San Francisco State University
ECON 715

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Fall 2016

Final Exam - Solution

December 15, 2016

Time: 120 minutes

Name: ______________________________

Instructions:
1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (25 points). Suppose that the market value of an asset, at time t, is given by $V(t)$ - a twice differentiable function of time. Suppose that the interest rate per period (year) is r.

   (a) The owner of the asset wishes to maximize the present value of the asset, by selling it at the right time. Write the optimization problem of the owner, and derive the first order necessary condition.

   Optimization problem is:

$$\max_t PV(t) = V(t) e^{-rt}$$

   First order necessary condition for maximum is:

$$\begin{aligned} \frac{d}{dt} PV(t) &= V'(t) e^{-rt} - rV(t) e^{-rt} = 0 \\ &\Rightarrow V'(t) - rV(t) = 0 \end{aligned}$$

   (b) Provide economics intuition of the first order necessary condition from the previous section.

   The first order condition can be written as

$$\frac{V'(t)}{V(t)} = r$$

   This means that the owner should keep the asset until the growth rate of its value equalizes to the interest rate.

(c) Suppose the value of the asset evolves according to $V(t) = Ke^{f(t)}$, where $f(t) = 0.08\sqrt{t}$, where t is time in years and the interest rate is $r = 1\%$. Find the optimal holding time, t^*, of the asset.

$$
\begin{aligned}
\frac{V'(t)}{V(t)} &= r \\
\frac{Ke^{f(t)} \cdot f'(t)}{Ke^{f(t)}} &= r \\
f'(t) &= 0.08 \cdot 0.5t^{-0.5} = r \\
t^* &= \left(\frac{0.04}{r}\right)^2 = \left(\frac{0.04}{0.01}\right)^2 = 4^2 = 16 \text{ years}
\end{aligned}
$$

(d) Suppose that $f(t)$ is unknown, but it is given that f is increasing and concave ($f'(t) > 0\ \forall t$, and $f''(t) < 0\ \forall t$). Prove that the optimal holding time, t^*, is decreasing in interest rate r.

Proof 1. Taking differential of both sides of the optimality condition $f'(t) = r$:

$$\begin{aligned} f''(t)\,dt &= dr \\ &\Rightarrow \frac{dt}{dr} = \frac{1}{f''(t)} < 0 \end{aligned}$$

The last inequality follows from the given that $f''(t) < 0$.

Proof 2. Write the F.O.N.C. as implicit function:

$$F(t, r) = f'(t) - r = 0$$

Then, by the implicit function theorem:

$$\frac{dt}{dr} = -\frac{F_r}{F_t} = -\frac{-1}{f''(t)} = \frac{1}{f''(t)} < 0$$

(e) Provide economic intuition for the result in the previous section.

The interest rate is the opportunity cost of holding the asset for additional time. The owner can sell the asset, and invest the money at interest rate r. Thus, higher interest rate, means that the opportunity cost of holding the asset is higher, and the seller would like to sell it earlier.

2. (10 points). Let the profit of some firm be $\pi(w,x) = R(x) - wx$, where w is $1 \times n$ vector of input prices, x is $n \times 1$ vector of inputs and R is revenue function. Prove that an increase in at least one input price, holding all else constant, must weakly decrease the firm's profit. In particular, let w_1, w_2 be two input price vectors, with $w_2 > w_1$. This means that for any input i we have $w_{2i} \geq w_{1i}$, with $w_{2i} > w_{1i}$ for at least one input. Let x_1 and x_2 be profit maximizing inputs under w_1, w_2. You need to prove that

$$\pi(w_1, x_1) \geq \pi(w_2, x_2)$$

Proof.

$$\begin{aligned} \pi(w_1, x_1) &\geq \pi(w_1, x_2) \quad (x_1 \text{ is profit maximizing quantity under } w_1) \\ &\geq \pi(w_2, x_2) \quad (\text{costs higher under } w_2, \text{ i.e. } w_2 x_2 \geq w_1 x_2) \end{aligned}$$

