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INTRODUCERE

In the period May 2013 – December 2015, the team members of the grant PNII-RU-TE-2012-3-0161, with title ”Algebraic study of some combinatorial problems and associated computational experiments”, have written eleven scientific papers [9], [22], [23], [17], [18], [19], [20], [13], [36], [27], [32] from which nine have been published or are accepted for publishing and two are submitted for publishing. At the same time, they have launched seven software packages (in several versions) which are available online [10], [4], [24]. Both for the scientific papers, as well as for the internet pages of the software packages one may find links from the grant web page, at the address https://dl.dropboxusercontent.com/s/lb9znyg9ov2gtkn/ASSCPACE.htm. In the following we present in detail the results obtained.

1. MAIN SCIENTIFIC RESULTS

1. In the paper [22], published in the journal Experimental Mathematics, we introduce an algorithm for computing the Hilbert depth of a finitely generated multigraded module M over the standard multigraded polynomial ring $R = K[X_1,\dots,X_n]$. The algorithm is based on the method presented in [21] and some extra improvements. It may also be adapted for computing the Stanley depth of M if $\dim_K M_a \le 1$ for all $a \in \mathbb{Z}^n$. Further, we provide an experimental implementation of the algorithm [22] in CoCoA [14] and we use it to find interesting examples. As a consequence, we give complete answers to the following open problems proposed by Herzog in [16]:

Problem 1. [16, Problem 1.66] *Find an algorithm to compute the Stanley depth for finitely generated multigraded R-modules M with* $\dim_K M_a \le 1$ *for all* $a \in \mathbb{Z}^n$.

Problem 2. [16, Problem 1.67] *Let M and N be finitely generated multigraded R-modules. Then*

$$\operatorname{sdepth}(M \oplus N) \ge \operatorname{Min}\{\operatorname{sdepth}(M), \operatorname{sdepth}(N)\}.$$

Do we have equality?

Problem 3. [16, Text following Problem 1.67] *In the particular case that* $I \subset R$ *is a monomial ideal, does* $\operatorname{sdepth}(R \oplus I) = \operatorname{sdepth} I$ *hold?*

2. In the paper [27], published in Springer Proceed. in Math. and Statistics, we study the Stanley's Conjecture [40] in the monomial square free case.

Let $S = K[x_1,\dots,x_n]$ be the polynomial algebra over the field K and $J \subset I \subset S$ two squarefree monomial ideals. Suppose that I is generated by some square free monomials $f_1,\dots,f_r$ of degrees d and a set of square free monomials E of degrees $\ge d+1$. Let B be the set of square free monomials from $I\setminus J$ of degrees $d+1$. An easy remark says that if $d=1$ and all monomials from $B\setminus E$ are some least common multiples of two monomials f_i then $\operatorname{depth}_S I/J=3$. It gives a weak form of this result for $d>1$.

The main result of this paper says that if $r\leq 3$ then a weak form of Stanley Conjecture holds, namely if the Stanley depth of I/J is $\leq d+1$ then $\operatorname{depth}_S I/J\leq d+1$. The used method is taken from [30] and uses some important results from [28], [29] and [38]. Finally, it states a weaker form as above in a special case when $r=4$.

3. In the paper [32], published in Bull. Math. Soc. Sci. Math. Roumanie, we extend the results on a weak form of the Stanley's Conjecture [40] from the monomial square free case (obtained in [28], [31]) to the monomial general case by essentially using a recent theorem obtained in [17].

Let $S=K[x_1,\dots,x_n]$ be the polynomial algebra over the field K and $J\subset I\subset S$ two monomial ideals. It is known that if I,J are square free and I is generated by monomials of degrees d then $\operatorname{depth}_S I/J\geq d$. The present paper extends this result for the case when the ideals are not square free.

The lower bound of depth given in this case is obtained from the polarization of a certain canonical form associated to I/J in [5]. Certainly after polarization we get a factor of square free monomial ideals where we may apply the usual lower bound given by degree.

Using this lower bound we get weak forms of Stanley Conjecture which extend those existing in the square free frame [28], [31]. Here it is essential the result of [17] concerning the behavior of the Stanley depth under polarization.

