\psi_j u \in H_{\sigma+j}$ for $0 \leq j < k$. We would like to prove that $\psi_{j=1} u \in H_{\sigma+j+1}$. To see this observe that

$$\begin{aligned} L(\psi_j u) &= \psi_{j+1}[L, \psi_{j+1}]u \\ &= \psi_{j+1} L u + [L, \psi_{j+1}]\psi_j u \\ &\in H_s + H_{\sigma+h-(m-1)} \\ &= H_{\sigma+j+1-m} \end{aligned}$$

where the inclusion follows from the inductive hypothesis. Moreover, we must have $s \geq \sigma + j - (m-1)$ and so $s = \sigma + k - m$. Now we can apply elliptic regularity in $\mathbb{R}^d$ and conclude the theroem.

$\square$