Math 308, Midterm 4 Name: ____________________

Instructions.

- Unless you are explicitly asked to find an inverse, it is okay to write $A^{-1}, P^{-1}, D^{-1}, \ldots$ wherever your calculation would use a matrix.
- It is also okay to introduce shorthand $A = [\vec{u}_1\vec{u}_2\vec{u}_3]$ to denote a matrix if the vectors are known.
- You are allowed a two-sided sheet of notes in your own handwriting.
- No calculators.
- There are 7 problems on 7 pages. Make sure your exam is complete.
- **Any cheating observed during, or noticed afterwards when comparing exams, will result in a 0 on this exam. Moreover, such an exam cannot be dropped. This means you lose 28% percent of your total raw grade. It will be difficult to pass the class if this occurs.**

Question	Points	Score
1	8	
2	7	
3	12	
4	9	
5	6	
6	8	
7	0	
Total:	50	

1. Chris, Matt and Abe are big pizza eaters, and also big on knowing their macro nutrients. They agree on the same kind of pizza dough, $\vec{d}$, but prefer different toppings.

   Matt likes feta cheese $\vec{f}$, and sunflower seeds, $\vec{s}$. Chris prefers egg, $\vec{e}$, and kale, $\vec{k}$. Abe likes pineapple, $\vec{p}$, and oysters, $\vec{o}$.

   These have macro profiles $\begin{bmatrix} \text{Fat} \\ \text{Carbs} \\ \text{Protein} \end{bmatrix} \in \mathbb{R}^3$ and induce bases as follows:

$$\vec{d} = \begin{bmatrix} 3 \\ 42 \\ 7 \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} 21 \\ 4 \\ 14 \end{bmatrix}, \quad \vec{s} = \begin{bmatrix} 50 \\ 24 \\ 19 \end{bmatrix}, \quad \vec{e} = \begin{bmatrix} 10 \\ 1 \\ 13 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}, \quad \vec{o} = \begin{bmatrix} 13 \\ 12 \\ 9 \end{bmatrix}.$$

$$\mathcal{M} = \{\vec{d}, \vec{f}, \vec{s}\},\ M = [\vec{d}\,\vec{f}\,\vec{s}], \qquad \mathcal{C} = \{\vec{d}, \vec{e}, \vec{k}\},\ C = [\vec{d}\,\vec{e}\,\vec{k}], \qquad \mathcal{A} = \{\vec{d}, \vec{p}, \vec{o}\},\ A = [\vec{d}\,\vec{p}\,\vec{o}].$$

   If you need to write an inverse, it is okay to write M^{-1}, C^{-1}, or A^{-1}.

[2 points] (a) What are the entries c_1, c_2, c_3 of $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{\mathcal{M}} = \vec{d}_{\mathcal{M}} + 2\vec{f}_{\mathcal{M}} - \vec{s}_{\mathcal{M}}$?

Solution: $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_{\mathcal{M}}$

[2 points] (b) Write a linear combination in the standard basis for $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{M}} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{C}} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{A}}$.

Solution:

$$\vec{d} + \vec{e} + \vec{o}$$

[2 points] (c) Matt offers Abe a pizza with profile $\vec{m}_{\mathcal{M}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{M}}$. Write an expression that translates this vector into the standard basis.

Solution: $M\vec{m}_{\mathcal{M}} = \vec{m}$

[2 points] (d) Chris wants in too. Write an expression that describes $\vec{m}_{\mathcal{M}}$ from the previous part in terms of Chris's basis.

Solution:

$$\vec{m}_{\mathcal{C}} = C^{-1}M\vec{m}_{\mathcal{M}}.$$

2. The hippo population in a hippo preserve has the following features

   - There are three life stages at which 0, 2 and 1 offspring are produced (on average).

- If $p_{i,j}$ is the probability of reaching stage j from stage i, then $p_{1,2} = 3/4$, and $p_{2,3} = 1/4$.

