5

CHAPTER

KINEMATICS OF FLOW AND IDEAL FLOW

A. KINEMATICS OF FLOW

▶ 5.1 INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

▶ 5.2 METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods. They are—(*i*) Lagrangian Method, and (*ii*) Eulerian Method. In the Lagrangian method, a single fluid particle is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, pressure, density etc., are described at a point in flow field. The Eulerian method is commonly used in fluid mechanics.

▶ 5.3 TYPES OF FLUID FLOW

The fluid flow is classified as :

(*i*) Steady and unsteady flows ;

(*ii*) Uniform and non-uniform flows ;

(*iii*) Laminar and turbulent flows ;

(*iv*) Compressible and incompressible flows ;

(*v*) Rotational and irrotational flows ; and

(*vi*) One, two and three-dimensional flows.

5.3.1 Steady and Unsteady Flows. Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

5.3.2 Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where ∂V = Change of velocity

∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} \neq 0.$$

5.3.3 Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$ called the Reynold number.

where D = Diameter of pipe

V = Mean velocity of flow in pipe

and ν = Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

5.3.4 Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant}.$$

5.3.5 Rotational and Irrotational Flows. Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

5.3.6 One-, Two- and Three-Dimensional Flows. **One-dimensional flow** is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only, say x. For a steady one-dimensional flow, the velocity is a function of one-space-co-ordinate only. The variation of velocities in other two mutually perpendicular directions is assumed negligible. Hence mathematically, for one-dimensional flow

$$u = f(x),\ v = 0 \text{ and } w = 0$$

where u, v and w are velocity components in x, y and z directions respectively.

Two-dimensional flow is that type of flow in which the velocity is a function of time and two rectangular space co-ordinates say x and y. For a steady two-dimensional flow the velocity is a function of two space co-ordinates only. The variation of velocity in the third direction is negligible. Thus, mathematically for two-dimensional flow

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$

Three-dimensional flow is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. But for a steady three-dimensional flow the fluid parameters are functions of three space co-ordinates (x, y and z) only. Thus, mathematically, for three-dimensional flow

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

▸ 5.4 RATE OF FLOW OR DISCHARGE (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

(*i*) For liquids the units of Q are m^3/s or litres/s

(*ii*) For gases the units of Q is kgf/s or Newton/s

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

V = Average velocity of fluid across the section

Then discharge $Q = A \times V$. ...(5.1)

▸ 5.5 CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Fig. 5.1.

Let V_1 = Average velocity at cross-section 1-1

ρ_1 = Density at section 1-1

A_1 = Area of pipe at section 1-1

and V_2, ρ_2, A_2 are corresponding values at section, 2-2.

Then rate of flow at section 1-1 $= \rho_1 A_1 V_1$

Rate of flow at section 2-2 $= \rho_2 A_2 V_2$

According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

or
$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad ...(5.2)$$

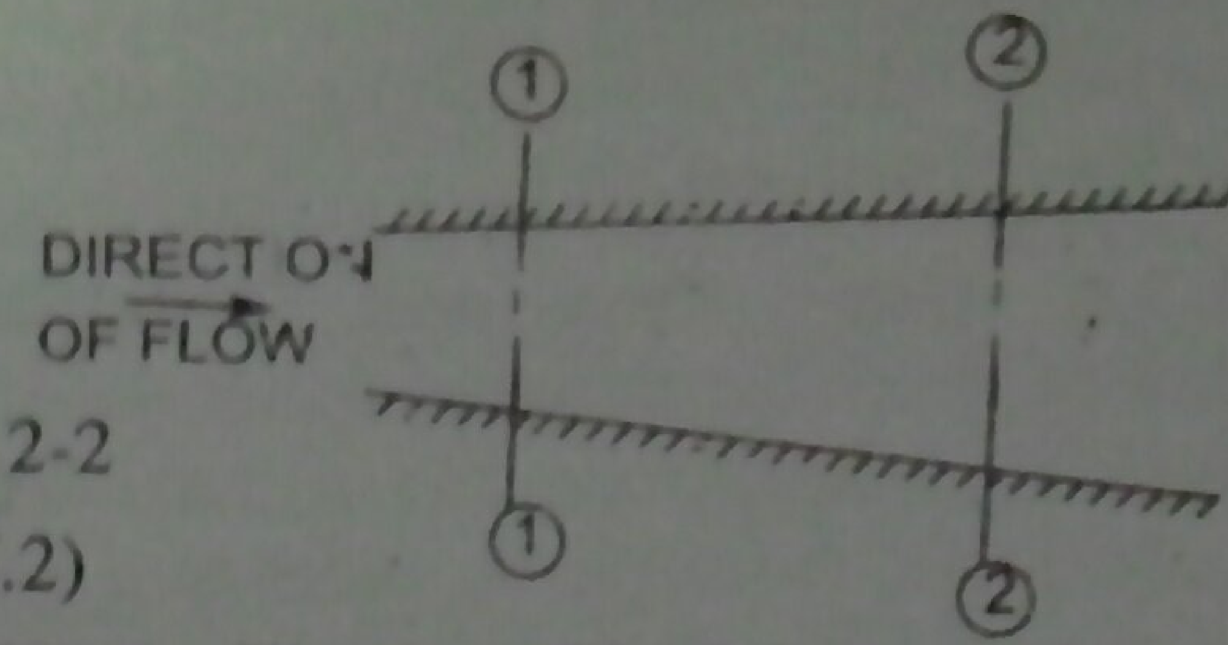

Fig. 5.1 *Fluid flowing through a pipe.*

Equation (5.2) is applicable to the compressible as well as incompressible fluids and is called **Continuity Equation**. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation (5.2) reduces to

$$A_1 V_1 = A_2 V_2 \qquad ...(5.3)$$

Problem 5.1 *The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5 m/s. Determine also the velocity at section 2.*

Solution. Given :

At section 1, $D_1 = 10$ cm $= 0.1$ m

$$A_1 = \frac{\pi}{4}(D_1^2) = \frac{\pi}{4}(.1)^2 = 0.007854 \text{ m}^2$$

$$V_1 = 5 \text{ m/s.}$$

At section 2, $D_2 = 15$ cm $= 0.15$ m

$$A_2 = \frac{\pi}{4}(.15)^2 = 0.01767 \text{ m}^2$$

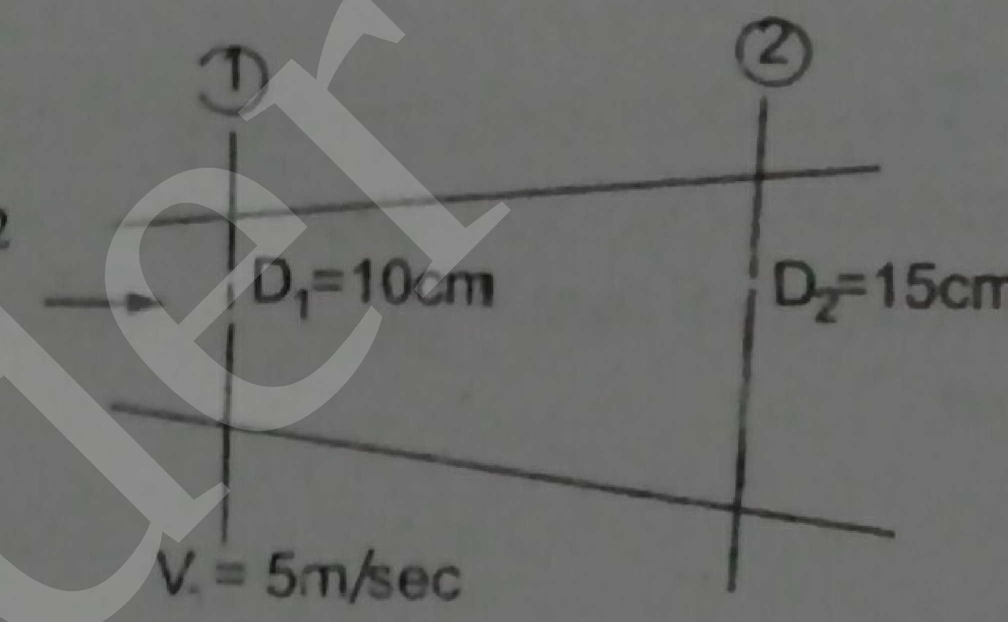

Fig. 5.2

(i) Discharge through pipe is given by equation (5.1)

or
$$Q = A_1 \times V_1$$
$$= 0.007854 \times 5 = 0.03927 \text{ m}^3\text{/s. } \textbf{Ans.}$$