Finally a weak form of Stanley Conjecture is stated for the case when I is generated by 6 variables and J is a square free monomial ideal.

4. In the paper [23], published in Proceedings of the Third Conference of Mathematical Society of Moldova IMCS-50, we make a brief presentation of the program [24].

5. The paper "The behavior of Stanley depth under polarization" (see [17]) is written in collaboration by Bogdan Ichim, Lukas Katthän and Julio Moyano-Fernández, and it was published in Journal of Combinatorial Theory A.

As noted by Stanley himself in [40, p. 191], the Stanley Conjecture was formulated such that "the question raised in (Garsia, Stanley) would follow affirmatively". This question was reformulated by Stanley [41, Conjecture 2.7], and asks whether every Cohen-Macaulay simplicial complex is partitionable. While it is clear that the Stanley Conjecture implies the Garsia-Stanley Conjecture on Cohen-Macaulay simplicial complexes, in this paper we show that the converse is also true, that is, the Garsia-Stanley Conjecture on Cohen-Macaulay simplicial complexes implies what is (arguably) the most important case of the Stanley Conjecture. More precisely, we show that Stanley's Conjecture on Cohen-Macaulay simplicial complexes is equivalent to [2, Conjecture 1].

6. The paper "Stanley depth and the lcm-lattice" (see [18]) is written in collaboration by Bogdan Ichim, Lukas Katthän and Julio Moyano-Fernández, and it is submitted for publication.

A particularly striking aspect of Stanley's conjecture is that it relates two numerical invariants coming from different fields. On the one hand, we have the depth, which is an algebraic invariant (homological in nature), very important in algebraic geometry since its introduction at the beginning of the last century; and on the other hand we have the Stanley depth, which is a purely combinatorial invariant. In this paper we propose the *lcm-lattice* as a common base for the study of both invariants.

In Section 5 the reader will find many applications which will illustrate the power of this approach.

7. The paper "Lcm-lattices and Stanley depth: a first computational approach" (see [18]) is written in collaboration by Bogdan Ichim, Lukas Katthän and Julio Moyano-Fernández, and it was published in the journal Experimental Mathematics.

Let $S = K[x_1,\dots,x_n]$ be the polynomial algebra over the field K. Let I be a monomial ideal of S with up to 5 generators. In this paper, we present a computational experiment which allows us to prove that

$$\operatorname{depth}_S S/I = \operatorname{sdepth}_S S/I < \operatorname{sdepth}_S I.$$

This shows that the Stanley conjecture is true for S/I and I, if I can be generated by at most 5 monomials. The result also brings additional computational evidence for a conjecture made by Herzog.

8. The paper "The power of pyramid decomposition in Normaliz" (see [9]) is written in collaboration by Winfried Bruns, Bogdan Ichim and Christof Söger, and it was published in Journal of Symbolic Computation.

In this paper we describe the use of pyramid decomposition in Normaliz, a software tool for the computation of Hilbert bases and enumerative data of rational cones and affine monoids. Pyramid decomposition in connection with efficient parallelization and streamlined evaluation of simplicial cones has enabled Normaliz to process triangulations of size $\approx 5\cdot 10^{11}$ that arise in the computation of Hilbert series related to combinatorial voting theory. In this article we document the mathematical ideas and the recent development resulting from them. It has extended the scope of Normaliz by several orders of magnitude.

Very recently, Normaliz was benchmarked against other top software available. We illustrate the data obtained with well-known classical examples proposed by independent authors in distinct scientific fields, as listed in the following.

(1) `CondPar`, `CEffPl` and `PlVsCut` come from combinatorial voting theory. `CondPar` represents the Condorcet paradox, `CEffPl` computes the Condorcet efficiency of plurality voting, and `PlVsCut` compares plurality voting to cutoff, all for 4 candidates. See Schürmann [37] for more details.

(2) `4x4`, `5x5` and `6x6` represent monoids of "magic squares": squares of size 4×4, 5×5 and 6×6 to be filled with nonnegative integers in such a way that all rows, columns and the two diagonals sum to the same "magic constant". They belong to the standard LattE distribution [25].

