بسم الله الرحمن الرحیم

مدل‌ها در ریاضیات مالی

Financial models

گردآورندگان:

مسعود (دریا)خسروتاش، نجمه قندالی، طیبه تاجیک

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مقدمه

سال‌ها پیش وقتی اروپا غرق در تاریکی بود، ما خیام‌ها، بوعلی‌ها، ابوریحان‌ها و ... داشتیم که متاسفانه به خاطر فرهنگ شفاهی‌مان بسیاری از کشفیات و اثبات‌ها و مطالب زیبای این دوره به نام اروپاییان ثبت شده است. حالیا چاره نیست مگر تن دادن به زبانی که امروزه علم را پیش می‌راند و می‌تازد. به امید روزی که دگر بار ایرانی بر چهره علم سایه افکند و افتخارات گذشته را بازآفرینی کند. بیایید از این نقطه آغاز کنیم حرکت به سمت ایرانی آباد و علمی را.

در این کتاب به معرفی مدل‌های ریاضیات مالی پرداخته شده است. کسانی که در رشته ریاضیات مالی کار می‌کنند، معمولاً در مطالعه کتب و مقالات به معادلات دیفرانسیل و یا مدل‌های معروفی بر می‌خورند که دانستن ویژگی‌های اولیه و چند خاصیت آنها باعث تسریع امر مطالعه و پژوهش می‌گردد. به طور معمول جمع کردن این اطلاعات برای یک نفر در زمان نیاز داشتن به آن چیزی شبیه به غیر ممکن است. اما سعی در این کتاب بر این بوده که نیازهای اولیه پژوهشگران را مرتفع سازد. ممکن است این سوال پیش آید که چرا ترجمه فارسی را نیاورده‌ایم. دلیل این امر این است که معمولاً کسانی که با این جملات و عبارات درگیر می‌شوند، این تعداد کلمه جزو دایره لغات آنها خواهد بود و اکثراً استفاده کنندگان این مطالب در مقاطع ارشد و دکتری خواهند بود.

بد نیست تا در اینجا به جایگاه ریاضیات مالی نیز اشاره کنیم. ریاضیات مالی شاخه‌ای از ریاضیات کاربردی و مرتبط با بازارهای مالی است. موضوع آن رابطه نزدیکی با مبانی اقتصاد مالی دارد که با بسیاری از تئوری‌ها در ارتباط است. به طور کلی ریاضیات مالی، مدل‌های ریاضی و عددی‌ای را که اقتصاد مالی پیشنهاد داده است، توسعه می‌دهد (به صورت خودمانی‌تر کاربردی می‌سازد). برای مثال، در حالی که یک اقتصاددان مالی بر روی دلایل ساختاری قیمت سهام یک شرکت مطالعه می‌کند، یک ریاضیدان مالی قیمت سهام را به عنوان داده می‌گیرد و سعی می‌کند با استفاده از حساب دیفرانسیل و انتگرال تصادفی (stochastic calculus) ارزش واقعی مشتقات آن سهام را به دست آورد. گاهی این طور به نظر می‌رسد که ریاضیات چه تاثیری می‌تواند بر داد و ستد، بورس، بازار، سود، زیان و ... داشته باشد. اما پاسخ صحیح این است که در واقع ابزارهای ریاضی می‌توانند به پیش‌بینی درست و تحلیل درست وقایع مالی کمک کنند. می‌توان گفت اکثر جوامع پیشرفته در مسایل مالی در طی این سال‌ها به جایگاه ریاضیات در مسایل مالی پی برده‌اند.

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Introduction

Mathematical finance, also known as **quantitative finance**, is a field of applied mathematics, concerned with financial markets. Generally, mathematical finance will derive and extend the mathematical or numerical models without necessarily establishing a link to financial theory, taking observed market prices as input. Mathematical consistency is required, not compatibility with economic theory. Thus, for example, while a financial economist might study the structural reasons why a company may have a certain share price, a financial mathematician may take the share price as a given, and attempt to use stochastic calculus to obtain the corresponding value of derivatives of the stock (*see: Valuation of options; Financial modeling*). The fundamental theorem of arbitrage-free pricing is one of the key theorems in mathematical finance, while the Black–Scholes equation and formula are amongst the key results.

Mathematical finance also overlaps heavily with the fields of computational finance and financial engineering. The latter focuses on applications and modeling, often by help of stochastic asset models (*see: Quantitative analyst*), while the former focuses, in addition to analysis, on building tools of implementation for the models. In general, there exist two separate branches of finance that require advanced quantitative techniques: derivatives pricing on the one hand, and risk- and portfolio management on the other.

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Black–Derman–Toy model (BDT)

In mathematical finance, the **Black–Derman–Toy model** (**BDT**) is a popular short rate model used in the pricing of bond options, swaptions and other interest rate derivatives; see Lattice model (finance) #Interest rate derivatives. It is a one-factor model; that is, a single stochastic factor – the short rate – determines the future evolution of all interest rates. It was the first model to combine the mean-reverting behaviour of the short rate with the lognormal distribution, and is still widely used.

The model was introduced by Fischer Black, Emanuel Derman, and Bill Toy. It was first developed for in-house use by Goldman Sachs in the 1980 s and was published in the *Financial Analysts Journal* in 1990. A personal account of the development of the model is provided in one of the chapters in Emanuel Derman's memoir "My Life as a Quant."

Under BDT, using a binomial lattice, one calibrates the model parameters to fit both the current term structure of interest rates (yield curve), and the volatility structure for interest rate caps (usually as implied by the Black-76-prices for each component caplet); see aside. Using the calibrated lattice one can then value a variety of more complex interest-rate sensitive securities and interest rate derivatives.

Although initially developed for a lattice-based environment, the model has been shown to imply the following continuous stochastic differential equation:

$$d\ln(r) = [\theta_t + \frac{\sigma'_t}{\sigma_t}\ln(r)]dt + \sigma_t dW_t$$

r_t = the instantaneous short rate at time

θ_t = value of the underlying asset at option expiry

σ_t = instant short rate volatility

w_t = a standard Brownian motion under a risk-neutral probability measure dw_t its differential.

For constant (time independent) short rate volatility, σ, the model is:

$$d\ln(r) = \theta_t\, dt + \sigma dW_t$$

One reason that the model remains popular, is that the "standard" Root-finding algorithms – such as Newton's method (the secant method) or bisection – are very easily applied to the calibration. Relatedly, the model was originally described in algorithmic language, and not using stochastic calculus or martingales.

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Black–Karasinski model

In financial mathematics, the **Black–Karasinski model** is a mathematical model of the term structure of interest rates; see short rate model. It is a one-factor model as it describes interest rate movements as driven by a single source of randomness. It belongs to the class of no-arbitrage models, i.e. it can fit today's zero-coupon bond prices, and in its most general form, today's prices for a set of caps, floors or European swaptions. The model was introduced by Fischer Black and Piotr Karasinski in 1991.

The main state variable of the model is the short rate, which is assumed to follow the stochastic differential equation (under the risk-neutral measure):

$$d\ln(r) = [\theta_t - \phi_t \ln(r)]dt + \sigma_t dW_t$$

where dW_t is a standard Brownian motion. The model implies a log-normal distribution for the short rate and therefore the expected value of the money-market account is infinite for any maturity.