Using equation (5.3), we have $A_1V_1 = A_2V_2$

(ii) $\therefore$
$$V_2 = \frac{A_1V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0 = 2.22 \text{ m/s. } \textbf{Ans.}$$

Problem 5.2 *A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.*

Solution. Given :

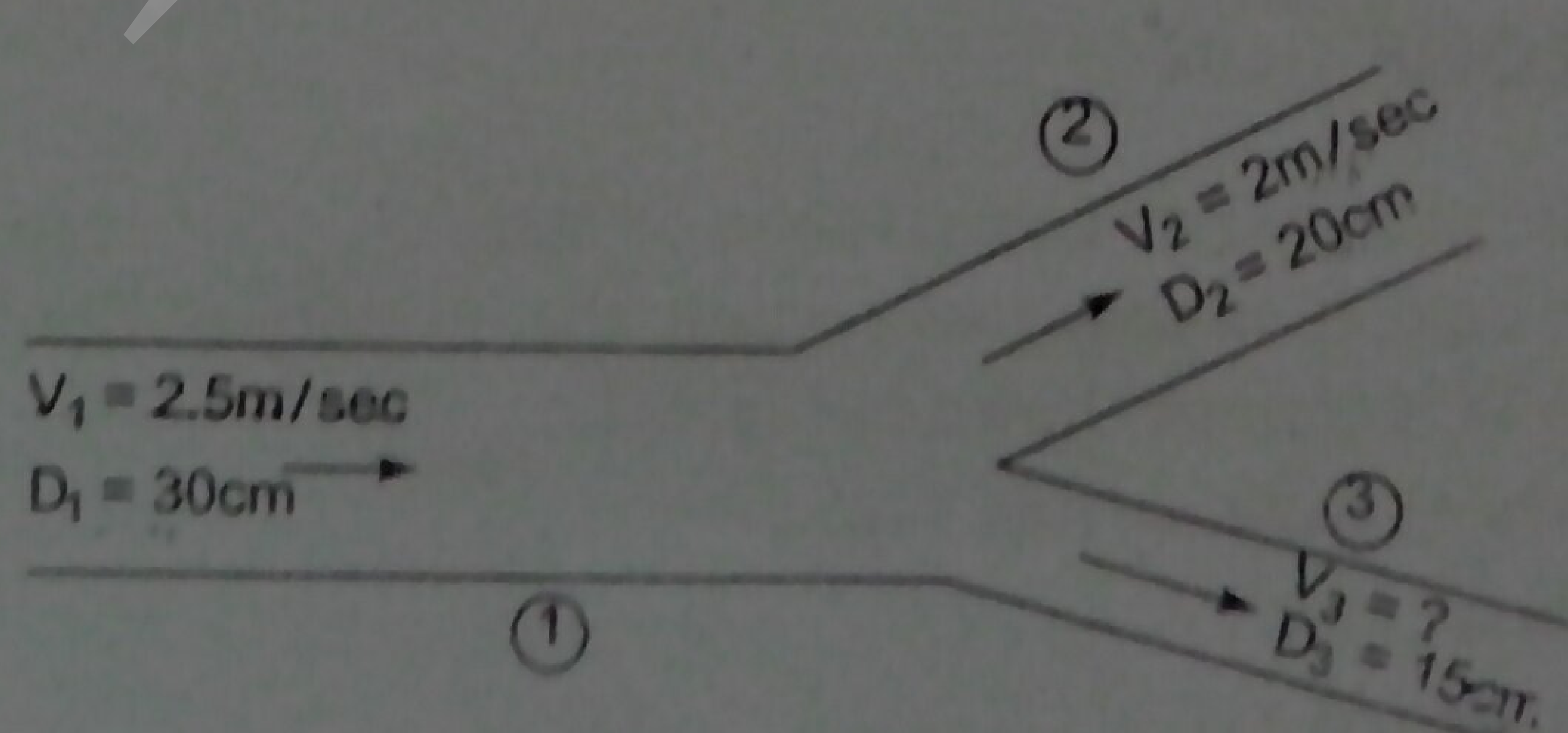

Fig. 5.3

$$D_1 = 30 \text{ cm} = 0.30 \text{ m}$$

$$\therefore \quad A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times .3^2 = 0.07068 \text{ m}^2$$

$$V_1 = 2.5 \text{ m/s}$$

$$D_2 = 20 \text{ cm} = 0.20 \text{ m}$$

$$\therefore \quad A_2 = \frac{\pi}{4} (.2)^2 = \frac{\pi}{4} \times .4 = 0.0314 \text{ m}^2,$$

$$V_2 = 2 \text{ m/s}$$

$$D_3 = 15 \text{ cm} = 0.15 \text{ m}$$

$$\therefore \quad A_3 = \frac{\pi}{4} (.15)^2 = \frac{\pi}{4} \times 0.225 = 0.01767 \text{ m}^2$$

Find (*i*) Discharge in pipe 1 or Q_1

(*ii*) Velocity in pipe of dia. 15 cm or V_3

Let Q_1, Q_2 and Q_3 are discharges in pipe 1, 2 and 3 respectively.

Then according to continuity equation

$$Q_1 = Q_2 + Q_3 \quad \ldots(1)$$

(*i*) **The discharge Q_1 in pipe 1 is given by**

$$Q_1 = A_1 V_1 = 0.07068 \times 2.5 \text{ m}^3\text{/s} = \mathbf{0.1767 \text{ m}^3\text{/s. Ans.}}$$

(*ii*) **Value of V_3**

$$Q_2 = A_2 V_2 = 0.0314 \times 2.0 = 0.0628 \text{ m}^3\text{/s}$$

Substituting the values of Q_1 and Q_2 in equation (1)

$$0.1767 = 0.0628 + Q_3$$

$$\therefore \quad Q_3 = 0.1767 - 0.0628 = 0.1139 \text{ m}^3\text{/s}$$

But $\quad Q_3 = A_3 \times V_3 = 0.01767 \times V_3 \quad \text{or} \quad 0.1139 = 0.01767 \times V_3$

$$\therefore \quad V_3 = \frac{0.1139}{0.01767} = \mathbf{6.44 \text{ m/s. Ans.}}$$

Problem 5.3 *Water flows through a pipe AB 1.2 m diameter at 3 m/s and then passes through a pipe BC 1.5 m diameter. At C, the pipe branches. Branch CD is 0.8 m in diameter and carries one-third of the flow in AB. The flow velocity in branch CE is 2.5 m/s. Find the volume rate of flow in AB, the velocity in BC, the velocity in CD and the diameter of CE.*

Solution. Given :

