HOMEWORK 2

Section 2.5: 3, 8, 11, 16, 24, 30, 31, 34, 37, 41, 44, 47, 50, 51

Answer these two questions also.

1. This question is about a concept called *thinning*—where we take a random variable and make it smaller.
   (a) Fix numbers $0 < q, p < 1$ and a positive integer $n > 0$. Let $Y = \text{binomial}(n, p) = \sum_1^n X_i$ with $X_i = \text{bernouli}(p)$. Suppose $W = \text{binomial}(Y, q)$ and $X_i' = \text{bernouli}(q)$ for $i = 1, 2, \ldots, n$. Use the random variables X_i and X_i' to explain why $W = \text{binomial}(n, pq)$. *Hint: Set $\tilde{X}_i = X_i X_i'$. What is the distribution of $\tilde{X}_i$?*
   (b) You are picking blackberries on a sunny day. It is bright, so you can't see the berries that well. You pick 100 of them. Each is ripe independently with probability 9/10. Each has a bit of spiderweb on it with probability 1/9. The presence of a bit of web is independent of the ripeness and webs on other berries. What is the probability you pick at least 98 ripe berries with no spiderwebs?
   (c) Set $Z = \text{Poisson}(\lambda)$ and suppose $M = \text{binomial}(Z, q)$. We can't prove it with the material covered in this class, but an important property called *Poisson thinning* is that $M = \text{Poisson}(\lambda q)$. The United States has on average 20 shark attacks per year. The probability of a shark attack being fatal is 1/5. What is the probability of at least 3 deaths by shark attack this year in the United States?
   (d) The most amazing thing (imo) about the Poisson random variable is that if we start with $Z = \text{Poisson}(\lambda)$, then apply a multinomial thinning of Z with parameters $p_1, \ldots, p_n$ then the number put in group i is distributed like $Z_i = \text{Poisson}(\lambda p_1)$. Moreover, the Z_i are **independent**. No other random variable has this independence property! Suppose we place $\text{Poisson}(n)$ balls uniformly randomly into n bins. What is the distribution for the number of balls inside of the first bin?
   (e) Let Q be the number of empty bins after placing all $\text{Poisson}(n)$ balls. Explain why Poisson thinning ensures $Q = \text{binomial}(n, p)$, and find p.
   (f) Suppose n balls are placed uniformly into n bins. Find the probability, q, that the first bin is empty after this. Explain why the number of empty bins is not a $\text{binomial}(n, q)$ random variable.

2. You keep playing a game with a 9/10 chance of losing \$1 and a 1/10 chance of winning a dollar. Let $X_i \in \{-1, 1\}$ be the amount of money you win on the ith try and $S_n = \sum_{i=1}^n X_i$ be your total winnings/losses.
   (a) Let E_n be the event that $S_{2n} = 0$. What is $P(E_n)$?
   (b) Let A be the event that there exists a time t such that $S_t = 0$. Explain why $A = \cup_{n=1}^\infty E_n$.
   (c) Use a union bound and the previous part to give an upper bound on $P(A)$.
   (d) Use the fact that $C_{2n,n} \leq 4^n$ and that $\sum_{n=1}^\infty a^n = \frac{1}{1-a} - 1$ to estimate the sum in the previous part.
   (e) Say in words what it means that $P(A) < 1$. Is this surprising to you?