In the original article by Fischer Black and Piotr Karasinski the model was implemented using a binomial tree with variable spacing, but a trinomial treeimplementation is more common in practice, typically a lognormal application of the Hull-White Lattice.

The model is used mainly for the pricing of exotic interest rate derivatives such as American and Bermudan bond options and swaptions, once its parameters have been calibrated to the current term structure of interest rates and to the prices or implied volatilities of caps, floors or European swaptions. Numerical methods(usually trees) are used in the calibration stage as well as for pricing.

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Black–Scholes

The **Black–Scholes** or **Black–Scholes–Merton** model is a mathematical model of a financial market containing derivative investment instruments. From the model, one can deduce the **Black–Scholes formula**, which gives a theoretical estimate of the price of European-style options. The formula led to a boom in options trading and legitimised scientifically the activities of the Chicago Board Options Exchange and other options markets around the world. It is widely used, although often with adjustments and corrections, by options market participants. Many empirical tests have shown that the Black–Scholes price is "fairly close" to the observed prices, although there are well-known discrepancies such as the "option smile".

The Black–Scholes model was first published by Fischer Black and Myron Scholes in their 1973 paper, "The Pricing of Options and Corporate Liabilities", published in the *Journal of Political Economy*. They derived a partial differential equation, now called the Black–Scholes equation, which estimates the price of the option over time. The key idea behind the model is to hedge the option by buying and selling the underlying asset in just the right way and, as a consequence, to eliminate risk. This type of hedging is called delta hedging and is the basis of more complicated hedging strategies such as those engaged in by investment banks and hedge funds.

Robert C. Merton was the first to publish a paper expanding the mathematical understanding of the options pricing model, and coined the term "Black–Scholesoptions pricing model". Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work. Though ineligible for the prize because of his death in 1995, Black was mentioned as a contributor by the Swedish Academy.

The model's assumptions have been relaxed and generalized in many directions, leading to a plethora of models that are currently used in derivative pricing andrisk management. It is the insights of the model, as exemplified in the Black-Scholes formula, that are frequently used by market participants, as distinguished from the actual prices. These insights include no-arbitrage bounds and risk-neutral pricing. The Black-Scholes equation, a partial differential equation that governs the price of the option, is also important as it enables pricing when an explicit formula is not possible.

The Black–Scholes formula has only one parameter that cannot be observed in the market: the average future volatility of the underlying asset. Since the formula is increasing in this parameter, it can be inverted to produce a "volatility surface" that is then used to calibrate other models, e.g. for OTC derivatives.

S , be the price of the stock, which will sometimes be a random variable and other times a constant (context should make this clear).

$v(S,t)$, the price of a derivative as a function of time and stock price.

$C(S,t)$ the price of a European call option and $P(S,t)$ the price of a European put option.

K, the strike price of the option.

r, the annualized risk-free interest rate, continuously compounded (the force of interest).

μ, the drift rate of S, annualized.

σ, the standard deviation of the stock's returns; this is the square root of the quadratic variation of the stock's log price process.

t, a time in years; we generally use: now=0, expiry=T.

$\prod$, the value of a portfolio.

Finally we will use $N(x)$ to denote the standard normal cumulative distribution function

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{z^2}{2}}\,dz$$

$$N'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Black–Scholes equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

The Black–Scholes formula calculates the price of European put and call options. This price is consistent with the Black–Scholes equation as above; this follows since the formula can be obtained by solving the equation for the corresponding terminal and boundary conditions.

The value of a call option for a non-dividend-paying underlying stock in terms of the Black–Scholes parameters is:

$$
\begin{aligned}
C(S,t) &= N(d_1)S - N(d_2)Ke^{-r(T-t)} \\
d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)\right] \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

The price of a corresponding put option based on put–call parity is:

$$
\begin{aligned}
P(S,t) &= Ke^{-r(T-t)} - S + C(S,t) \\
&= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S
\end{aligned}
$$

$N(\cdot)$ is the cumulative distribution function of the standard normal distribution

$T-t$ is the time to maturity

S is the spot price of the underlying asset

K is the strike price

r is the risk free rate (annual rate, expressed in terms of continuous compounding)

σ is the volatility of returns of the underlying asset

"The Greeks" measure the sensitivity of the value of a derivative or a portfolio to changes in parameter value(s) while holding the other parameters fixed. They arepartial derivatives of the price with respect to the parameter values. One Greek, "gamma" (as well as others not listed here) is a partial derivative of another Greek, "delta" in this case.

The Greeks are important not only in the mathematical theory of finance, but also for those actively trading. Financial institutions will typically set (risk) limit values for each of the Greeks that their traders must not exceed. Delta is the most important Greek since this usually confers the largest risk. Many traders will zero their delta at the end of the day if they are speculating and following a delta-neutral hedging approach as defined by Black–Scholes.

The Greeks for Black–Scholes are given in closed form below. They can be obtained by differentiation of the Black–Scholes formula.

		Calls	**Puts**
Delta	$\frac{\partial C}{\partial S}$	$N(d_1)$	$-N(-d_1)=N(d_1)-1$
Gamma	$\frac{\partial^2 C}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	
Vega	$\frac{\partial C}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	
Theta	$\frac{\partial C}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}}$ $-rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}}$ $+rKe^{-r(T-t)}N(-d_2)$
Rho	$\frac{\partial C}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

Note that from the formulae, it is clear that the gamma is the same value for calls and puts and so too is the vega the same value for calls and put options. This can be seen directly from put–call parity, since the difference of a put and a call is a forward, which is linear in *S* and independent of *σ* (so a forward has zero gamma and zero vega). N' is the standard normal probability density function.

In practice, some sensitivities are usually quoted in scaled-down terms, to match the scale of likely changes in the parameters. For example, rho is often reported divided by 10,000 (1 basis point rate change), vega by 100 (1 vol point change), and theta by 365 or 252 (1 day decay based on either calendar days or trading days per year).

(Vega is not a letter in the Greek alphabet; the name arises from reading the Greek letter ν (nu) as a V.)

American option

The problem of finding the price of an American option is related to the optimal stopping problem of finding the time to execute the option. Since the American option can be exercised at any time before the expiration date, the Black–Scholes equation becomes an inequality of the form

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV\le 0$$

with the terminal and (free) boundary conditions:

$V(S,T)=H(S)$ and

$V(S,t)\ge H(S)$

Where

$H(S)$ denotes the payoff at stock price S .

In general this inequality does not have a closed form solution, though an American call with no dividends is equal to a European call and the Roll-Geske-Whaley method provides a solution for an American call with one dividend.

Barone-Adesi and Whaley is a further approximation formula. Here, the stochastic differential equation (which is valid for the value of any derivative) is split into two components: the European option value and the early exercise premium. With some assumptions, a quadratic equation that approximates the solution for the latter is then obtained. This solution involves finding the critical value, $S*$, such that one is indifferent between early exercise and holding to maturity.

Bjerksund and Stensland provide an approximation based on an exercise strategy corresponding to a trigger price. Here, if the underlying asset price is greater than or equal to the trigger price it is optimal to exercise, and the value must equal $S - X$, otherwise the option "boils down to: (i) a European up-and-out call option… and (ii) a rebate that is received at the knock-out date if the option is knocked out prior to the maturity date." The formula is readily modified for the valuation of a put option, using put call parity. This approximation is computationally inexpensive and the method is fast, with evidence indicating that the approximation may be more accurate in pricing long dated options than Barone-Adesi and Whaley.