Diameter of pipe *AB*, $D_{AB} = 1.2$ m

Velocity of flow through *AB*, $V_{AB} = 3.0$ m/s

Dia. of pipe *BC*, $D_{BC} = 1.5$ m

Dia. of branched pipe *CD*, $D_{CD} = 0.8$ m

Velocity of flow in pipe *CE*, $V_{CE} = 2.5$ m/s

Let the flow rate in pipe $AB = Q$ m^3/s

Velocity of flow in pipe $BC = V_{BC}$ m/s

Velocity of flow in pipe $CD = V_{CD}$ m/s

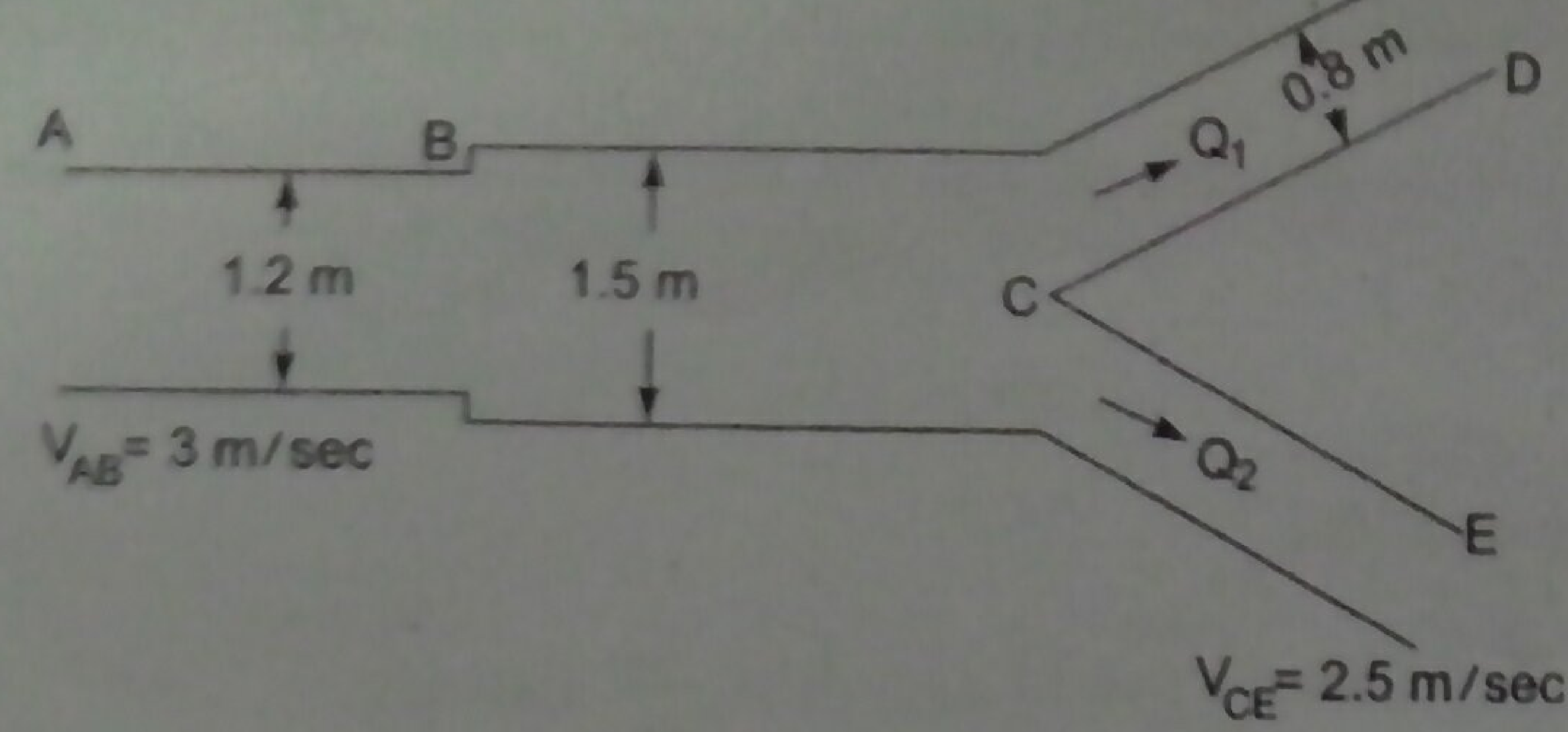

Fig. 5.4

Diameter of pipe $CE = D_{CE}$

Then flow rate through $CD = Q/3$

and flow rate through $CE = Q - Q/3 = \dfrac{2Q}{3}$

(*i*) Now volume flow rate through $AB = Q = V_{AB} \times$ Area of AB

$$= 3.0 \times \frac{\pi}{4}(D_{AB})^2 = 3.0 \times \frac{\pi}{4}(1.2)^2 = \mathbf{3.393\ m^3/s.\ Ans.}$$

(*ii*) Applying continuity equation to pipe AB and pipe BC,

$$V_{AB} \times \text{Area of pipe } AB = V_{BC} \times \text{Area of pipe } BC$$

or $$3.0 \times \frac{\pi}{4}(D_{AB})^2 = V_{BC} \times \frac{\pi}{4}(D_{BC})^2$$

or $$3.0 \times (1.2)^2 = V_{BC} \times (1.5)^2 \qquad \left[\text{Divide by } \frac{\pi}{4}\right]$$

or $$V_{BC} = \frac{3 \times 1.2^2}{1.5^2} = \mathbf{1.92\ m/s.\ Ans.}$$

(*iii*) The flow rate through pipe

$$CD = Q_1 = \frac{Q}{3} = \frac{3.393}{3} = 1.131\ m^3/s$$

∴ $$Q_1 = V_{CD} \times \text{Area of pipe } CD \times \frac{\pi}{4}(D_{CD})^2$$

or $$1.131 = V_{CD} \times \frac{\pi}{4} \times 0.8^2 = 0.5026\ V_{CD}$$

∴ $$V_{CD} = \frac{1.131}{0.5026} = \mathbf{2.25\ m/s.\ Ans.}$$

(*iv*) Flow rate through CE,

$$Q_2 = Q - Q_1 = 3.393 - 1.131 = 2.262\ m^3/s$$

∴ $$Q_2 = V_{CE} \times \text{Area of pipe } CE = V_{CE}\frac{\pi}{4}(D_{CE})^2$$

or $$2.263 = 2.5 \times \frac{\pi}{4} \times (D_{CE})^2$$

or $$D_{CE} = \sqrt{\frac{2.263 \times 4}{2.5 \times \pi}} = \sqrt{1.152} = 1.0735\ m$$

∴ Diameter of pipe CE = **1.0735 m. Ans.**

Problem 5.4 *A 25 cm diameter pipe carries oil of sp. gr. 0.9 at a velocity of 3 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil.*

Solution. Given :

at section 1, $D_1 = 25$ cm $= 0.25$ m

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.049 \text{ m}^2$$

$$V_1 = 3 \text{ m/s}$$

at section 2, $D_2 = 20$ cm $= 0.2$ m

$$A_2 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$$

$$V_2 = ?$$

Mass rate of flow of oil = ?

Applying continuity equation at sections 1 and 2,

$$A_1V_1 = A_2V_2$$

or $0.049 \times 3.0 = 0.0314 \times V_2$

$$\therefore \quad V_2 = \frac{0.049 \times 3.0}{0.0314} = \textbf{4.68 m/s. Ans.}$$

Mass rate of flow of oil $= \text{Mass density} \times Q = \rho \times A_1 \times V_1$

Sp. gr. of oil $= \dfrac{\text{Density of oil}}{\text{Density of water}}$

$\therefore$ **Density of oil** = **Sp. gr. of oil × Density of water**

$= 0.9 \times 1000 \text{ kg/m}^3 = \dfrac{900 \text{ kg}}{\text{m}^3}$

$\therefore$ **Mass rate of flow** $= 900 \times 0.049 \times 3.0$ kg/s $=$ **132.23 kg/s. Ans.**

Problem 5.5 *A jet of water from a 25 mm diameter nozzle is directed vertically upwards. Assuming that the jet remains circular and neglecting any loss of energy, that will be the diameter at a point 4.5 m above the nozzle, if the velocity with which the jet leaves the nozzle is 12 m/s.*

Solution. Given :

Dia. of nozzle, $D_1 = 25$ mm $= 0.025$ m

Velocity of jet at nozzle, $V_1 = 12$ m/s

Height of point A, $h = 4.5$ m

Let the velocity of the jet at a height 4.5 m $= V_2$

Consider the vertical motion of the jet from the outlet of the nozzle to the point A (neglecting any loss of energy).