(3) `bo5` and `lo6` belong to the area of statistical ranking; see Sturmfels and Welker [42]. `bo5` represents the boolean model for the symmetric group S_5 and `lo6` represents the linear order model for S_6.

(4) `small` and `big` are test examples used in the development of Normaliz without further importance. `small` has already been discussed in [8].

(5) `cyclo36`, `cyclo38`, `cyclo42` and `cyclo60` represent the cyclotomic monoids of orders 36, 38, 42 and 60. They have been discussed by Beck and Hoşten [6].

(6) `A443` and `A553` represent monoids defined by dimension 2 marginal distributions of dimension 3 contingency tables of sizes $4\times 4\times 3$ and $5\times 5\times 3$. They had been open cases in the classification of Ohsugi and Hibi [33] and were finished in [7].

(7) `cross10`, `cross15` and `cross20` are (the monoids defined by) the cross polytopes of dimensions 10, 15 and 20 contained in the LattE distribution [25].

For Hilbert bases computations we have used the software package 4ti2 [1] developed by a team from the Technische Universität München. The following table contains data from these tests.

We stopped the computations when the time had exceeded 150 h (T) or the memory usage had exceeded 100 GB (R).

Input	4ti2	Nmz -d 1x	Nmz -d 20x	Nmz -N 1x	Nmz -N 20x
CondPar	0.024 s	0.014 s	0.026 s	2.546 s	0.600 s
PlVsCut	6.672 s	0.820 s	0.476 s	–	–
CEffPl	6:08 m	28.488 s	3.092 s	–	–
4x4	0.008 s	0.003 s	0.011 s	0.005 s	0.016 s
5x5	3.823 s	1.004 s	0.339 s	1:06 m	23.714 s
6x6	115:26:31 h	14:19:39 h	1:19:34 h	–	–
bo5	T	–	–	0.273 s	0.174 s
lo6	31:09 m	1:46 m	39.824 s	1:08 m	13:369 s
small	48:19 m	18:45 m	3:25 m	1.935 s	1.878 s
big	T	–	–	1:45 m	15.636 s
cyclo36	T	–	–	0.774 s	0.837 s
cyclo38	R	–	–	6:32:50 h	1:04:04 h
cyclo60	R	–	–	2:55 m	1:02 m
A443	T	–	–	1.015 s	0.270 s
A553	R	–	–	44:11 m	4:24 m

TABELA 1. Computation times for Hilbert bases

For Hilbert series computations we have used the software package [25] developed by a team from the University of California at Davis. The following table contains data from these tests.

Input	LattE ES	LattE+M ES	LattE EP	Nmz 1x	Nmz 20x
CondPar	O	S	–	18.085 s	8.949 s
PlVsCut	O	S	–	–	145:43:03 h
CEffPl	O	S	–	–	197:45:10 h
4x4	0.329 s	4.152 s	–	0.006 s	0.018 s
5x5	O	72:39:23 h	–	3:59 m	1:12 m
bo5	T	T	T	82:40:18 h	6:41:12 h
lo6	R	R	T	13:02:44 h	1:21:52 h
small	46.266 s	30:15 m	22.849 s	0.233 s	0.095 s
big	R	R	10.246 s	1.473 s	0.148 s
cyclo36	R	R	23:03 m	1.142 s	1.106 s
cyclo38	R	R	R	26.442 s	22.789 s
cyclo42	R	R	1:44:07 h	3.942 s	1.521 s
cyclo60	R	R	T	5:57 m	1:44 m
A443	R	R	R	49.541 s	18.519 s
A553	R	R	T	88:21:18 h	6:29:05 h
cross10	T	T	9.550 s	0.016 s	0.022 s
cross15	R	R	21:48 m	0.536 s	0.533 s
cross20	R	R	R	26.678 s	26.029 s

TABELA 2. Computation times for Hilbert series and Hilbert polynomials

We have stopped the computations when the time exceeded 150 hours (T), the memory usage was more than 100 GB RAM (R) or it has produced more than 400 GB of output (O). These limitation were imposed by the system available for testing. In two cases it has exceeded the system stack limit; this is marked by S.

9. The paper ”Binomial fibers and indispensable binomials” (see [13]) is written in collaboration by Hara Charalambous, Apostolos Thoma and Marius Vladoiu and it was published in Journal of Symbolic Computation.