Recall that the Leslie matrix has the form: $\begin{bmatrix} b_1 & b_2 & b_3 \\ p_{1,2} & 0 & 0 \\ 0 & p_{2,3} & 0 \end{bmatrix}$. Where b_i is the average number of offspring for the stage i hippo, and $p_{i,j}$ is the probability of going from stage i to stage j.

[2 points] (a) Write the Leslie matrix, A, that models the hippo population.

Solution: $A = \begin{pmatrix} 0 & 2 & 1 \\ \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$.

[2 points] (b) Find the characteristic polynomial of A.

Solution: $\det(A - \lambda I) = -\lambda^3 - 2(4/3)\lambda + 1(3/4)(1/4)$

[2 points] (c) The roots are $\lambda_1 = -1.46, \lambda_2 = -.08$ and $\lambda_3 = 1.54$ these have eigenvectors:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -0.52 \\ 0.09 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ -9 \\ 27 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 0.5 \\ 0.08 \end{bmatrix}$$

Write a formula for $A^{13}(1\vec{u}_1 + 2\vec{u}_2 + 3\vec{u}_3)$.

Solution: Let $P = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$ and D be the diagonal matrix of eigenvalues. Then $A^{100} = PD^{100}P^{-1}$.

[1 point] (d) At what rate is the hippo population growing?

Solution:
1.54^k where k is the number of generations.

3. Suppose that $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

[2 points] (a) What are the eigenvalues of A?

Solution: The characteristic polynomial is $(\lambda - 2)(\lambda - 3)(\lambda - 4)$. So $2, 3,$ and 4 are the eigenvalues.

[6 points] (b) For each eigenvalue find a basis for it's eigenspace.

λ_1 :

Solution: $A-2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ So, the eigenspace has dimension 1 and is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which is an eigenvector.

λ_2 :

Solution: $A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ So, the eigenspace has dimension 1 and is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, which is an eigenvector.

λ_3 :

Solution: $A - 4I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ So, the eigenspace has dimension 1 and is spanned by $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$, which is an eigenvector.

[2 points] (c) Write the diagonalized decomposition of e^A. If you need to compute an inverse, it is acceptable to write P^{-1} in place of calculating it, so long as it's clear what P is.

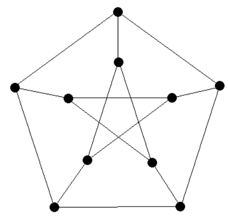
Solution: Let $P = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$ with $\vec{u}_i$ the eigenvectors from the previous part, this gives

$$A = P \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^4 \end{bmatrix} P^{-1}.$$

[2 points] (d) Let $B = e^A$. Write down a basis for the subspace $S = \text{ColumnSpace}(B)$ that uses the eigenvectors of B.

Solution: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3$. It is the same as the eigenvectors from before, because it is the same matrix P for the diagonalization of e^A.

4. A group of 10 websites are linked as:



Let A be the adjacency matrix. Let I denote the 10×10 identity matrix. Let "$B \sim C$" mean "B is row equivalent to C". A computer outputs:

$$A - I \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A - 3I \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A + 2I \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A - 4I \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

[2 points] (a) Just by looking at the matrices above list all of the eigenvalues of A.

Solution: $1, 3, -2$

[3 points] (b) Explain why your list includes every eigenvalue.

Solution: The rank of the the eigenspace of 1 is 5, for 3 it is 1 and for -2 it is 4. Since $5+1+4 = 10$ is the dimension of the matrix, we know we have every eigenvalue.

[2 points] (c) What is an eigenvector of the largest eigenvalue of A? (Write it horizontally.)

Solution: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

[2 points] (d) Explain what your answer to the previous part says about the importance of the websites.

Solution: Every site is equally well connected, so they should be ranked equally well. This is why each entry is a 1.

5. Suppose $A = PDP^{-1}$ is diagonalizable, and that we don't (yet) know P. So,

$$P = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{u}_4], \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Recall Cramer's Rule says that, for C_{ij} the i,jth cofactor of A,

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix}$$

Double recall $C_{i,j}$ is obtained by deleting the ith row and jth column from A then taking $(-1)^{i+j}$ times the the determinant of the resulting submatrix.