Initial velocity, $u = V_1 = 12$ m/s

Final velocity, $V = V_2$

Value of $g = -9.81$ m/s^2 and $h = 4.5$ m

Using, $V^2 - u^2 = 2gh$, we get

$$V_2^2 - 12^2 = 2 \times (-9.81) \times 4.5$$

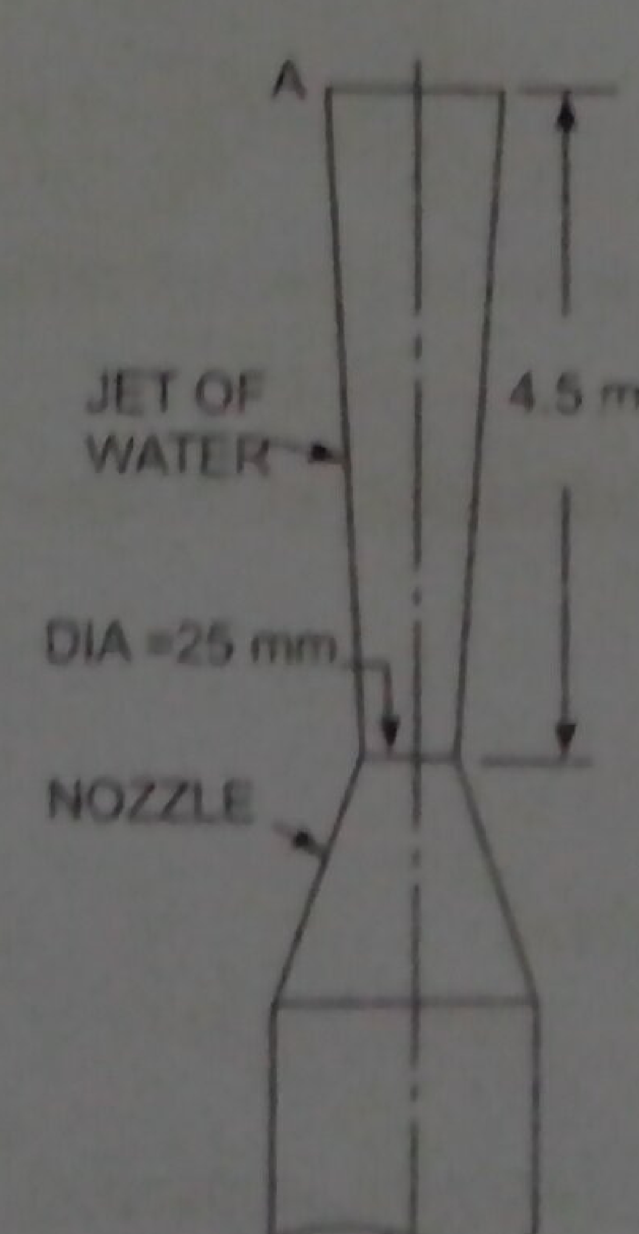

Fig. 5.5

$\therefore \qquad V_2 = \sqrt{12^2 - 2 \times 9.81 \times 4.5} = \sqrt{144 - 88.29} = 7.46$ m/s

Now applying continuity equation to the outlet of nozzle and at point A, we get

$$A_1V_1 = A_2V_2$$

or $$A_2 = \frac{A_1V_1}{V_2} = \frac{\frac{\pi}{4} D_1^2 \times V_1}{V_2} = \frac{\pi \times (0.025)^2 \times 12}{4 \times 7.46} = 0.0007896$$

Let $\qquad D_2$ = Diameter of jet at point A.

Then $\qquad A_2 = \frac{\pi}{2} D_2^2$ or $0.0007896 = \frac{\pi}{4} \times D_2^2$

$\therefore \qquad D_2 = \sqrt{\frac{0.0007896 \times 4}{\pi}} = 0.0317$ m = **31.7 mm. Ans.**

▶ 5.6 CONTINUITY EQUATION IN THREE-DIMENSIONS

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$
$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u\, dydz + \frac{\partial}{\partial x}(\rho u\, dydz)\, dx$

$\therefore$ **Gain of mass in x-direction**

= **Mass through $ABCD$ – Mass through $EFGH$ per second**

$$= \rho u\, dydz - \rho u\, dydz - \frac{\partial}{\partial x}(\rho u\, dydz) dx$$

$$= -\frac{\partial}{\partial x}(\rho u\, dydz)\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dydz \qquad \{\because dydz \text{ is constant}\}$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dxdydz$$

and in z-direction

$$= -\frac{\partial}{\partial z}(\rho w)\, dxdydz$$

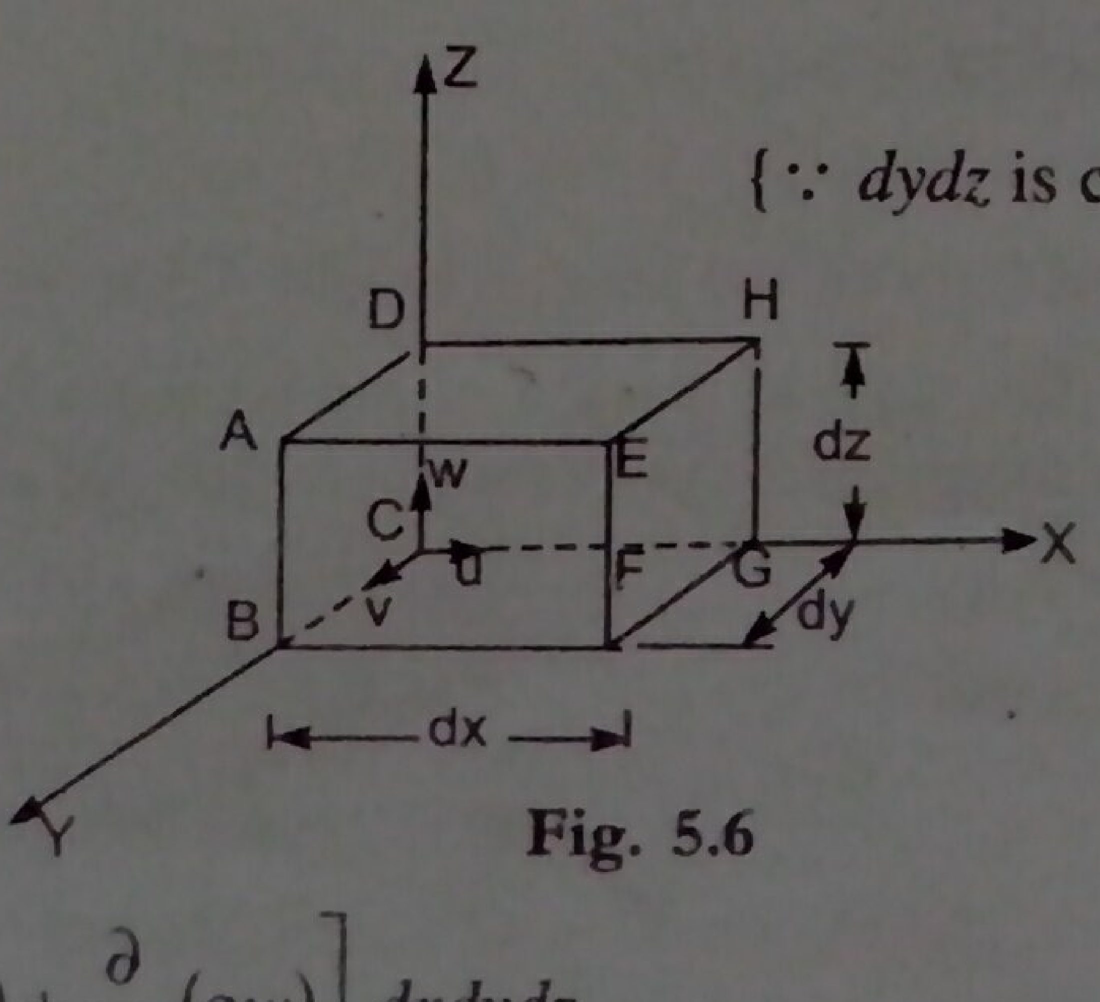

Fig. 5.6

$\therefore$ Net gain of masses $= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho.\ dx.\ dy.\ dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\ dx.\ dy.\ dz)$ or

$\frac{\partial \rho}{\partial t}.\ dx\ dy\ dz.$

Equating the two expressions,

or $$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\ dxdydz$$

or $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \text{ [Cancelling } dx.dy.dz \text{ from both sides]} \quad ...(5.3A)$$

Equation (5.3*A*) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,

(*ii*) Uniform and non-uniform flow, and

(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.3*A*) becomes as

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \quad ...(5.3B)$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \quad ...(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \quad ...(5.5)$$

5.6.1 Continuity Equation in Cylindrical Polar Co-ordinates. The continuity equation in cylindrical polar co-ordinates (*i.e.*, r, θ, z co-ordinates) is derived by the procedure given below.

Consider a two-dimensional incompressible flow field. The two-dimensional polar co-ordinates are r and θ. Consider a fluid element *ABCD* between the radii r and $r + dr$ as shown in Fig. 5.7. The angle subtended by the element at the centre is $d\theta$. The components of the velocity V are u_r in the radial direction and u_θ in the tangential direction. The sides of the element are having the lengths as

Side $AB = rd\theta$, $BC = dr$, $DC = (r + dr)\ d\theta$, $AD = dr$.