The last inequality says that the firm's costs must weakly increase, when inputs are more expensive, and the firm employs the same inputs. The revenue is the same. Notice that in the first step, the object $\pi(w_1, x_2)$ is well defined, since the firm can afford to buy x_2 under w_2, so it can definitely afford x_2 when inputs are cheaper under w_1. Also notice the generality of the last theorem. The only assumption we made about the firm is that it maximizes profit. The firm can use any number of inputs, produce any number of products, and operate under any possible market structure and government regulation. We proved that the firm's profit cannot increase when the price of inputs goes up. We can also prove that input demand is decreasing in their prices.

3. (15 points). Consider a monopoly that sells a single product to n segmented markets. The revenue function in market i, as a function of quantity of product sold in that market, is $R_i(Q_i)$, with $R_i'(Q_i) > 0$ and $R_i''(Q_i) < 0$. Assume that the total cost function, $C(Q) = C\left(\sum_{i=1}^{n} Q_i\right)$, is increasing and strictly convex.

   (a) Write the optimization problem of the monopoly, and derive the first order necessary condition.

   Profit maximization problem:

$$\max_{Q_1,\dots,Q_n} \pi(Q_1,\dots,Q_n) = \sum_{i=1}^{n} R_i(Q_i) - C(Q)$$

   First order necessary condition:

$$\frac{\partial}{\partial Q_i}\pi(Q_1,\dots,Q_n) = R_i'(Q_i) - C'(Q) = 0 \quad \forall i = 1,\dots,n$$

   (b) Provide economics interpretation of the first order necessary condition from the previous section.

   The above condition states that the marginal revenue in all markets must be equal to the common marginal cost.

(c) Prove that the critical value of the profit function is a unique global maximum. Clearly state the theorems used in your proof.

Proof. It is given that $R_i(Q_i)$ is strictly concave $\forall i$, and therefore $\sum_{i=1}^{n} R_i(Q_i)$ is strictly concave (sum of strictly concave functions is str. concave). The cost function is given to be strictly convex, so $-C(Q)$ is strictly concave (f is concave if and only is $-f$ is convex, strict or not). Thus, the profit function is a sum of strictly concave functions and therefore strictly concave. Consequently, the critical point of a strictly concave function is automatically a unique global maximum.

4. (20 points). Suppose a consumer derives utility from quantities of 2 goods, x, y, and his utility function is $u(x, y)$. The prices of the goods are p_x, p_y and consumer's income is I.

   (a) Write the consumer's utility maximization problem, assuming that he must spend all his income on the 2 goods.

   Consumer's problem:

$$\max_{x,y} u(x, y)$$
$$s.t.$$
$$p_x x + p_y y = I$$

   (b) Write the Lagrange function associated with the consumer's problem, and derive the first order necessary conditions for constrained optimum.

   The Lagrange function:

$$\mathcal{L} = u(x, y) - \lambda [p_x x + p_y y - I]$$

   The first ordered necessary conditions are:

$$\begin{aligned} \mathcal{L}_\lambda &= -[p_x x + p_y y - I] = 0 \\ \mathcal{L}_x &= u_x(x, y) - \lambda p_x = 0 \\ \mathcal{L}_y &= u_y(x, y) - \lambda p_y = 0 \end{aligned}$$

(c) Suppose that $u(x, y) = xy$, the optimal bundle is $(x^*, y^*) = (10, 20)$, and $p_x = 10$. Find the value of the lagrange multiplier and the price of good y, i.e. p_y.