[1 point] (a) What is $\det(A)$? *Hint: You may use the fact that* $\det A = $ (*product of the eigenvalues*).

Solution: The diagonal entries of D are the eigenvalues, thus $\det A = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 4 = 4$.

[1 point] (b) What is largest eigenvalue of A?

Solution: 4

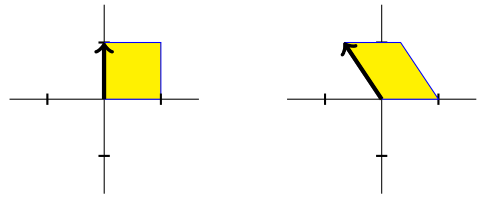
[2 points] (c) Use Cramer's rule to write a formula, in terms of the cofactors, for an eigenvector corresponding to the largest eigenvalue. Don't compute anything. You can write C_{ij} for any cofactors you use in the formula.

Solution: $\vec{u}_4 = \frac{1}{\det P^{-1}} \begin{bmatrix} C_{41} \\ C_{42} \\ C_{43} \\ C_{44} \end{bmatrix}$.

[2 points] (d) What is the third entry of $A\vec{u}_4$ equal to? Actually write a number for this question.

Solution: $A\vec{u}_4 = 4\frac{1}{12} \begin{bmatrix} C_{41} \\ C_{42} \\ C_{43} \\ C_{44} \end{bmatrix}$. So $\frac{1}{3}C_{43} = \frac{1}{3}(-1)^{4+3} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \frac{-1}{3}(3[2]+0+0) = -2$.

6. An animator performs a linear transformation $T(\vec{x}) = A\vec{x}$ and it has the following effect on the unit square. (Note T of the bolded vector on the left is the bolded vector on the right.)



Answer both questions using geometry only. DO NOT do any calculations. DO NOT find A.

[2 points] (a) What is an eigenvalue of A and an eigenvector of A?

Solution:
We see that the x axis is fixed and not distorted. So 1 is an eigenvalue with eigen-vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

[2 points] (b) Is A diagonalizable? ◯ Yes √ **No** ◯ Not enough information

Solution: No. Only the x axis is fixed by this transformation, so the eigenvectors cannot form a basis for $\mathbb{R}^2$.

[2 points] (c) Is A invertible? √ **Yes** ◯ No ◯ Not enough information

[2 points] (d) Which of the following is true?
√ $\det(A) > 0$ ◯ $\det(A) = 0$ ◯ $\det(A) < 0$ ◯ Not enough information

[0 points] 7. Fill in or '✓' whether the statement is true or false. Each correct answer is worth 1 EC point.

(a) If N is an invertible matrix with eigenvector $\vec{v}$, then $\vec{v}$ is also an eigenvector of N^{-1}.
√ **True** ◯ False

(b) If $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation given by $T(\vec{x}) = A\vec{x}$ that fixes a plane in $\mathbb{R}^3$, then A cannot be diagonalizable.
◯ True √ **False**

(c) If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases for $\mathbb{R}^n$, then the change of basis matrix from $\mathcal{B}_2$ to $\mathcal{B}_1$ is invertible.
√ **True** ◯ False

(d) If $\det(M) = 0$, then 0 is an eigenvalue of M.
√ **True** ◯ False

(e) If A and B are equivalent matrices, then it must be the case that $\det(A) = \det(B)$.
◯ True √ **False**

(f) If C is an invertible matrix, then its eigenvalues are distinct.
◯ True √ **False**

(g) For any $n \times n$ matrix J, we can find a basis for $\mathbb{R}^n$ consisting of eigenvectors of J.
◯ True √ **False**

(h) If A is an $n \times n$ matrix such that $\det(A) > 0$, then $\det(-A) < 0$.
◯ True √ **False**

(i) If $\vec{u}$ and $\vec{v}$ are eigenvectors of a matrix W with eigenvalue λ, then so is $\vec{u} + \vec{v}$.
$\surd$ **True** ◯ False

(j) A square matrix A and its transpose will always have the same eigenvalues and eigenvectors.
$\surd$ **True** ◯ False