The thickness of the element perpendicular to the plane of the paper is assumed to be unity.

Consider the flow in radial direction

Mass of fluid entering the face AB per unit time

$= \rho \times$ Velocity in r-direction $\times$ Area

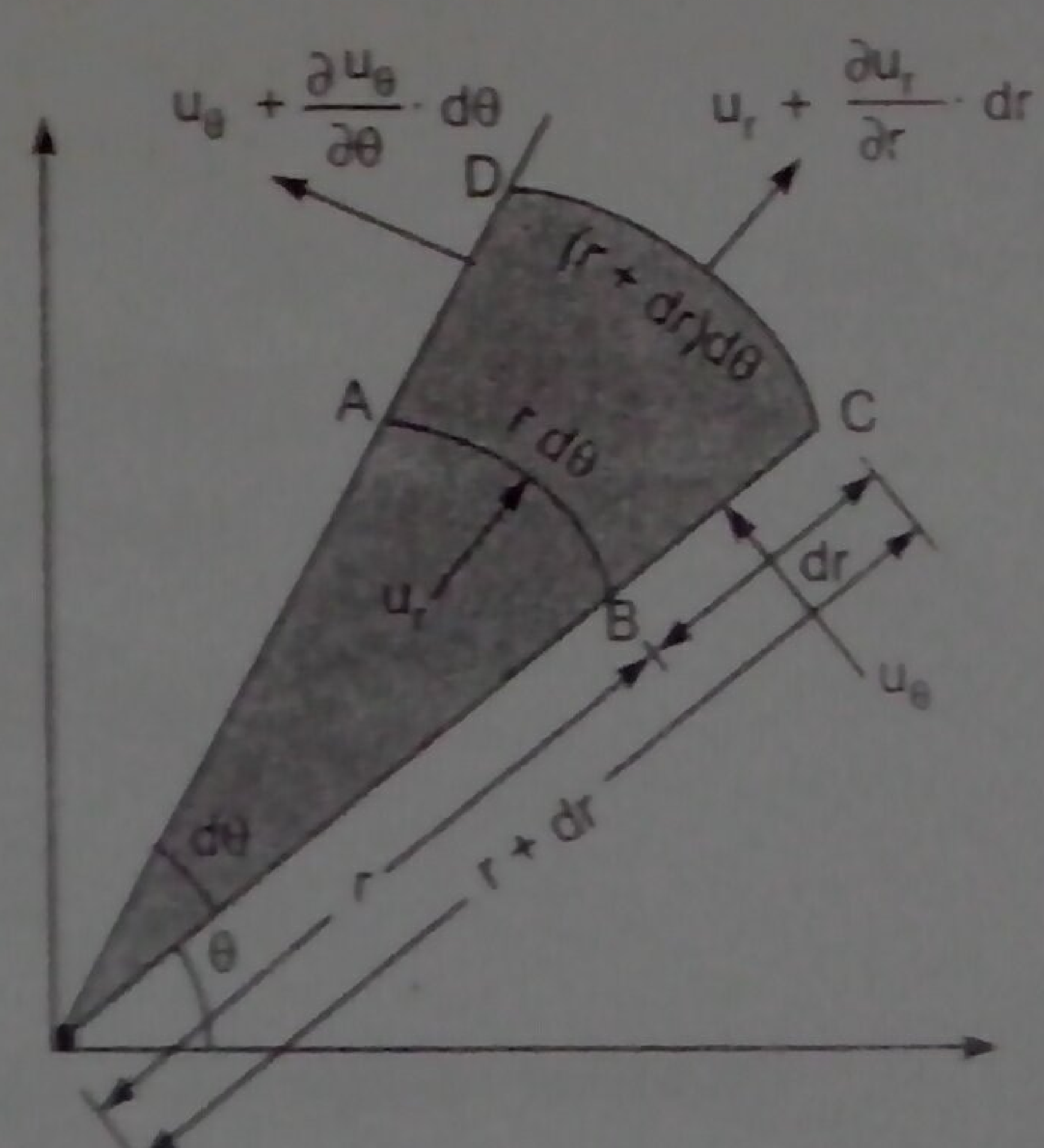

Fig. 5.7

$$= \rho \times u_r \times (AB \times 1) \quad (\because \text{Area} = AB \times \text{Thickness} = rd\theta \times 1)$$

$$= \rho \times u_r \times (rd\theta \times 1) = \rho \,.\, u_r.\, rd\theta$$

Mass of fluid leaving the face CD per unit time

$$= \rho \times \text{Velocity} \times \text{Area}$$

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}.dr\right) \times (CD \times 1) \quad (\because \text{Area} = CD \times 1)$$

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}\, dr\right) \times (r + dr)d\theta \quad [\because CD = (r - dr)\, d\theta]$$

$$= \rho \times \left[u_r \times r + u_r dr + r\frac{\partial u_r}{\partial r}\, dr + \frac{\partial u_r}{\partial r}(dr)^2\right] d\theta$$

$$= \rho\left[u_r \times r + u_r \times dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

[The term containing $(dr)^2$ is very small and has been neglected]

∴ Gain of mass in r-direction per unit time

$$= (\text{Mass through } AB - \text{Mass through } CD) \text{ per unit time}$$

$$= \rho.\, u_r.\, rd\theta - \rho\left[u_r.r + u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= \rho.\, u_r.\, rd\theta - \rho.\, u_r.\, r.\, d\theta - \rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= -\rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right].d\theta$$

$$= -\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] r.\, dr.\, d\theta$$

[This is written in this form because $(r.\, d\theta.\, dr.\, 1)$ is equal to volume of element]

Now consider the flow in θ-direction

Gain in mass in θ-direction per unit time

$$= (\text{Mass through } BC - \text{Mass through } AD) \text{ per unit time}$$

$$= [\rho \times \text{Velocity through } BC \times \text{Area} - \rho \times \text{Velocity through } AD \times \text{Area}]$$

$$= \left[\rho.u_\theta.dr \times 1 - \rho\left(u_\theta + \frac{\partial u_\theta}{\partial \theta}.d\theta\right) \times dr \times 1\right]$$

$$= -\rho\left(\frac{\partial u_\theta}{\partial \theta}.d\theta\right) dr \times 1 \quad (\because \text{Area} = dr \times 1)$$

$$= -\rho\frac{\partial u_\theta}{\partial \theta}.\frac{r.\, d\theta.\, dr}{r} \quad [\text{Multiplying and dividing by } r]$$

∴ Total gain in fluid mass per unit time

$$= -\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right].\, r.\, dr.\, d\theta - \rho\frac{\partial u_\theta}{\partial \theta}.\frac{rd\theta.\, dr}{r} \quad \text{...(5.5A)}$$

But mass of fluid element $= \rho \times$ Volume of fluid element

$= \rho \times [rd\theta \times dr \times 1]$

$= \rho \times rd\theta \, . \, dr$

Rate of increase of fluid mass in the element with time

$$= \frac{\partial}{\partial t} [\rho \, . \, rd\theta \, . \, dr] = \frac{\partial \rho}{\partial t} . \, rd\theta \, dr \qquad \text{...(5.5B)}$$

($\because$ $rd\theta \, . \, dr \, . \, 1$ is the volume of element and is a constant quantity)

Since the mass is neither created nor destroyed in the fluid element, hence net gain of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Hence equating the two expressions given by equations (5.5 *A*) and (5.5 *B*), we get

$$-\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] r \, . \, dr. \, d\theta - \rho \, \frac{\partial u_\theta}{\partial \theta} \frac{rd\theta \, . \, dr}{r} = \frac{\partial \rho}{\partial t} \, rd\theta \, dr$$

or $$-\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] - \rho \frac{\partial u_\theta}{\partial \theta} . \frac{1}{r} = \frac{\partial \rho}{\partial t} \qquad \text{[Cancelling } rdr \, . \, d\theta \text{ from both sides]}$$

or $$\frac{\partial \rho}{\partial t} + \rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] + \rho \frac{\partial u_\theta}{\partial \theta} . \frac{1}{r} = 0 \qquad \text{...(5.5C)}$$

Equation (5.5 *C*) is the continuity equation in polar co-ordinates for two-dimensional flow.