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Chen model

In finance, the **Chen model** is a mathematical model describing the evolution of interest rates. It is a type of "three-factor model" (short rate model) as it describes interest rate movements as driven by three sources of market risk. It was the first stochastic mean and stochastic volatility model and it was published in 1994 by Lin Chen, financial economist and environmental economist, professor of American University, Yonsei University and Nanyang Tech University of Singapore.

The dynamics of the instantaneous interest rate are specified by the stochastic differential equations:

$$dr_t = (\theta_t - \alpha_t)dt + \sqrt{r_t}\,\sigma_t\,dW_t$$

$$d\alpha_t = (\zeta_t - \alpha_t)dt + \sqrt{\alpha_t}\,\sigma_t\,dW_t$$

$$d\sigma_t = (\beta_t - \sigma_t)dt + \sqrt{\sigma_t}\,\eta_t\,dW_t$$

In an authoritative review of modern finance (*Continuous-Time Methods in Finance: A Review and an Assessment*), Chen model is listed along with the models ofRobert C. Merton, Oldrich Vasicek, John C. Cox, Stephen A. Ross, Darrell Duffie, John Hull, Robert A. Jarrow, and Emanuel Derman as a major term structure model.

Different variants of Chen model are still being used in financial institutions worldwide. James and Webber devote a section to discuss Chen model in their book; Gibson et al. devote a section to cover Chen model in their review article. Andersen et al. devote a paper to study and extend Chen model. Gallant et al. devote a paper to test Chen model and other models; Wibowo and Cai, among some others, devote their PhD dissertations to testing Chen model and other competing interest rate models.

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constant elasticity of variance model

In mathematical finance, the **CEV** or constant elasticity of variance model is a stochastic volatility model, which attempts to capture stochastic volatility and the leverage effect. The model is widely used by practitioners in the financial industry, especially for modelling equities and commodities. It was developed by John Cox in 1975

The **CEV** model describes a process which evolves according to the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t^{\gamma} dW_t$$

$\sigma \geq 0, \gamma \geq 0$

The parameter γ controls the relationship between volatility and price, and is the central feature of the model. When $\gamma < 1$ we see the so-called leverage effect, commonly observed in equity markets, where the volatility of a stock increases as its price falls. Conversely, in commodity markets, we often observe $\gamma > 1$, the so-called inverse leverage effect, whereby the volatility of the price of a commodity tends to increase as its price increases.

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Cox–Ingersoll–Ross model

In mathematical finance, the **Cox–Ingersoll–Ross model** (or **CIR model**) describes the evolution of interest rates. It is a type of "one factor model" (short rate model) as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives. It was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model.

The CIR model specifies that the instantaneous interest rate follows the stochastic differential equation, also named the CIR Process:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}\,dW_t$$

where W_t is a Wiener process (modelling the random market risk factor) and a, b, and σ are the parameters. The parameter a corresponds to the speed of adjustment, b to the mean and σ to volatility. The drift factor, $a(b - r_t)$, is exactly the same as in the Vasicek model. It ensures mean reversion of the interest rate towards the long run value b, with speed of adjustment governed by the strictly positive parameter a.

The standard deviation factor, $\sigma\sqrt{r_t}$, avoids the possibility of negative interest rates for all positive values of a and b. An interest rate of zero is also precluded if the condition

$$2ab \geq \sigma^2$$

is met. More generally, when the rate is at a low level (close to zero), the standard deviation also becomes very small, which dampens the effect of the random shock on the rate. Consequently, when the rate gets close to zero, its evolution becomes dominated by the drift factor, which pushes the rate upwards (towardsequilibrium).

This process can be defined as a sum of squared Ornstein–Uhlenbeck process. The CIR is an ergodic process, and possesses a stationary distribution. The same process is used in the Heston model to model stochastic volatility.

Properties

Mean reversion

(In finance, **mean reversion** is the assumption that a stock's price will tend to move to the average price over time)

Level dependent volatility $\sigma\sqrt{r_t}$

For given positive r_0 the process will never touch zero, if $2ab \geq \sigma^2$; otherwise it can occasionally touch the zero point

A solution to the CIR stochastic differential equation is given by:

$$r_t = \theta + (r_0 - \theta)e^{-\kappa t} + \sigma e^{-\kappa t}\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u.$$

$$E[r_t \mid r_0] = r_0 e^{-at} + b(1 - e^{-at})$$

$$Var[r_t \mid r_0] = r_0\frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \frac{b\sigma^2}{2a}(1 - e^{-at})^2$$

Bond pricing

Under the no-arbitrage assumption, a bond may be priced using this interest rate process. The bond price is exponential affine in the interest rate:

$$P(t,T) = A(t,T)\exp(-B(t,T)r_t)$$

Where

$$A(t,T) = \left(\frac{2h\exp((a+h)(T-t)/2)}{2h + (a+h)(\exp((T-t)h) - 1)}\right)^{2ab/\sigma^2}$$

$$B(t,T) = \frac{2(\exp((T-t)h) - 1)}{2h + (a+h)(\exp((T-t)h) - 1)}$$

$$h = \sqrt{a^2 + 2\sigma^2}$$

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Garman–Kohlhagen model (foreign exchange option)

In finance, a **foreign exchange option** (commonly shortened to just **FX option** or **currency option**) is a derivativefinancial instrument that gives the right but not the obligation to exchange money denominated in one currency into another currency at a pre-agreed exchange rate on a specified date.

The foreign exchange options market is the deepest, largest and most liquid market for options of any kind. Most trading is over the counter (OTC) and is lightly regulated, but a fraction is traded on exchanges like the International Securities Exchange, Philadelphia Stock Exchange, or the Chicago Mercantile Exchange for options on futures contracts. The global market for exchange-traded currency options was notionally valued by the Bank for International Settlements at $158.3 trillion in 2005.

As in the Black–Scholes model for stock options and the Black model for certain interest rate options, the value of a European option on an FX rate is typically calculated by assuming that the rate follows a log-normal process.

In 1983 Garman and Kohlhagen extended the Black–Scholes model to cope with the presence of two interest rates (one for each currency). Suppose that r_d is therisk-free interest rate to expiry of the domestic currency and r_f is the foreign currency risk-free interest rate (where domestic currency is the currency in which we obtain the value of the option; the formula also requires that FX rates – both strike and current spot be quoted in terms of "units of domestic currency per unit of foreign currency"). The results are also in the same units and to be meaningful need to be converted into one of the currencies.

Then the domestic currency value of a call option into the foreign currency is

$$c = S_0 e^{-r_f T} \mathcal{N}(d_1) - K e^{-r_d T} \mathcal{N}(d_2)$$

The value of a put option has value

$$p = K e^{-r_d T} \mathcal{N}(-d_2) - S_0 e^{-r_f T} \mathcal{N}(-d_1)$$

Where

$$d_1 = \frac{\ln(S_0 / K) + (r_d - r_f + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S_0 is the current spot rate

K is the strike price

$\mathcal{N}(x)$ is the cumulative normal distribution function

r_d is domestic risk free simple interest rate

r_f is foreign risk free simple interest rate

T is the time to maturity (calculated according to the appropriate day count convention)

and σ is the volatility of the FX rate.