Let $I \subset R = K[x_1,\ldots,x_n]$ be an arbitrary ideal generated by binomials, with K a field. The authors show that certain equivalence classes of fibers are associated to any minimal binomial generating set of I. They provide a polynomial-time algorithm (see [13, Algorithm 1]) to compute the indispensable binomials of a binomial ideal from a given generating set of binomials and an algorithm [13, Algorithm 2] to detect whether a binomial ideal is generated by indispensable binomials. To this end we recall that a binomial is called *indispensable* if (up to a nonzero constant) it belongs to every minimal generating set of I consisting of binomials. This implies of course that (up to a nonzero constant) it belongs to every binomial generating set of I.

A related question that attracted a lot of interest in the recent years is whether there is a unique minimal binomial generating set for a binomial ideal. One of the first papers to deal with this question for lattice ideals from a purely theoretical point of view was [35] and it turns out that the positive answer has applications to Algebraic Statistics. The authors prove that an arbitrary system of binomial generators of I gives all necessary information to decide whether a given binomial is indispensable, see [13, Theorem 3.3]. As an immediate application of [13, Theorem 3.3] the authors obtain a polynomial-time algorithm which computes the indispensable elements of a binomial ideal I, given any system of binomial generators of I, see [13, Algorithm 1]. This algorithm bypasses the computation of a reduced Gröbner basis unlike the previous methods. We recall that the other methods were:

- the algorithm in [33, Theorem 2.4] implies computation of $n!$ reduced Gröbner bases with respect to the lexicographic orders,
- the algorithm in [12, Theorem 3.4] implies the computation of one Gröbner basis and the knowledge of the minimal elements in the set of I-fibers,
- the algorithm in [34, Theorem 13] implies computation of n reduced Gröbner basis with respect to n degree reverse lexicographic orders.

Finally the authors also show, that to decide whether a minimal system of binomial generators of a binomial ideal is in fact a system of indispensable binomials it is enough to compute the cardinality of the minimal generating set of an associated monomial ideal, see [13, Corollary 3.6] and the resulting algorithm [13, Algorithm 2].

10. The paper ”Bouquet algebra of toric ideals” (see [36]) is written in collaboration by Sonja Petrović, Apostolos Thoma and Marius Vladoiu and it is submitted for publication.

To any toric ideal I_A, encoded by an integer matrix A, we associate a matroid structure called *the bouquet graph* of A and introduce another toric ideal called *the bouquet ideal* of A. It is shown how these objects capture the essential combinatorial and algebraic information about I_A. Passing from the toric ideal to its bouquet ideal reveals a structure that allows us to classify several cases. For example, on the one end of the spectrum, there are ideals that we call *stable*, for which bouquets capture the complexity of various generating sets and the minimal free resolution. On the other end of the spectrum lie toric ideals whose various bases (e.g. minimal Markov, Gröbner, Graver bases) coincide.

Apart from allowing for classification-type results, bouquets provide a way to construct families of examples of toric ideals with various interesting properties, e.g., robust, generic, unimodular. The new bouquet framework can be used to provide a characterization of toric ideals whose Graver basis, the universal Gröbner basis, any reduced Gröbner basis and any minimal generating set coincide. The authors also show that the toric ideal of a general matrix A can be encoded by that of a $0/1$ matrix while preserving complexity of its bases. Along the way, the authors answer two open problems for toric ideals of hypergraphs.

11. The paper "How to compute the Stanley depth of a module" (see [20]) is written in collaboration by Bogdan Ichim, Lukas Katthän and Julio Moyano-Fernández, and it was accepted for publication in the journal Mathematics of Computation.