For steady flow $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.5 *C*) reduces to

$$\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] + \rho \frac{\partial u_\theta}{\partial \theta} . \frac{1}{r} = 0$$

or $$\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} . \frac{1}{r} = 0$$

or $$u_r + r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0$$

or $$\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial \theta}(u_\theta) = 0 \qquad \left[\because \frac{\partial}{\partial r}(r \, . \, u_r) = r \, . \, \frac{\partial u_r}{\partial r} + u_r\right] \qquad \text{...(5.5D)}$$

Equation (5.5 *D*) represents the continuity equation in polar co-ordinates for two-dimensional steady incompressible flow.

Problem 5.5A *Examine whether the following velocity components represent a physically possible flow ?*

$$u_r = r \sin \theta, \; u_\theta = 2r \cos \theta.$$

Solution. Given : $u_r = r \sin \theta$ and $u_\theta = 2r \cos \theta$

For physically possible flow, the continuity equation,

$$\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial \theta}(u_\theta) = 0 \text{ should be satisfied.}$$

Now $u_r = r \sin \theta$

Multiplying the above equation by r, we get

$$ru_r = r^2 \sin \theta$$

Differentiating the preceding equation w.r.t. r, we get

$$\frac{\partial}{\partial r}(ru_r) = \frac{\partial}{\partial r}(r^2 \sin\theta)$$

$$= 2r \sin\theta \qquad (\because \ \sin\theta \text{ is constant w.r.t. } r)$$

Now $\quad u_\theta = 2r\cos\theta$

Differentiating the above equation w.r.t. θ, we get

$$\frac{\partial}{\partial \theta}(u_\theta) = \frac{\partial}{\partial \theta}(2r\cos\theta)$$

$$= 2r(-\sin\theta) \qquad (\because \ 2r \text{ is constant w.r.t. } \theta)$$

$$= -2r\sin\theta$$

$$\therefore \quad \frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial \theta}(u_\theta) = 2r\sin\theta - 2r\sin\theta = 0$$

Hence the continuity equation is satisfied. Hence the given velocity components represent a physically possible flow.

▸ 5.7 VELOCITY AND ACCELERATION

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity, $\quad V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the total acceleration in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\quad \frac{dx}{dt} = u, \frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$

$$\therefore \quad a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly,

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \qquad \text{...(5.6)}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity or $$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x, y and z directions becomes

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right] \quad \text{...(5.7)}$$

Acceleration vector
$$\left.\begin{aligned} A &= a_x i + a_y j + a_z k \\ &= \sqrt{a_x^2 + a_y^2 + a_z^2}. \end{aligned}\right] \quad \text{...(5.8)}$$

5.7.1 Local Acceleration and Convective Acceleration. Local acceleration is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In the equation given by (5.6), the expression $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ or $\frac{\partial w}{\partial t}$ is known as local acceleration.

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expressions other than $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ in equation (5.6) are known as convective acceleration.

Problem 5.6 *The velocity vector in a fluid flow is given*

$$V = 4x^3 i - 10x^2 y j + 2tk.$$

Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time t = 1.

Solution. The velocity components u, v and w are $u = 4x^3$, $v = -10x^2 y$, $w = 2t$

For the point (2, 1, 3), we have $x = 2$, $y = 1$ and $z = 3$ at time $t = 1$.

Hence velocity components at (2, 1, 3) are

$$u = 4 \times (2)^3 = 32 \text{ units}$$
$$v = -10(2)^2(1) = -40 \text{ units}$$
$$w = 2 \times 1 = 2 \text{ units}$$

∴ Velocity vector V at (2, 1, 3) $= 32i - 40j + 2k$

or Resultant velocity $= \sqrt{u^2 + v^2 + w^2}$

$$= \sqrt{32^2 + (-40)^2 + 2^2} = \sqrt{1024 + 1600 + 4} = \textbf{51.26 units. Ans.}$$

Acceleration is given by equation (5.6)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \frac{\partial v}{\partial y} = -10x^2, \frac{\partial v}{\partial z} = 0 \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \frac{\partial w}{\partial y} = 0, \frac{\partial w}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2, 1, 3) at time $t = 1$ are

$$a_x = 4x^3(12x^2) + (-10x^2y)(0) + 2t \times (0) + 0$$
$$= 48x^5 = 48 \times (2)^5 = 48 \times 32 = 1536 \text{ units}$$
$$a_y = 4x^3(-20xy) + (-10x^2y)(-10x^2) + 2t(0) + 0$$
$$= -80x^4y + 100x^4y$$
$$= -80(2)^4(1) + 100(2)^4 \times 1 = -1280 + 1600 = 320 \text{ units.}$$
$$a_z = 4x^3(0) + (-10x^2y)(0) + (2t)(0) + 2.1 = 2.0 \text{ units}$$

∴ Acceleration is $A = a_x i + a_y j + a_z k =$ **1536i + 320j + 2k. Ans.**

or Resultant $A = \sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

$= \sqrt{2359296 + 102400 + 4} =$ **1568.9 units. Ans.**

Problem 5.7 *The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation :*

(i) $u = x^2 + y^2 + z^2 \; ; v = xy^2 - yz^2 + xy$

(ii) $v = 2y^2, w = 2xyz.$

Solution. The continuity equation for incompressible fluid is given by equation (5.4) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Case I. $u = x^2 + y^2 + z^2 \quad \therefore \quad \frac{\partial u}{\partial x} = 2x$

$v = xy^2 - yz^2 + xy \quad \therefore \quad \frac{\partial v}{\partial y} = 2xy - z^2 + x$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation.

$$2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

or $$\frac{\partial w}{\partial z} = -3x - 2xy + z^2 \text{ or } \partial w = (-3x - 2xy + z^2)\,\partial z$$

Integration of both sides gives $\int dw = \int(-3x - 2xy - z^2)\,dz$

or
$$w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + \text{Constant of integration,}$$

where constant of integration cannot be a function of z. But it can be a function of x and y that is $f(x, y)$.

$$\therefore \qquad w = \left(-3xz - 2xyz + \frac{z^3}{3}\right) + f(x, y). \textbf{ Ans.}$$

Case II.
$$v = 2y^2 \qquad \therefore \qquad \frac{\partial v}{\partial y} = 4y$$
$$w = 2xyz \qquad \therefore \qquad \frac{\partial w}{\partial z} = 2xy$$

Substituting the values of $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ in continuity equation, we get

$$\frac{\partial u}{\partial x} + 4y + 2xy = 0$$

or
$$\frac{\partial u}{\partial x} = -4y - 2xy \text{ or } du = (-4y - 2xy)\,dx$$

Integrating, we get
$$u = -4xy - 2y\frac{x^2}{2} + f(y, z) = -4xy - x^2y + f(y, z). \textbf{ Ans.}$$

Problem 5.8 *A fluid flow field is given by*

$$V = x^2y\,i + y^2z\,j - (2xyz + yz^2)\,k$$

Prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point (2, 1, 3).

