The General Binomial Theorem: Part 2

In the previous post we established the general binomial theorem using Taylor's theorem which uses derivatives in a crucial manner. In this post we present another approach to the general binomial theorem by studying more about the properties of the *binomial series* itself. Needless to say, this approach requires some basic understanding about infinite series and we will assume that the reader is familiar with ideas of convergence/divergence of an infinite series and some of the tests for convergence of a series.

Apart from the basic understanding of infinite series we will require a fundamental theorem concerning the multiplication of two infinite series. This we present next.

Multiplication of Infinite Series

Consider two infinite series

$$\sum a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

and

$$\sum b_n = b_1 + b_2 + \cdots + b_n + \cdots$$

If we multiply them term by term (i.e multiply each term of one series by first term of another series, then multiply each term of the first series by second term of the second series and so on) we get a sum (whose meaning we are yet to define) like the following

$$\begin{aligned}
&a_1b_1 + a_1b_2 + a_1b_3 + \cdots +\\
&a_2b_1 + a_2b_2 + a_2b_3 + \cdots +\\
&a_3b_1 + a_3b_2 + a_3b_3 + \cdots +\\
&\quad\cdots + \cdots +\\
&a_nb_1 + a_nb_2 + \cdots +\\
&\quad\cdots + \cdots +
\end{aligned}$$

In order to give this 2 dimensional expression a meaning it is better to regroup the terms and arrange them in linear fashion to make a new infinite series. One particularly nice way to group terms is to add the diagonal entries and get terms like $(a_1b_1), (a_1b_2 + a_2b_1), \cdots$ and in general the n^{th} group is

$$c_n = a_1b_n + a_2b_{n-1} + \cdots + a_{n-1}b_2 + a_nb_1$$

and then we have another series

$$\sum c_n = c_1 + c_2 + \cdots + c_n + \cdots$$

and since the series c_n eventually captures all the terms which are obtained by multiplying the terms of series $\sum a_n$ and $\sum b_n$ it is reasonable to expect that the sum of series $\sum c_n$ will be the product of sums of $\sum a_n$ and $\sum b_n$. This is in fact true under very general circumstances and we will establish a result to that effect. Before we proceed to do that we first establish certain lemmas on sequences.

Lemma 1: *If* $\lim_{n\to\infty} a_n = A$ *then*

$$\lim_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n}=A$$

Let $t_n = a_n - A$ so that $t_n \to 0$ and we can see that

$$\frac{a_1+a_2+\cdots+a_n}{n}=\frac{t_1+t_2+\cdots+t_n}{n}+A$$

and hence it suffices to prove that

$$\frac{t_1+t_2+\cdots+t_n}{n}\to 0$$

under the condition that $t_n \to 0$. Consider $s_n = [\sqrt{n}]$ (greatest integer not exceeding $\sqrt{n}$) so that s_n is an integer and $s_n \to \infty, s_n/n \to 0$ as $n \to \infty$. Let $\epsilon > 0$ be given. Since $t_n \to 0$ there is a positive integer n_0 such that $|t_n| < \epsilon/2$ for all $n \geq n_0$. Further since $s_n \to \infty$ it follows that there is a positive integer m_1 such that $s_n > n_0$ for $n \geq m_1$. Thus it follows that $|t_n| < \epsilon/2$ for $n \geq s_{m_1}$. Again $t_n \to 0$ hence the sequence t_n is bounded and let $|t_n| < K$ for all n. Further note that $s_n/n \to 0$ hence there is a positive integer m_2 such that $|s_n/n| < \epsilon/2K$ for all $n \geq m_2$. Let

$$T_n=\frac{t_1+t_2+\cdots+t_n}{n}$$

Then for $n \geq m = \max(m_1, m_2)$

$$\begin{aligned}|T_n| &= \left|\frac{t_1+t_2+\cdots+t_n}{n}\right|\\ &\leq \frac{|t_1|+|t_2|+\cdots+|t_{s_m}|}{n}+\frac{|t_{s_m+1}|+|t_{s_m+2}|+\cdots+|t_n|}{n}\\ &< \frac{s_m K}{n}+\frac{n-s_m}{n}\cdot\frac{\epsilon}{2}\\ &< \frac{s_n}{n}\cdot K+\frac{\epsilon}{2}\\ &< \frac{\epsilon}{2K}\cdot K+\frac{\epsilon}{2}\\ &= \epsilon\end{aligned}$$

and therefore $T_n \to 0$ as $n \to \infty$. This completes the proof of the lemma.

Using this lemma we prove that:

Lemma 2: *If* $a_n \to A, b_n \to B$ *as* $n \to \infty$ *then*

$$\lim_{n\to\infty}\frac{a_1b_n+a_2b_{n-1}+\cdots+a_{n-1}b_2+a_nb_1}{n}=AB$$

Let $a_n = A + t_n$ so that $t_n \to 0$ as $n \to \infty$. Then

$$\begin{aligned}c_n &= \frac{a_1b_n+a_2b_{n-1}+\cdots+a_{n-1}b_2+a_nb_1}{n}\\ &= A\cdot\frac{b_1+b_2+\cdots+b_n}{n}+\frac{t_1b_n+t_2b_{n-1}+\cdots+t_{n-1}b_2+t_nb_1}{n}\end{aligned}$$

Since $b_n \to B$ it is bounded by some K so that $|b_n| < K$ for all n. Now the first term in the above equation tends to AB (by Lemma 1) and the absolute value of second fraction is less than

$$K \cdot \frac{|t_1| + |t_2| + \cdots + |t_n|}{n}$$

which also tends to $K \cdot 0 = 0$ by Lemma 1. Hence it follows that $c_n \to AB$ as $n \to \infty$.

