1. (a) Find $y(x)$ if $\frac{dy}{dx} = x^2 y \sec^2(x^3)$, and $y(0) = -3$.
Simplify to obtain the integral equation.

$$\int \frac{1}{y} dy = \int x^2 \sec^2(x^3) dx.$$

Make the u-sub $u = x^3$ and we get

$$\ln y = \frac{1}{3}\tan(x^3) + C.$$

Exponentiate both sides and

$$y = Ae^{\frac{1}{3}\tan(x^3)}$$

Since $y(0) = -3$ we must have $A = -3$.

(b) Find a function $y(x)$ such that $y(1) = -1$ and $xy' + y = y^2$.
Write $y' = \frac{dy}{dx}$ and we can simplify the differential equation to be

$$\frac{1}{y^2 - y} dy = \frac{1}{x} dx.$$

Integrate using partial fractions to write $\frac{1}{y^2-y} = \frac{1}{y-1} - \frac{1}{y}$ and we get

$$\ln(y-1) - \ln(y) = \ln(x) + C.$$

Combine $\ln(y-1) - \ln(y) = \ln\frac{y-1}{y}$ and exponentiate to obtain

$$\frac{y-1}{y} = xe^C.$$

Call $A = e^C$ and solve for y and we have

$$y = \frac{1}{1 - Ax}.$$

Solve the initial condition $y(1) = -1 = \frac{1}{1-Ax}$ and we get $A = 2$.

2. When a cake is removed from an oven, the temperature of the cake is 210 F. The cake is left to cool at room temperature (70 F) and after 30 minutes the temperature of the cake is 140 F. According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the temperature difference between the body and the environment. Set up and solve a differential equation to find when the temperature of the cake will be 100 F.

Let $y(t)$ be the cake temperature t minutes after being removed from the oven. This means $y(0) = 210$ and $y(30) = 140$. We want to solve the differential equation

$$\frac{dy}{dt} = k(y - 70).$$

This means that

$$\int \frac{1}{y-70} dy = \int k dt.$$

Integrating we have

$$\ln(y - 70) = kt + C.$$

Exponentiate and let $A = e^C$ to obtain

$$y - 70 = Ae^{kt}$$

Thus $y = Ae^{kt} + 70$. We still need to solve for k and A. First we find A via $y(0) = 210 = Ae^{k\cdot 0} + 70 = A$. So $A = 140$. We can use this to find k since

$$y(30) = 140 = 140e^{k(30)} + 70.$$

Hence $k = \frac{1}{30}\ln(\frac{1}{2})$. Our final answer is then

$$y(t) = 140e^{\frac{1}{30}\ln(\frac{1}{2})t} + 70.$$

3. At time $t = 0$ a 5000 liter tank is full of pure water. Starting at that moment salt is added to it at a steady rate of 40 grams per hour. Assume that the salt is thoroughly mixed in the water. Meanwhile, pure water is entering the tank at 50 liters per hour and the salty water in the tank is leaving at the same rate. Let $y(t)$ be the amount of salt in grams in the water in the tank after t hours. Compute $\lim_{t\to\infty} y(t)$.

   Let $y(t)$ equal the amount of grams of salt in the tank at time t. So $y(0) = 0$. We obtain the differential equation

   $$\frac{dy}{dt} = 40 - \frac{y \cdot 50}{5000}.$$

   If we separate the variables this gives the integral equation

   $$\int \frac{1}{40 - \frac{y}{100}} dt = \int dt.$$

   Solving with a u-sub $u = 40 - \frac{y}{100}$ we have

   $$-100\ln(40 - \frac{y}{100}) = t + C.$$

   Dividing by (-100) and exponentiating yields

   $$40 - \frac{y}{100} = Ae^{-\frac{t}{100}}.$$

   Solve for y and

   $$y(t) = 40 \cdot 100 - Ae^{-\frac{t}{100}}.$$

   Since $y(0) = 0$ we must have $A = 4000$. We are asked to compute the limit

   $$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} 4000 - 4000e^{-\frac{t}{100}} \\ &= 4000 - 0 \\ &= 4000.\end{aligned}$$

4. A vat contains 100 liters of yogurt. Pure yogurt flows in at 5 L/min. A bacteria colony is growing inside the vat (assume it is mixed uniformly at all times). Mixed yogurt flows out at 5 L/min. Initially there are 1000 bacteria. Let $P(t)$ represent the number of bacteria at time t. Left alone, the bacteria grow at a rate proportional to P (i.e. kP for some constant k).

(a) Find an equation for $\frac{dP}{dt}$ in terms of k and t.

$$\frac{dP}{dt} = kP - \frac{5P}{100} = P(k - \frac{1}{20}).$$

(b) Solve for $P(t)$.

$$P(t) = 1000e^{(k-\frac{1}{20})t}$$

(c) Suppose that left alone the bacteria population doubles every hour. Compute $\lim_{t\to\infty} P(t)$.
Let $B(t)$ be the left alone bacteria after t minutes. We are told that $\frac{dB}{dt} = kB$ and so $B(t) = 1000e^{kt}$. Since the population doubles after 60 minutes we have $B(60) = 2000 = 1000e^{60k}$ and so $k = \frac{\ln 2}{60} \approx .012$. We then have

$$P(t) = 1000e^{(.012-\frac{1}{20})t} = 1000e^{-.038t}$$

And so $\lim_{t\to\infty} P(t) = 0$.

(d) Let k be as in (c). Instead of pure yogurt, say we let bacteria filled yogurt flow in at 5 L/min. What concentration of bacteria flowing in (call it b bacteria/liter) will result in $\lim_{t\to\infty} P(t) = 2000$?
We now have the rate of increase of the bacteria is $kP + 5b$. This means

$$\frac{dP}{dt} = kP + 5b - \frac{5P}{100} = P(k - \frac{1}{20}) + 5b.$$

This gives an integral equation

$$\int \frac{dP}{P(k-\frac{1}{20}) + 5b} = \int dt.$$

This integrates to

$$\frac{1}{k - \frac{1}{20}} \ln(P(k - \frac{1}{20}) + 5b) = t + C.$$

If we exponentiate and find the constant we obtain

$$P(t) = \underbrace{Ae^{(k-\frac{1}{20})t}}_{(I)} - \underbrace{\frac{5b}{k - \frac{1}{20}}}_{(II)}.$$

We know from part (c) that (I) goes to zero as $t \to \infty$. So we want to solve for b in (II)

$$-\frac{5b}{k - \frac{1}{20}} = 2000.$$

This has solution $b = \frac{-2000(k-\frac{1}{20})}{5} \approx 14$ bacteria/liter.