Solution. For the given fluid flow field $u = x^2y$ $\qquad \therefore \quad \frac{\partial u}{\partial x} = 2xy$

$$v = y^2z \qquad \therefore \qquad \frac{\partial v}{\partial y} = 2yz$$
$$w = -2xyz - yz^2 \qquad \therefore \qquad \frac{\partial w}{\partial z} = -2xy - 2yz.$$

For a case of possible steady incompressible fluid flow, the continuity equation (5.4) should be satisfied.

i.e.,
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2xy + 2yz - 2xy - 2yz = 0$$

Hence the velocity field $V = x^2yi + y^2zj - (2xyz + yz^2)\,k$ is a possible case of fluid flow. **Ans.**

Velocity at (2, 1, 3)

Substituting the values $x = 2$, $y = 1$ and $z = 3$ in velocity field, we get

$$V = x^2yi + y^2zj - (2xyz + yz^2)\,k$$
$$= 2^2 \times 1i + 1^2 \times 3j - (2 \times 2 \times 1 \times 3 + 1 \times 3^2)\,k$$
$$= \mathbf{4i + 3j - 21k.}\ \textbf{Ans.}$$

and Resultant velocity $= \sqrt{4^2 + 3^2 + (-21)^2} = \sqrt{16 + 9 + 441} = \sqrt{466} = $ **21.587 units. Ans.**

Acceleration at (2, 1, 3)

The acceleration components a_x, a_y and a_z for steady flow are

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$u = x^2y, \frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 \text{ and } \frac{\partial u}{\partial z} = 0$$

$$v = y^2z, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 2yz, \frac{\partial v}{\partial z} = y^2$$

$$w = -2xyz - yz^2, \frac{\partial w}{\partial x} = -2yz, \frac{\partial w}{\partial y} = -2xz - z^2, \frac{\partial w}{\partial z} = -2xy - 2yz.$$

Substituting these values in acceleration components, we get acceleration at (2, 1, 3)

$$a_x = x^2y\,(2xy) + y^2z\,(x^2) - (2xyz + yz^2)\,(0)$$
$$= 2x^3y^2 + x^2y^2z$$
$$= 2(2)^3 1^2 + 2^2 \times 1^2 \times 3 = 2 \times 8 + 12$$
$$= 16 + 12 = 28 \text{ units}$$

$$a_y = x^2y\,(0) + y^2z\,(2yz) - (2xyz + yz^2)\,(y^2)$$
$$= 2y^3z^2 - 2xy^3z - y^3z^2$$
$$= 2 \times 1^3 \times 3^2 - 2 \times 2 \times 1^3 \times 3 - 1^3 \times 3^2 = 18 - 12 - 9 = -3 \text{ units}$$

$$a_z = x^2y\,(-2yz) + y^2z\,(-2xz - z^2) - (2xyz + yz^2)\,(-2xy - 2yz)$$
$$= -2x^2y^2z - 2xy^2z^2 - y^2z^3 + [4x^2y^2z + 2xy^2z^2 + 4xy^2z^2 + 2y^2z^3]$$
$$= -2 \times 2^2 \times 1^2 \times 3 - 2 \times 2 \times 1^2 \times 3^2 - 1^2 \times 3^3$$
$$+ [4 \times 2^2 \times 1^2 \times 3 + 2 \times 2 \times 1^2 \times 3^2 + 4 \times 2 \times 1^2 \times 3^2 + 2 \times 1^2 \times 3^3]$$
$$= -24 - 36 - 27 + [48 + 36 + 72 + 54]$$
$$= -24 - 36 - 27 + 48 + 36 + 72 + 54 = 123$$

$\therefore$ Acceleration $= a_xi + a_yj + a_zk = $ **28i − 3j + 123k. Ans.**

or Resultant acceleration $= \sqrt{28^2 + (-3)^2 + 123^2} = \sqrt{784 + 9 + 15129}$

$= \sqrt{15922} =$ **126.18 units. Ans.**

Problem 5.9 *Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m diameter to 0.2 m diameter over 2 m length. The rate of flow is 20 lit/s. If the rate of flow changes uniformly from 20 l/s to 40 l/s in 30 seconds, find the total acceleration at the middle of the pipe at 15th second.*

Solution. Given :

Diameter at section 1, $D_1 = 0.4$ m ; $D_2 = 0.2$ m, $L = 2$ m, $Q = 20$ l/s $= 0.02\ m^3/s$ as one litre $= 0.001 m^3 = 1000\ cm^3$

Find (*i*) Convective acceleration at middle *i.e.*, at *A* when $Q = 20$ *l*/s.

(*ii*) Total acceleration at *A* when Q changes from 20 *l*/s to 40 *l*/s in 30 seconds.

Case I. In this case, the rate of flow is constant and equal to $0.02\ m^3/s$. The velocity of flow is in *x*-direction only. Hence this is one-dimensional flow and velocity components in *y* and *z* directions are zero or $v = 0$, $z = 0$.

$$\therefore \quad \text{Convective acceleration} = u\frac{\partial u}{\partial y} \text{ only} \quad \ldots(i)$$

Let us find the value of u and $\frac{\partial u}{\partial x}$ at a distance x from inlet

The diameter (D_x) at a distance x from inlet or at section *X-X* is given by,

$$D_x = 0.4 - \frac{0.4 - 0.2}{2} \times x$$

$$= (0.4 - 0.1\,x)\ \text{m}$$

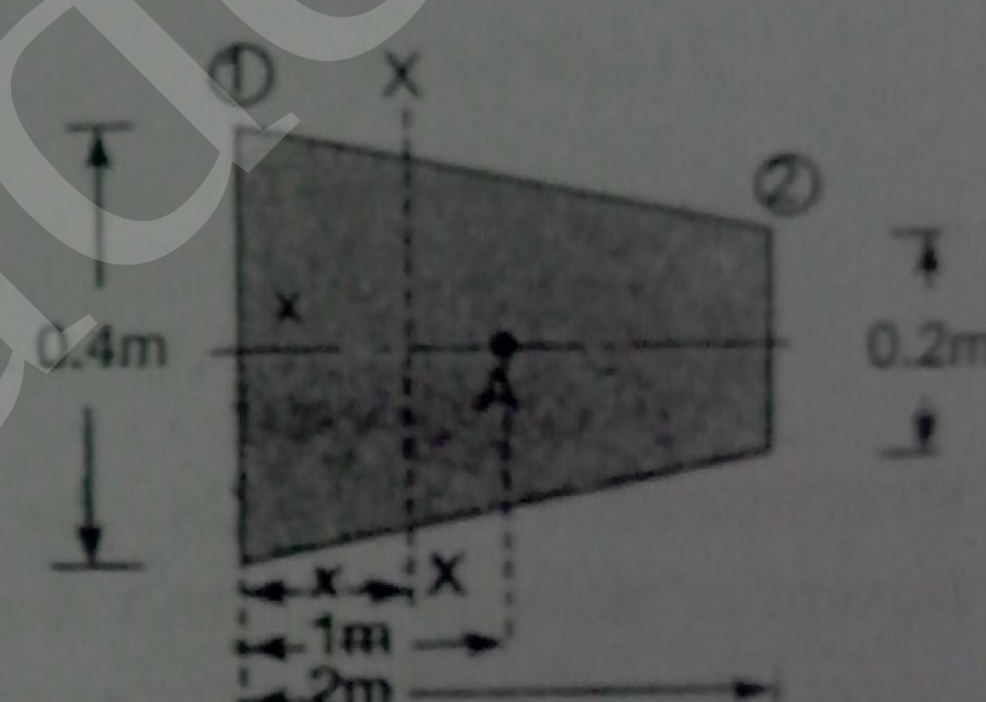

Fig. 5.8

The area of cross-section (A_x) at section *X-X* is given by,

$$A_x = \frac{\pi}{4} D_x^2 = \frac{\pi}{4}(0.4 - 0.1\,x)^2$$

Velocity (u) at the section *X-X* in terms of Q (*i.e.*, in terms of rate of flow)

$$u = \frac{Q}{\text{Area}} = \frac{Q}{A_x} = \frac{Q}{\frac{\pi}{4} D_x^2} = \frac{4Q}{\pi\,(0.4 - 0.1\,x)^2}$$

$$= \frac{1.273Q}{(0.4 - 0.1\,x)^2} = 1.273\,Q\,(0.4 - 0.1\,x)^{-2}\ \text{m/s} \quad \ldots(ii)$$

To find $\frac{\partial u}{\partial x}$, we must differentiate equation (*ii*) with respect to *x*.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[1.273\,Q\,(0.4 - 0.1\,x)^{-2}]$$

$$= 1.273\,Q\,(-2)\,(0.4 - 0.1\,x)^{-1} \times (-0.1) \quad \text{[Here Q is constant]}$$

$$= 0.2546\,Q\,(0.4 - 0.1\,x)^{-1} \quad \ldots(iii)$$

Substituting the value of u and $\frac{\partial u}{\partial x}$ in equation (*i*), we get

Convective acceleration $= [1.273\,Q\,(0.4 - 0.1\,x)^{-2}] \times [0.2546\,Q\,(0.4 - 0.1\,x)^{-1}]$

$= 1.273 \times 0.2546 \times Q^2 \times (0.4 - 0.1\,x)^{-3}$

$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1\,x)^{-3}$ [$\because Q = 0.02$ m^3/s]

$\therefore$ Convective acceleration at the middle (where $x = 1$ m)

$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4 - 0.1 \times 1)^{-3}$ m/s^2

$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.3)^{-3}$ m/s^2

$= 0.0048$ m/s^2. **Ans.**

Case II. When Q changes from 0.02 m^3/s to 0.04 m^3/s in 30 seconds, find the total acceleration at $x = 1$ m and $t = 15$ seconds.