In this paper we introduce an algorithm for computing the Stanley depth of a finitely generated multigraded module M over the standard multigraded polynomial ring $R = K[X_1,\dots,X_n]$. The algorithm is based on the methods presented in [21], [22] and some extra improvements. It may also be adapted for computing the Stanley depth of M if $\dim_K M_a \le 1$ for all $a \in \mathbb{Z}^n$. As a consequence, we give complete answers to the following open questions proposed by Herzog in [16]:

Question 4. [16, Question 1.65] *Does there exist an algorithm to compute the Stanley depth of finitely generated multigraded R-modules?*

Question 5. [16, Question 1.63] *Let M be a finitely generated multigraded R-module with syzygy module Z_k for $k = 1,2,\dots$. Is it true that* $\operatorname{sdepth} Z_{k+1} \ge \operatorname{sdepth} Z_k$*?*

2. Parallel Scientific Results

In June 2013 we have released the version 2.10 of the program Normaliz (see [10]) together with the graphical interface jNormaliz version 1.4 (see [4]). In November 2013 we have released the first version of the program Hdepth [24], this being the first program available for computing the multigraded Hilbert depth. In April 2014 we have released the version 2.11 of the program Normaliz (see [10]) together with the graphical interface jNormaliz version 1.5 (see [4]). In October 2014 we have released the version 2.12 of the program Normaliz (see [10]) together with the graphical interface jNormaliz version 1.6 (see [4]). In September 2015 we have released the graphical interface jNormaliz version 1.7 (see [4]).

3. Results Presentation

In order to disseminate the scientific results more than twenty presentations were held by this grant team within the algebra seminars of the Simion Stoilow Institute of Mathematics of the Romanian Academy. Four presentations were held at the National School on Algebra "Algebraic Methods in Combinatorics", Bucuresti, 01–07 Septembrie 2013; two presentations were held at the National School on Algebra "Algebraic and Combinatorial Applications of Toric Ideals", Bucuresti, 31 August – 06 Septembrie 2014 by Marius Vladoiu and Andrei Zarojanu (they were also members of the organizing committee) and one presentations was held at the National School on Algebra "Interactions of Computer Algebra with Commutative Algebra, Combinatorics and Algebraic Statistics", Bucuresti, 30 August – 05 Septembrie 2015 by Marius Vladoiu (he was also member of the organizing committee). Four presentations were held at The Eighth Congress of Romanian Mathematicians, Iasi, 26 June – 01 July 2015 by Bogdan Ichim, Dorin Popescu, Marius Vladoiu and Andrei Zarojanu.

Bogdan Ichim has participated at the conference "Joint International Meeting of the American Mathematical Society and the Romanian Mathematical Society" where he gave a talk with the title "How to compute the multigraded Hilbert depth of a module" at the University of Alba Iulia on 28-06-2013 and at the conference "AMS–EMS–SPM International Meeting" where he gave a talk with the title "How to compute the Stanley depth of a module" at the University of Porto on 11-06-2015;

Bogdan Ichim has participated at five research stages abroad, four in Spain and one in Germany, which ended up with the writing of five articles in collaboration with professors from the host universities [9], [17], [18], [19] and [20], all of them being published or submitted for publication. He gave a talk with the title ”An algorithm for computing the multigraded Hilbert depth of a module” at the University of Osnabrück on 05-11-2013, a talk with the title ”Recent results in Computational Voting Theory” at the conference ”Trends in Commutative Algebra” at the Jaume I University on 18-09-2014. a talk with the title ”How to compute the Stanley depth of a module?” at the Jaume I University on 18-03-2014 and a talk with the title ”On Hilbert decompositions” at the conference ” Topics in Monoid Theory and its applications,” at the Jaume I University on 24-09-2015.

Dorin Popescu presented his results at the conference in the honor of Constantin Nastasescu organized by the University of Bucharest in Mai 2013, at the conference ”Experimental and Theoretical Methods in Algebra, Geometry, and Topology”, Eforie Nord, 20–24 June, 2013 and at the 150 years anniversary conference organized by the University of Bucharest, 30 August – 1 September, 2013.

Marius Vladoiu gave a talk with the title ”Markov bases of lattice ideals” at the conference ”Meeting On Combinatorial Commutative Algebra 2014” on 12-09-2014 and a talk withe the title ”Bouquet Algebra of Toric Ideals” at the University of Bilkent on 30-03-2015.

Andrei Zarojanu gave a talk with the title ”An introduction to Hilbert depth” at the conference ”The Third Conference of Mathematical Society of the Republic of Moldova ICMS-50” on 20-08-2014.

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