The first order conditions determine the unique global maximum because the constrain set is convex and objective function is strictly quasiconcave. Using the first order condition for x, gives:

$$\lambda = \frac{u_x(x, y)}{p_x} = \frac{y}{p_x} = \frac{20}{10} = 2$$

Using the first order condition for y, gives:

$$\begin{aligned} \frac{u_y(x, y)}{p_y} &= \lambda \\ \frac{x}{p_y} &= \frac{10}{p_y} = 2 \\ &\Rightarrow p_y = 5 \end{aligned}$$

Remark: if we want, we can find the income by $I = p_x x + p_y y = 10 \cdot 10 + 5 \cdot 20 = 200$.

(d) Write the economic interpretation of the lagrange multiplier you found in the previous section.

The Lagrange multiplier λ gives the marginal increase in the maximal utility due to \$1 added to the income I, i.e. the marginal utility of income. Thus, a \$1 increase in income, will increase the maximal utility by 2 units.

5. (20 points). Suppose that you are trying to maximize some objective function $u(x, y)$, subject to a constraint set C, in $\mathbb{R}_+^2$.

(a) Suppose you found a critical point (x^*, y^*) of u, subject to constraint set C. Circle the correct answer.

i. If C is convex and $u(x, y)$ is quasiconcave, then (x^*, y^*) is global minimum.
ii. If C is convex and $u(x, y)$ is concave, then (x^*, y^*) is global minimum.
iii. If C is convex and $u(x, y)$ is quasiconvex, then (x^*, y^*) is global maximum.
iv. If C is convex and $u(x, y)$ is convex, then (x^*, y^*) is global maximum.
v. If C is convex and $u(x, y)$ is quasiconcave, then (x^*, y^*) is global maximum.

(b) Prove that any Cobb-Douglas function $f(x, y) = x^\alpha y^\beta$, with $\alpha, \beta > 0$, is quasiconcave. Clearly state the theorems used in your proof.

Proof 1. One way is to use the theorem that any monotone increasing transformation of a concave function is quasiconcave. Thus, we can write

$$f(x, y) = \exp(\alpha \ln(x) + \beta \ln(y))$$

We proved that weighted sum (with positive weights) of concave functions is concave. Since $\ln(\cdot)$ is concave ($\frac{d}{dx}\ln(x) = x^{-1}$, $\frac{d^2}{dx^2}\ln(x) = -x^{-2} < 0$), and $\alpha, \beta > 0$, the function $h(x, y) = \alpha \ln(x) + \beta \ln(y)$ is concave. The exponential function $\exp(\cdot)$ is monotone increasing, so $f(x, y) = \exp(h(x, y))$ is quasiconcave.

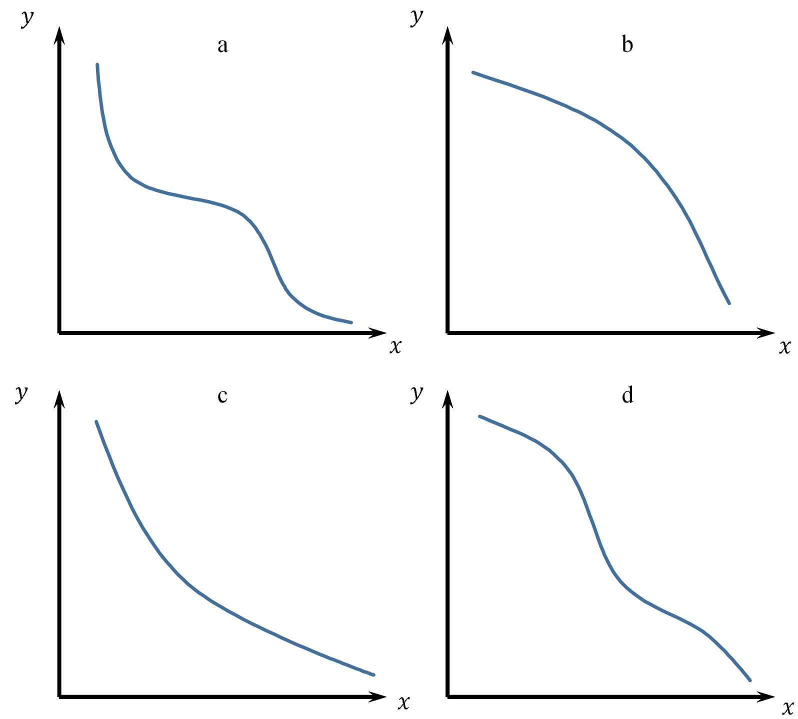
Proof 2. An alternative way to prove the above is to use the theorem that level curves of quasiconcave functions are convex functions. Thus, a level curve of f is $x^\alpha y^\beta = \bar{u}$. Solving for y, gives