Risk management

A wide range of techniques are in use for calculating the options risk exposure, or Greeks (as for example the Vanna-Volga method). Although the option prices produced by every model agree (with Garman–Kohlhagen), risk numbers can vary significantly depending on the assumptions used for the properties of spot price movements, volatility surface and interest rate curves.

After Garman–Kohlhagen, the most common models are SABR and local volatility , although when agreeing risk numbers with a counterparty (e.g. for exchanging delta, or calculating the strike on a 25 delta option) Garman–Kohlhagen is always used.

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Heath–Jarrow–Morton (HJM) framework

The **Heath–Jarrow–Morton (HJM) framework** is a general framework to model the evolution of interest rate curve – instantaneous forward rate curve in particular (as opposed to simple forward rates). When the volatility and drift of the instantaneous forward rate are assumed to be deterministic, this is known as the **Gaussian Heath–Jarrow–Morton (HJM) model** of forward rates. The HJM framework originates from the work of David Heath, Robert A. Jarrow, and Andrew Morton in the late 1980s, especially *Bond pricing and the term structure of interest rates: a new methodology* (1987) – working paper, Cornell University, and *Bond pricing and the term structure of interest rates: a new methodology* (1989) – working paper (revised ed.), Cornell University. It has its critics, however, with Paul Wilmott describing it as "...actually just a big rug for [mistakes] to be swept under".

Framework

The key to these techniques is the recognition that the drifts of the no-arbitrage evolution of certain variables can be expressed as functions of their volatilities and the correlations among themselves. In other words, no drift estimation is needed.

Models developed according to the HJM framework are different from the so-called short-rate models in the sense that HJM-type models capture the full dynamics of the entire forward rate curve, while the short-rate models only capture the dynamics of a point on the curve (the short rate).

However, models developed according to the general HJM framework are often non-Markovian and can even have infinite dimensions. A number of researchers have made great contributions to tackle this problem. They show that if the volatility structure of the forward rates satisfy certain conditions, then an HJM model can be expressed entirely by a finite state Markovian system, making it computationally feasible. Examples include a one-factor, two state model (O. Cheyette, "Term Structure Dynamics and Mortgage Valuation", *Journal of Fixed Income,* 1, 1992; P. Ritchken and L. Sankarasubramanian in "Volatility Structures of Forward Rates and the Dynamics of Term Structure", *Mathematical Finance*, 5, No. 1, Jan 1995), and later multi-factor versions.

Mathematical formulation

The class of models developed by Heath, Jarrow and Morton (1992) is based on modelling the forward rates, yet it does not capture all of the complexities of an evolving term structure.

The model begins by introducing the instantaneous forward rate $f(t,T)$, $t \le T$, which is defined as the continuous compounding rate available at time T as seen from time t. The relation between bond prices and the forward rate is also provided in the following way:

$$p(t,T) = e^{-\int_t^T f(t,s)ds}$$

Here $P(t,T)$ is the price at time t of a zero-coupon bond maturing at time $T \ge t$. The risk-free money market account is also defined as

$$\beta(t) = e^{\int_0^t f(u,u)du}$$

This last equation lets us define $f(t,t) \triangleq r(t)$, the risk free short rate. The HJM framework assumes that the dynamics of $f(t,s)$ under a risk-neutral pricing measure $\mathbb{Q}$ are the following:

$$df(t,s) = \mu(t,s)dt + \mathbf{\Sigma}(t,s)dW_t$$

Let's define the following process:

$$Y_t \triangleq \log P(t,s) = -\int_t^s f(t,u)du$$

The dynamics of Y_t can be obtained through Leibniz's rule:

$$\begin{aligned} dY_t &= f(t,t)dt - \int_t^s d\,f(t,u)du \\ &= r_t dt - \int_t^s \mu(t,u)dt + \mathbf{\Sigma}(t,u)dW_t du \end{aligned}$$

If we define $\mu(t,s)^* = \int_t^s \mu(t,u)du$, $\mathbf{\Sigma}(t,s)^* = \int_t^s \mathbf{\Sigma}(t,u)du$ and assume that the conditions for Fubini's Theorem are satisfied in the formula for the dynamics of Y_t , we get:

$$dY_t = \left(r_t - \mu(t,s)^*\right)dt - \mathbf{\Sigma}(t,s)^* dW_t$$

By Itō's lemma, the dynamics of $P(t,T)$ are then

$$\frac{dP(t,s)}{P(t,s)} = \left(r_t - \mu(t,s)^* + \frac{1}{2}\mathbf{\Sigma}(t,s)^*\mathbf{\Sigma}(t,s)^{*T}\right)dt - \mathbf{\Sigma}(t,s)^* dW_t$$

But $\dfrac{P(t,s)}{\beta(t)}$ must be a martingale under the pricing measure $\mathbb{Q}$, so we require that $\mu(t,s)^* = \dfrac{1}{2}\mathbf{\Sigma}(t,s)^*\mathbf{\Sigma}(t,s)^{*T}$. Differentiating this with respect to S we get:

$$\mu(t,u) = \mathbf{\Sigma}(t,u)\int_t^u \mathbf{\Sigma}(t,s)^T ds$$

Which finally tells us that the dynamics of f must be of the following form:

$$df(t,u) = \left(\mathbf{\Sigma}(t,u)\int_t^u \mathbf{\Sigma}(t,s)^T ds\right)dt + \mathbf{\Sigma}(t,u)dW_t$$

Which allows us to price bonds and interest rate derivatives based on our choice of Σ.

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Heston model

In finance, the **Heston model**, named after Steven Heston, is a mathematical model describing the evolution of the volatility of an underlying asset. It is astochastic volatility model: such a model assumes that the volatility of the asset is not constant, nor even deterministic, but follows a random process.

Basic heston model

The basic Heston model assumes that S_t, the price of the asset, is determined by a stochastic process:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S$$

where v_t, the instantaneous variance, is a CIR process:

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}\, dW_t^v$$

and dW_t^S, dW_t^v are Wiener processes (i.e., random walks) with correlation ρ, or equivalently, with covariance ρ dt.

The parameters in the above equations represent the following:

- μ is the rate of return of the asset.
- θ is the **long variance**, or long run average price variance; as *t* tends to infinity, the expected value of v_t tends to θ.
- κ is the rate at which v_t reverts to θ.
- ξ is the volatility of the volatility, or **vol of vol**, and determines the variance of v_t.