Total acceleration = Convective acceleration + Local acceleration at $t = 15$ seconds.

The rate of flow at $t = 15$ seconds is given by

$$Q = Q_1 + \frac{Q_2 - Q_1}{30} \times 15 \text{ where } Q_2 = 0.04 \text{ m}^3\text{/s and } Q_1 = 0.02 \text{ m}^3\text{/s}$$

$$= 0.02 + \frac{(0.04 - 0.02)}{30} \times 15 = 0.03 \text{ m}^3\text{/s}$$

The velocity (u) and gradient $\left(\frac{\partial u}{\partial x}\right)$ in terms of Q are given by equations (*ii*) and (*iii*) respectively

$\therefore$ Convective acceleration $= u.\dfrac{\partial u}{\partial x}$

$= [1.273\,Q\,(0.4 - 0.1\,x)^{-2}] \times [0.2546\,Q\,(0.4 - 0.1\,x)^{-1}]$

$= 1.273 \times 0.2546\,Q^2 \times (0.4 - 0.1 \times 1)^{-3}$

$\therefore$ Convective acceleration (when $Q = 0.03$ m^3/s and $x = 1$ m)

$= 1.273 \times 0.2546 \times (0.03)^2 \times (0.4 - 0.1 \times 1)^{-3}$

$= 1.273 \times 0.2546 \times (0.03)^2 \times (0.3)^{-3}$ m/s^2

$= 0.0108$ m/s^2 ...(*iv*)

Local acceleration $= \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial t}[1.273\,Q\,(0.4 - 0.1\,x)^{-2}]$

[$\because u$ from equation (*ii*) is $u = 1.273\,Q\,(0.4 - 0.1\,x)^{-2}$]

$= 1.273 \times (0.4 - 0.1\,x)^{-2} \times \dfrac{\partial Q}{\partial t}$

[$\because$ Local acceleration is at a point where x is constant but Q is changing]

Local acceleration (at $x = 1$ m)

$= 1.273 \times (0.4 - 0.1 \times 1)^{-2} \times \dfrac{\partial Q}{\partial t}$

$= 1.273 \times (0.3)^{-2} \times \dfrac{0.02}{30}$ $\left[\because \dfrac{\partial Q}{\partial t} = \dfrac{Q_2 - Q_1}{t} = \dfrac{0.04 - 0.02}{30} = \dfrac{0.02}{30}\right]$

$= 0.00943$ m/s^2 ...(*v*)

Hence adding equations (*iv*) and (*v*), we get total acceleration.

$\therefore$ Total acceleration = Convective acceleration + Local acceleration

$= 0.0108 + 0.00943 = $ **0.02023 m/s^2. Ans.**

5.8 VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

5.8.1 Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \quad \text{...(5.9)}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential function are given by

$$\left.\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \\ u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{aligned}\right\} \quad \text{...(5.9A)}$$

where u_r = velocity component in radial direction (*i.e.*, in r direction)

and u_θ = velocity component in tangential direction (*i.e.*, in θ direction)

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.

Substituting the values of u, v and w from equation (5.9), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad \text{...(5.10)}$$

Equation (5.10) is a Laplace equation.

For two-dimension case, equation (5.10) reduces to $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$ ...(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

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* Please, refer to equation (5.17) on page 192.

(i) $$Q_1 = A_1V_1 = \frac{\pi}{4}(0.45^2) \times 3 = \mathbf{0.477\ m^3/s.}$$

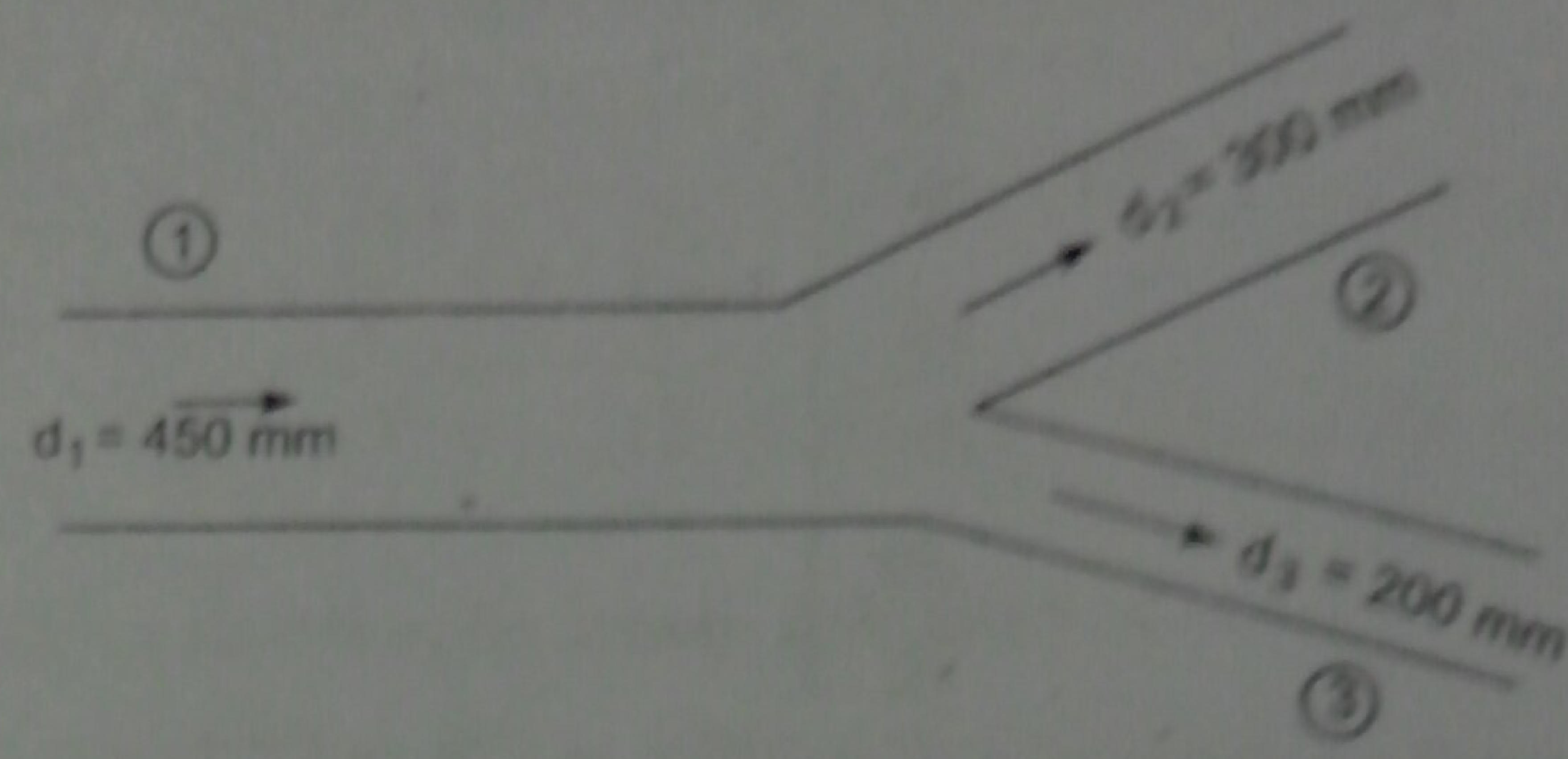

Fig. 5.57

(ii) $$Q_2 = A_2V_2 = \frac{\pi}{4}(.3^2) \times 2.5 = 0.176\ m^3/s$$

But $$Q_1 = Q_2 + Q_3 \quad \therefore \quad Q_3 = Q_1 - Q_2 = 0.477 - 0.176 = 0.301$$

Also $$Q_3 = A_3 \times V_3 = \frac{\pi}{4}(0.2^2) \times V_3$$

$\therefore$ $$V_3 = \frac{Q_3}{\frac{\pi}{4}(0.2^2)} = \frac{0.301}{0.0314} = \mathbf{9.6\ m/s.]}$$