$$y = \frac{\bar{u}^{1/\beta}}{x^{\alpha/\beta}}$$

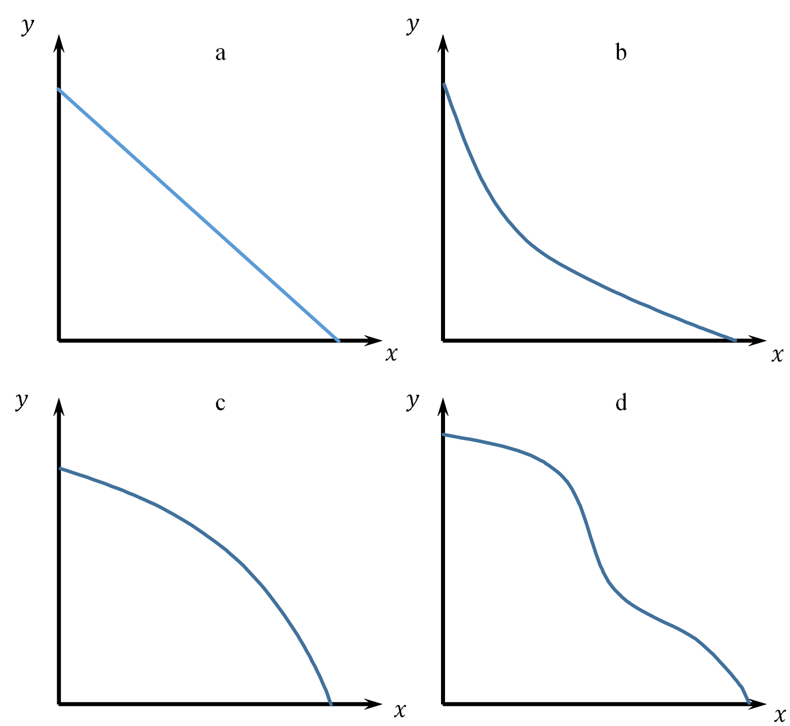
Here y is a convex function of x, implying that $f(x, y)$ is quasiconcave.

Note: the second way of proving the result is easy for functions of two variables, but becomes messy for functions with 3 or more variables.

(c) Which of the following graphs display indifference curve of a quasiconcave utility function? Circle all the correct answers: a, b, $\boxed{\text{c}}$, d.



(d) Which of the following graphs display constraint sets (area on and below the curves) that are convex? Circle all the correct answers: $\boxed{\text{a}}$, b, $\boxed{\text{c}}$, d.



6. (10 points). For the following Matlab commands, write in words what they do.

(a) `syms x y`

Declaring symbolic variables x and y.

(b) `diff(f(x,y),x)`

Partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial}{\partial x} f(x, y)$$

(c) `diff(f(x,y),y)`

Partial derivative of $f(x, y)$ with respect to y:

$$\frac{\partial}{\partial y} f(x, y)$$

(d) `solve('x^2 = 16',x)`

Solving the equation $x^2 = 16$ for the unknown x. Matlab will give the solution

```
ans =
-4
 4
```

(e) `t = linspace(-2,3,6)`

Creating vector t with 6 equally spaced values from -2 to 3. In other words, `t = [-2,-1,0,1,2,3]`.