If the parameters obey the following condition (known as the Feller condition) then the process ν_t is strictly positive

$$2\kappa\theta > \xi^2.$$

In order to take into account all the features from the volatility surface, the Heston model may be a too rigid framework It may be necessary to add degrees of freedom to the original model. A first straightforward extension is to allow the parameters to be time-dependent. The model dynamics are then written as:

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S.$$

$$d\nu_t = \kappa_t(\theta_t - \nu_t)dt + \xi_t \sqrt{\nu_t} dW_t^\nu$$

Another approach is to add a second process of variance, independent of the first one.

$$dS_t = \mu S_t dt + \sqrt{\nu_t^1} S_t dW_t^{S,1} + \sqrt{\nu_t^2} S_t dW_t^{S,2}$$

$$d\nu_t^1 = \kappa^1(\theta^1 - \nu_t^1)dt + \xi^1 \sqrt{\nu_t^1} dW_t^{\nu^1}$$

$$d\nu_t^2 = \kappa^2(\theta^2 - \nu_t^2)dt + \xi^2 \sqrt{\nu_t^2} dW_t^{\nu^2}$$

A significant extension of Heston model to make both volatility and mean stochastic is given by Lin Chen (1996). In the Chen model the dynamics of the instantaneous interest rate are specified by

$$dr_t = (\theta_t - \alpha_t)dt + \sqrt{r_t}\,\sigma_t\, dW_t$$
$$d\alpha_t = (\zeta_t - \alpha_t)dt + \sqrt{\alpha_t}\,\sigma_t\, dW_t$$
$$d\sigma_t = (\beta_t - \sigma_t)dt + \sqrt{\sigma_t}\,\eta_t\, dW_t.$$

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Ho–Lee model

In financial mathematics, the **Ho–Lee model** is a short rate model widely used in the pricing of bond options, swaptions and other interest rate derivatives, and in modeling future interest rates. It was developed in 1986 by Thomas Ho and Sang Bin Lee. It was the first arbitrage free model of interest rates.

Under this model, the short rate follows a normal process:

$$dr_t = \theta_t \, dt + \sigma \, dW_t$$

The model can be calibrated to market data by implying the form of θ_t from market prices, meaning that it can exactly return the price of bonds comprising the yield curve. This calibration, and subsequent valuation of bond options, swaptions and other interest rate derivatives, is typically performed via a binomial lattice based model. Closed form valuations of bonds, and "Black-like" bond option formulae are also available.

As the model generates a symmetric ("bell shaped") distribution of rates in the future, negative rates are possible. Further, it does not incorporate mean reversion. For both of these reasons, models such as Black–Derman–Toy (lognormal and mean reverting) and Hull–White (mean reverting with lognormal variant available) are often preferred. The Kalotay–Williams–Fabozzi model is a lognormal analogue to the Ho–Lee model, although is less widely used than the latter two.

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Hull–White model

In financial mathematics, the **Hull–White model** is a model of future interest rates. In its most generic formulation, it belongs to the class of no-arbitrage models that are able to fit today's term structure of interest rates. It is relatively straightforward to translate the mathematical description of the evolution of future interest rates onto a tree or lattice and so interest rate derivatives such as bermudan swaptions can be valued in the model.

The first Hull–White model was described by John C. Hull and Alan White in 1990. The model is still popular in the market today.

One-factor model

The model is a short-rate model. In general, it has dynamics

$$dr(t) = \left[\theta(t) - \alpha(t) r(t)\right] dt + \sigma(t) dW(t)$$

There is a degree of ambiguity amongst practitioners about exactly which parameters in the model are time-dependent or what name to apply to the model in each case. The most commonly accepted hierarchy has

θ and α constant – **the Vasicek model**

θ has t dependence – **the Hull-White model**

θ and α also time-dependent – **the extended Vasicek model**

two-factor model

The two-factor Hull–White model (Hull 2006 :657–658) contains an additional disturbance term whose mean reverts to zero, and is of the form:

$$d\,f(r(t)) = \left[\theta(t) + u - \alpha(t)\,f(r(t))\right]dt + \sigma_1(t)\,dW_1(t)$$

where u has an initial value of ٠ and follows the process:

$$du = -bu\,dt + \sigma_2\,dW_2(t)$$

Analysis of one-factor model

For the rest of this article we assume only θ has t-dependence. Neglecting the stochastic term for a moment, notice that the change in r is negative if r is currently "large" (greater than $\frac{\theta(t)}{\alpha}$) and positive if the current value is small. That is, the stochastic process is a mean-reverting Ornstein–Uhlenbeck process.

θ is calculated from the initial yield curve describing the current term structure of interest rates. Typically α is left as a user input (for example it may be estimated from historical data). σ is determined via calibration to a set of caplets and swaptions readily tradeable in the market.

When α, θ, and σ are constant, Itô's lemma can be used to prove that

$$r(t) = e^{-\alpha t} r(0) + \frac{\theta}{\alpha}\left(1 - e^{-\alpha t}\right) + \sigma e^{-\alpha t} \int_0^t e^{\alpha u}\, dW(u)$$

which has distribution

$$r(t) \sim \mathcal{N}\left(e^{-\alpha t} r(0) + \frac{\theta}{\alpha}\left(1 - e^{-\alpha t}\right), \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right) \right).$$

where $\mathcal{N}(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2.

When $\theta(t)$ is time dependent,

$$r(t) = e^{-\alpha t} r(0) + \int_0^t e^{\alpha(s-t)} \theta(s) ds + \sigma e^{-\alpha t} \int_0^t e^{\alpha u}\, dW(u)$$

which has distribution

$$r(t) \sim \mathcal{N}\left(e^{-\alpha t} r(0) + \int_0^t e^{\alpha(s-t)} \theta(s) ds, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right) \right).$$

Bond pricing using Hull-White model

It turns out that the time-*S* value of the *T*-maturity discount bond has distribution (note the *affine term* structure here!)

$$P(S,T) = A(S,T) \exp(-B(S,T) r(S))$$

Where

$$B(S,T) = \frac{1 - \exp(-\alpha(T-S))}{\alpha}$$

$$A(S,T)=\frac{P(0,T)}{P(0,S)}\exp\left(\begin{array}{l}-B(S,T)\dfrac{\partial\log(P(0,S))}{\partial S}-\\ \dfrac{\sigma^2(\exp(-\alpha T)-\exp(-\alpha S))^2(\exp(2\alpha S)-1)}{4\alpha^3}\end{array}\right)$$

Note that their terminal distribution for $P(S,T)$ is distributed log-normally.

Derivitative pricing

By selecting as numeraire the time-S bond (which corresponds to switching to the S-forward measure), we have from the fundamental theorem of arbitrage-free pricing, the value at time ٠ of a derivative which has payoff at time S.

$$V(t)=P(t,S)\mathbb{E}_S[V(S)\,|\,\mathcal{F}(t)].$$

Here, $\mathbb{E}_S$ is the expectation taken with respect to the forward measure. Moreover that standard arbitrage arguments show that the time T forward price $F_V(t,T)$ for a payoff at time T given by $V(T)$ must satisfy $F_V(t,T)=V(t)/P(t,T)$, thus

$$F_V(t,T)=\mathbb{E}_T[V(T)\,|\,\mathcal{F}(t)].$$

Thus it is possible to value many derivatives V dependent solely on a single bond $P(S,T)$ analytically when working in the Hull–White model. For example in the case of a bond put

$$V(S)=(K-P(S,T))^+.$$ ٣٩

Because $P(S,T)$ is lognormally distributed, the general calculation used for Black-Scholes shows that

$$E_S[(K-P(S,T))^+]=KN(-d_2)-F(t,S,T)N(d_1)$$

Where

$$d_1 = \frac{\log(F/K) + \sigma_P^2 S/2}{\sigma_P \sqrt{S}}$$

And

$$d_2 = d_1 - \sigma_P \sqrt{S}.$$

Thus today's value (with the $P(0,S)$ multiplied back in) is:

$$P(0,S)KN(-d_2) - P(0,T)N(-d_1)$$

Here σ_P is the standard deviation of the log-normal distribution for $P(S,T)$. A fairly substantial amount of algebra shows that it is related to the original parameters via

$$\sqrt{S}\sigma_P = \frac{\sigma}{\alpha}(1 - \mathrm{e}^{(-\alpha(T-S))})\sqrt{\frac{1-\mathrm{e}^{(-2\alpha S)}}{2\alpha}}$$

Note that this expectation was done in the S-bond measure, whereas we did not specify a measure at all for the original Hull-White process. This does not matter — the volatility is all that matters and is measure-independent.

Because interest rate caps/floors are equivalent to bond puts and calls respectively, the above analysis shows that caps and floors can be priced analytically in the Hull–White model. Jamshidian's trick applies to Hull-White (as today's value of a swaption in HW is a monotonic function of today's short rate). Thus knowing how to price caps is also sufficient for pricing swaptions.

The swaptions can also be priced directly as described in Henrard (2003). The direct implementation is usually more efficient.

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Online utilities

http://lombok.demon.co.uk/financialTap/montecarlo/hullwhite

http://lombok.demon.co.uk/financialTap/interestrates/hwtrinomialtree

LIBOR market model

The **LIBOR market model**, also known as the **BGM Model** (**Brace Gatarek Musiela Model**, in reference to the names of some of the inventors) is a financial model of interest rates. It is used for pricing interest rate derivatives, especially exotic derivatives like Bermudan swaptions, ratchet caps and floors, target redemption notes, autocaps, zero coupon swaptions, constant maturity swaps and spread options, among many others. The quantities that are modeled, rather than the short rate or instantaneous forward rates (like in the Heath-Jarrow-Morton framework) are a set of forward rates (also called forward LIBORs), which have the advantage of being directly observable in the market, and whose volatilities are naturally linked to traded contracts. Each forward rate is modeled by a lognormalprocess under its forward measure, i.e. a Black model leading to a Black formula for interest rate caps. This formula is the market standard to quote cap prices in terms of implied volatilities, hence the term "market model". The LIBOR market model may be interpreted as a collection of forward LIBOR dynamics for different forward rates with spanning tenors and maturities, each forward rate being consistent with a Black interest rate caplet formula for its canonical maturity. One can write the different rates dynamics under a common pricing measure, for example the forward measure for a preferred single maturity, and in this case forward rates will not be lognormal under the unique measure in general, leading to the need for numerical methods such as monte carlo simulation or approximations like the frozen drift assumption.

Dynamic model

The LIBOR market model models a set of n forward rates $L_j, j=1,\ldots,n$ as lognormal processes. Under the respective T_j -Forward measure $Q_{T_{j+1}}$

$$dL_j(t) = \sigma_j(t) L_j(t) dW^{Q_{T_{j+1}}}(t).$$

Here, L_j denotes the forward rate for the period $[T_j, T_{j+1}]$. For each single forward rate the model corresponds to the Black model. The novelty is that, in contrast to the Black model, the LIBOR market model describes the dynamic of a whole family of forward rates under a common measure. The question now is how to switch between the different T -Forward measures. By means of the multivariate Girsanov's theorem one can show that

$$dW^{Q_{T_j}}(t)=\begin{cases} dW^{Q_{T_p}}(t)-\sum_{k=j}^{p-1}\frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_k(t)dt & j<p \\ dW^{Q_{T_p}}(t) & j=p \\ dW^{Q_{T_p}}(t)+\sum_{k=p}^{j-1}\frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_k(t)dt & j>p \end{cases}$$

$$dL_j(t)=\begin{cases} L_j(t)\sigma_j(t)dW^{Q_{T_p}}(t)-L_j(t)\sum_{k=j}^{p-1}\frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_j(t)\sigma_k(t)\rho_{jk}dt & j<p \\ L_j(t)\sigma_j(t)dW^{Q_{T_p}}(t) & j=p \\ L_j(t)\sigma_j(t)dW^{Q_{T_p}}(t)+L_j(t)\sum_{k=p}^{j-1}\frac{\delta L_k(t)}{1+\delta L_k(t)}\sigma_j(t)\sigma_k(t)\rho_{jk}dt & j>p \end{cases}$$

References

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- D. Papaioannou (2011):"Applied Multidimensional Girsanov Theorem", SSRN
- "An accompaniment to a course on interest rate modeling: with discussion of Black -76, Vasicek and HJM models and a gentle introduction to the multivariate LIBOR Market Model"

Useful links

http://christian-fries.de/finmath/applets/

http://finmath.net/topics/libormarketmodel/

Rendleman–Bartter model

The **Rendleman–Bartter model** (Richard J. Rendleman, Jr. and Brit J. Bartter) in finance is a short rate model describing the evolution of interest rates. It is a "one factor model" as it describes interest rate movements as driven by only one source of market risk. It can be used in the valuation of interest rate derivatives. It is astochastic asset model.

The model specifies that the instantaneous interest rate follows a geometric Brownian motion:

$$dr_t = \theta r_t dt + \sigma r_t dW_t$$

where W_t is a Wiener process modelling the random market risk factor. The drift parameter, θ, represents a constant expected instantaneous rate of change in the interest rate, while the standard deviation parameter, σ, determines the volatility of the interest rate.

This is one of the early models of the short-term interest rates, using the same stochastic process as the one already used to describe the dynamics of the underlying price in stock options. Its main disadvantage is that it does not capture the mean reversion of interest rates (their tendency to revert toward some value or range of values rather than wander without bounds in either direction).

References

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- *Rendleman, R. and B. Bartter (1980). "The Pricing of Options on Debt Securities". Journal of Financial and Quantitative Analysis 15: 11–24.doi:*10,2307 / 2979016

SABR model

In mathematical finance, the **SABR** model is a stochastic volatility model, which attempts to capture the volatility smile in derivatives markets. The name stands for "stochastic alpha, beta, rho", referring to the parameters of the model. The **SABR** model is widely used by practitioners in the financial industry, especially in theinterest rate derivative markets. It was developed by Patrick S. Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward.

Dynamics

The **SABR** model describes a single forward F, such as a LIBOR forward rate, a forward swap rate, or a forward stock price. The volatility of the forward F is described by a parameter σ σ. **SABR** is a dynamic model in which both F and σ are represented by stochastic state variables whose time evolution is given by the following system of stochastic differential equations:

$$dF_t = \sigma_t F_t^{\beta} dW_t$$

$$d\sigma_t = \alpha\sigma_t \, dZ_t,$$

with the prescribed time zero (currently observed) values F_0 and σ_0. Here, W_t and Z_t are two correlated Wiener processes with correlation coefficient $-1 < \rho < 1$:

$$dW_t dZ_t = \rho dt$$

The constant parameters satisfy the conditions $0 \le \beta \le 1, \alpha \ge 0$

Asymptotic solution

We consider a European option (say, a call) on the forward F struck at K, which expires T years from now. The value of this option is equal to the suitably discounted expected value of the payoff $\max(F_T - K, 0)$ under the probability distribution of the process F_t.

Except for the special cases of $\beta = 0, \beta = 1$ no closed form expression for this probability distribution is known. The general case can be solved approximately by means of an asymptotic expansion in the parameter $\varepsilon = T\alpha^2$. Under typical market conditions, this parameter is small and the approximate solution is actually quite accurate. Also significantly, this solution has a rather simple functional form, is very easy to implement in computer code, and lends itself well to risk management of large portfolios of options in real time.

It is convenient to express the solution in terms of the implied volatility of the option. Namely, we force the SABR model price of the option into the form of the Black model valuation formula. Then the implied volatility, which is the value of the lognormal volatility parameter in Black's model that forces it to match the SABR price, is approximately given by:

$$\sigma_{\text{impl}} = \alpha \frac{\log(F_0/K)}{D(\zeta)} \left\{ 1 + \left[\begin{array}{l} \frac{2\gamma_2 - \gamma_1^2 + 1/F_{\text{mid}}^2}{24} \left(\frac{\sigma_0 C(F_{\text{mid}})}{\alpha} \right)^2 + \\ \frac{\rho\gamma_1}{4} \frac{\sigma_0 C(F_{\text{mid}})}{\alpha} + \frac{2 - 3\rho^2}{24} \end{array} \right] \varepsilon \right\},$$

where, for clarity, we have set $C(F)=F^{\beta}$ The value F_{mid} denotes a conveniently chosen midpoint between F_0, *and* K (such as the geometric average $\sqrt{F_0 K}$ or the arithmetic average $\frac{(F_0+K)}{2}$). We have also set

$$\zeta=\frac{\alpha}{\sigma_0}\int_K^{F_0}\frac{dx}{C(x)}=\frac{\alpha}{\sigma_0(1-\beta)}\left(F_0^{1-\beta}-K^{1-\beta}\right),$$

$$\gamma_1=\frac{C'(F_{\text{mid}})}{C(F_{\text{mid}})}=\frac{\beta}{F_{\text{mid}}}$$

$$\gamma_2=\frac{C''(F_{\text{mid}})}{C(F_{\text{mid}})}=-\frac{\beta(1-\beta)}{F_{\text{mid}}^2}.$$

The function $D(\zeta)$ entering the formula above is given by

$$D(\zeta)=\log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right).$$

Alternatively, one can express the SABR price in terms of the normal Black's model. Then the implied normal volatility can be asymptotically computed by means of the following expression

$$\sigma_{\text{impl}}^{\text{n}}=\alpha\frac{F_0-K}{D(\zeta)}\left\{1+\left[\begin{array}{c}\frac{2\gamma_2-\gamma_1^2}{24}\left(\frac{\sigma_0 C(F_{\text{mid}})}{\alpha}\right)^2+\\ \frac{\rho\gamma_1}{4}\frac{\sigma_0 C(F_{\text{mid}})}{\alpha}+\frac{2-3\rho^2}{24}\end{array}\right]\varepsilon\right\}.$$

It is worth noting that the normal SABR implied volatility is generally somewhat more accurate than the lognormal implied volatility.

SABR for Negative the interest rates

A **SABR** model extension for Negative interest rates that has gained popularity in the recent years is the shifted SABR model, where shifted forward rate is assumed to follow a SABR process

$$dF_t = \sigma_t (F_t + s)^{\beta} dW_t$$
$$d\sigma_t = \alpha \sigma_t dZ_t,$$

for some positive shift s. Since shifts are included in a market quotes, and there is an intuitive soft boundary for how negative rates can become, shifted SABR has become market best practice to accommodate negative rates.

The **SABR** model can also be modified to cover Negative interest rates by:

$$dF_t = \sigma_t |F_t|^{\beta} dW_t$$
$$d\sigma_t = \alpha \sigma_t dZ_t$$

for $0 \le \beta \le 1/2$ and a **free** boundary condition for $F = 0$. Its exact solution for the zero correlation as well as an efficient approximation for a general case are available.

An obvious drawback of this approach is the a priori assumption of potential highly negative interest rates via the free boundary.

References

- Managing Smile Risk, P. Hagan et al. – The original paper introducing the SABR model.
- Probability Distribution in the SABR Model of Stochastic Volatility, P. Hagan et al. - Introduced the normal SABR model, heat kernel expansion, and asymptotic probability distribution.
- Hedging under SABR Model, B. Bartlett – Refined risk management under the SABR model.

- LIBOR market model with SABR style stochastic volatility, P. Hagan and A. Lesniewski – LMM extension of SABR for term structure modeling.
- Arbitrage Free SABR, P. Hagan et al. - Refined treatment of near zero forwards.
- Fine Tune Your Smile – Correction to Hagan et al.
- A summary of the approaches to the SABR model for equity derivatives smile
- Unifying the BGM and SABR models: a short ride in hyperbolic geometry
- Asymptotic Approximations to CEV and SABR Models
- Test SABR (with calibration) online
- SABR calibration
- Advanced Analytics for the SABR Model - Includes **exact** formula for zero correlation case
- Small-Strike Implied Volatility Expansion in the SABR Model - Arbitrage-free asymptotic formula for small strikes and for long-dated options
- The Free Boundary SABR: Natural Extension to Negative Rates - SABR for the negative rates
- Antonov, Alexandre and Konikov, Michael and Spector, Michael, The Free Boundary SABR: Natural Extension to Negative Rates (January 28, 2015). Available at SSRN:http://ssrn.com/abstract= 2557046

Vasicek model

In finance, the **Vasicek model** is a mathematical model describing the evolution of interest rates. It is a type of one-factorshort rate model as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives, and has also been adapted for credit markets, although its use in the credit market is in principle wrong, implying negative probabilities (see for example Brigo and Mercurio (2006), Section 21.1.1). It was introduced in 1977 by Oldřich Vašíček and can be also seen as a stochastic investment model.

The model specifies that the instantaneous interest rate follows the stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

where W_t is a Wiener process under the risk neutral framework modelling the random market risk factor, in that it models the continuous inflow of randomness into the system. The standard deviation parameter, σ, determines the volatility of the interest rate and in a way characterizes the amplitude of the instantaneous randomness inflow. The typical parameters b, a and σ, together with the initial condition r_0, completely characterize the dynamics, and can be quickly characterized as follows, assuming a to be non-negative:

- b : "long term mean level". All future trajectories of r will evolve around a mean level b in the long run;
- a : "speed of reversion". a characterizes the velocity at which such trajectories will regroup around b in time;

- σ : "instantaneous volatility", measures instant by instant the amplitude of randomness entering the system. Higher σ implies more randomness

The following derived quantity is also of interest,

- $\sigma^2 / (2a)$: "long term variance". All future trajectories of r will regroup around the long term mean with such variance after a long time.

a and σ tend to oppose each other: increasing σ increases the amount of randomness entering the system, but at the same time increasing a amounts to increasing the speed at which the system will stabilize statistically around the long term mean b with a corridor of variance determined also by a. This is clear when looking at the long term variance,

$$\frac{\sigma^2}{2a}$$

which increases with σ but decreases with a.This model is an Ornstein–Uhlenbeck stochastic process. Making the long term mean stochastic to another SDE is a simplified version of the cointelation SDE.

Vasicek's model was the first one to capture mean reversion, an essential characteristic of the interest rate that sets it apart from other financial prices. Thus, as opposed to stock prices for instance, interest rates cannot rise indefinitely. This is because at very high levels they would hamper economic activity, prompting a decrease in interest rates. Similarly, interest rates do not usually decrease below 0. As a result, interest rates move in a limited range, showing a tendency to revert to a long run value. The drift factor $a(b-r_t)$ represents the expected instantaneous change in the interest rate at time t. The parameter b represents the long-run equilibrium value towards which the interest rate reverts. Indeed, in the absence of shocks ($dW_t = 0$), the interest rate remains constant when $r_t = b$. The parameter a, governing the speed of adjustment, needs to be positive to ensure stability around the long term value. For example, when r_t is below b, the drift term $a(b-r_t)$ becomes positive for positive a, generating a tendency for the interest rate to move upwards (toward equilibrium).

The main disadvantage is that, under Vasicek's model, it is theoretically possible for the interest rate to become negative, an undesirable feature under pre-crisis assumptions. This shortcoming was fixed in the Cox–Ingersoll–Ross model, exponential Vasicek model, Black–Derman–Toy model and Black–Karasinski model, among many others. The Vasicek model was further extended in the Hull–White model. The Vasicek model is also a canonical example of the affine term structure model, along with the Cox–Ingersoll–Ross model.

We solve the stochastic differential equation to obtain

$$r(t) = r(0)e^{-at} + b\left(1-e^{-at}\right) + \sigma e^{-at}\int_0^t e^{as}\,dW_s.$$

Using similar techniques as applied to the Ornstein–Uhlenbeck stochastic process we get that state variable is distributed normally with mean

$$\mathrm{E}[r_t] = r_0 e^{-at} + b(1-e^{-at})$$

and variance

$$\mathrm{Var}[r_t] = \frac{\sigma^2}{2a}(1-e^{-2at}).$$

Consequently, we have

$$\lim_{t\to\infty} \mathrm{E}[r_t] = b$$

$$\lim_{t\to\infty} \mathrm{Var}[r_t] = \frac{\sigma^2}{2a}.$$

Online utilities

http://www.quantcalc.net/ZCB_Vasicek.html

references

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- *Damiano Brigo, Fabio Mercurio (2001). Interest Rate Models – Theory and Practice with Smile, Inflation and Credit (2nd ed. 2006 ed.). Springer Verlag.ISBN* 978 - 3 - 540 - 22149 - 4 .
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Wilkie investment model

The **Wilkie investment model**, often just called **Wilkie model**, is a stochastic asset model developed by A. D. Wilkie that describes the behavior of various economics factors as stochastic time series. These time series are generated by autoregressive models. The main factor of the model which influences all asset prices is the consumer price index. The model is mainly in use for actuarial work and asset liability management. Because of the stochastic properties of that model it is mainly combined with Monte Carlo methods.

Wilkie first proposed the model in 1986, in a paper published in the *Transactions of the Faculty of Actuaries*.It has since been the subject of extensive study and debate. Wilkie himself updated and expanded the model in a second paper published in 1995. He advises to use that model to determine the "funnel of doubt", which can be seen as an interval of minimum and maximum development of a corresponding economic factor.

Components

- price inflation
- wage inflation
- share yield
- share dividend
- consols yield (long-term interest rate)
- bank rate (short-term interest rate)

references

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- Şahin, Şule; Cairns, Andrew; Kleinow, Torsten; Wilkie, A. D. (12 June 2008). *Revisiting the Wilkie Investment Model* (PDF). International Actuarial Association, AFIR/ERM Sectional Colloquium, Rome, 2008
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- Wilkie, A.D. (1986). "A stochastic investment model for Actuarial Use" (PDF). *Transactions of the Faculty of Actuaries* **39**: 341–403.

Stochastic processes (brief reference letter)	
Discrete time	Bernoulli process Branching process Chinese restaurant process Galton–Watson process Independent and identically distributed random variables Markov chain Moran process Random walk Loop-erased Self-avoiding
Continuous time	Bessel process Birth–death process Brownian motion Bridge Excursion Fractional Geometric Meander Cauchy process Contact process Continuous-time random walk Cox process

	Diffusion process
	Empirical process
	Feller process
	Fleming–Viot process
	Gamma process
	Hunt process
	Interacting particle systems
	Itô diffusion
	Itô process
	Jump diffusion
	Jump process
	Lévy process
	Local time
	Markov additive process
	McKean–Vlasov process
	Ornstein–Uhlenbeck process
	Poisson process
	        Compound
	        Non-homogeneous
	        Point process
	Schramm–Loewner evolution
	Semimartingale
	Sigma-martingale
	Stable process
	Superprocess
	Telegraph process
	Variance gamma process
	Wiener process

	Wiener sausage

Both	Branching process
	Gaussian process
	Hidden Markov model (HMM)
	Markov process
	Martingale
	    Differences
	    Local
	    Sub-
	    Super-
	Random dynamical system
	Regenerative process
	Renewal process
	White noise

Fields and other	Dirichlet process
	Gaussian random field
	Gibbs measure
	Hopfield model
	Ising model
	    Potts model
	    Boolean network
	Markov random field
	Percolation
	Pitman–Yor process
	Point process
	    Cox
	    Poisson
	Random field
	Random graph

Time series models	Autoregressive conditional heteroskedasticity (ARCH) model Autoregressive integrated moving average (ARIMA) model Autoregressive (AR) model Autoregressive–moving-average (ARMA) model Generalized autoregressive conditional heteroskedasticity (GARCH) model Moving-average (MA) model
Financial models	Black–Derman–Toy Black–Karasinski Black–Scholes Chen Constant elasticity of variance (CEV) Cox–Ingersoll–Ross (CIR) Garman–Kohlhagen Heath–Jarrow–Morton (HJM) Heston Ho–Lee Hull–White LIBOR market Rendleman–Bartter SABR volatility Vašíček Wilkie
Actuarial models	Bühlmann Cramér–Lundberg

	Risk process Sparre–Anderson
Queueing models	Bulk Fluid Generalized queueing network M/G/۱ M/M/۱ M/M/c
Properties	Càdlàg paths Continuous Continuous paths Ergodic Exchangeable Feller-continuous Gauss–Markov Markov Mixing Piecewise deterministic Predictable Progressively measurable Self-similar Stationary Time-reversible
Limit theorems	Central limit theorem Donsker's theorem Doob's martingale convergence theorems Ergodic theorem Fisher–Tippett–Gnedenko theorem Large deviation principle

	Law of large numbers (weak/strong) Law of the iterated logarithm Maximal ergodic theorem Sanov's theorem
Inequalities	Burkholder–Davis–Gundy Doob's martingale Kunita–Watanabe
Tools	Cameron–Martin formula Convergence of random variables Doléans-Dade exponential Doob decomposition theorem Doob–Meyer decomposition theorem Doob's optional stopping theorem Dynkin's formula Feynman–Kac formula Filtration Girsanov theorem Infinitesimal generator Itô integral Itô's lemma Kolmogorov continuity theorem Kolmogorov extension theorem Lévy–Prokhorov metric Malliavin calculus Martingale representation theorem Optional stopping theorem Prokhorov's theorem

	Quadratic variation Reflection principle Skorokhod integral Skorokhod's representation theorem Skorokhod space Snell envelope Stochastic differential equation Tanaka Stopping time Stratonovich integral Uniform integrability Usual hypotheses Wiener space Classical Abstract
Disciplines	Actuarial mathematics Econometrics Ergodic theory Extreme value theory (EVT) Large deviations theory Mathematical finance Mathematical statistics Probability theory Queueing theory Renewal theory Ruin theory Statistics Stochastic analysis Time series analysis Machine learning

Financial models

Compilers

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