We are now ready to prove Abel's theorem on multiplication of infinite series:

Abel's Theorem: *Let a_n, b_n be any two sequences and let*

$$\begin{aligned} A_n &= a_1 + a_2 + \cdots + a_n \\ B_n &= b_1 + b_2 + \cdots + b_n \\ c_n &= a_1 b_n + a_2 b_{n-1} + \cdots + a_{n-1} b_2 + a_n b_1 \\ C_n &= c_1 + c_2 + \cdots + c_n \end{aligned}$$

If $A_n \to A, B_n \to B, C_n \to C$ as $n \to \infty$ then $C = AB$.

The theorem says that if $\sum a_n, \sum b_n$ are two convergent series and $\sum c_n$ is the series formed by term by term multiplication of $\sum a_n, \sum b_n$ (with terms grouped in a manner explained above) and further if $\sum c_n$ is convergent then $\sum c_n = (\sum a_n)(\sum b_n)$.

The proof of the theorem is based on the simple idea that we can write $a_n = A_n - A_{n-1}, b_n = B_n - B_{n-1}$ for $n > 1$ and $a_1 = A_1, b_1 = B_1$. We make the convention that $A_0 = B_0 = 0$ so that the above relations hold for all n. Using these relations it is easily proven that

$$\begin{aligned} C_n &= a_1 B_n + a_2 B_{n-1} + \cdots + a_n B_1 \\ &= b_1 A_n + b_2 A_{n-1} + \cdots + b_n A_1 \\ \sum_{i=1}^{n} C_i &= A_1 B_n + A_2 B_{n-1} + \cdots + A_n B_1 \end{aligned}$$

By lemma 2 it is clear that

$$\frac{1}{n} \cdot \sum_{i=1}^{n} C_i = \frac{A_1 B_n + A_2 B_{n-1} + \cdots + A_n B_1}{n} \to AB$$

and further $C_n \to C$ therefore by lemma 1

$$\frac{1}{n} \cdot \sum_{i=1}^{n} C_i \to C$$

and therefore $C = AB$.

Multiplication of Power Series

The above rule for multiplication of infinite series can be used to multiply special kinds of series called power series. A *power series* is of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

where a_n is a sequence and x is a variable. If for some values of x lying in a certain set A the above series is convergent then its sum defines a function from $A \to \mathbb{R}$. Clearly the series is convergent if $x = 0$ and it is possible that it may be convergent for other values of x. Power series have a special property that if they are convergent for $x = r$ then they are convergent for all values of x with $|x| < |r|$. This is not difficult to prove.

Let the series (1) be convergence for $x = r \neq 0$. Then $a_n r^n \to 0$ and hence $|a_n r^n| < K$ for all n and some K. Let $|x| = r' < |r|$ then

$$|a_n x^n| = |a_n| r|^n| \left(\frac{r'}{|r|}\right)^n < K \left(\frac{r'}{|r|}\right)^n$$

and the series is convergent by comparison with the geometric series $\sum (r'/|r|)^n$.

Thus the region of convergence of a power series is always an interval of the form $(-R, R)$ where R may also be ∞. Moreover the power series may or may not converge for $x = \pm R$ and then we say that R is the *radius of convergence* (the word radius comes due the reason that if x is treated as a complex variable then the region of convergence turns out to be a circle with radius R).

Using the rule for multiplication of infinite series we can multiply two power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, g(x) = \sum_{n=0}^{\infty} b_n x^n$$

to get another power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0$$

Whether this new series is convergent or not needs to be determined separately.

Exponential Property of the Binomial Series

We will now study a very important property of the binomial series

$$f(x, n) = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots = \sum_{m=0}^{\infty} \binom{n}{m} x^m \tag{2}$$

where n is a real number. If p, q are real and we multiply the two binomial series $f(x, p), f(x, q)$ then we get another series

$$h(x) = \sum_{m=0}^{\infty} c_m x^m$$

where

$$c_m = \binom{p}{0}\binom{q}{m} + \binom{p}{1}\binom{q}{m-1} + \cdots + \binom{p}{m-1}\binom{q}{1} + \binom{p}{m}\binom{q}{0} \tag{3}$$

The challenge is to evaluate the coefficient c_m in terms of a simple formula.

Note that if p, q are positive integers then both the series $f(x, p), f(x, q)$ turn out to be finite as $\binom{p}{m}, \binom{q}{m}$ become 0 as m exceeds p and q. Moreover in this case we know that

$$f(x, p) = (1 + x)^p, f(x, q) = (1 + x)^q$$

and therefore

$$f(x, p)f(x, q) = (1 + x)^{p+q}$$

and the coefficient of x^m in $(1 + x)^{p+q}$ is $\binom{p+q}{m}$ (it will become 0 as soon as m exceeds $p + q$). Thus it follows from multiplication of series that

$$\binom{p}{0}\binom{q}{m} + \binom{p}{1}\binom{q}{m-1} + \cdots + \binom{p}{m-1}\binom{q}{1} + \binom{p}{m}\binom{q}{0} = \binom{p+q}{m}$$

for all positive integers p, q and all non-negative integers m. Note that by definition the expression $\binom{p}{m}$ is a polynomial in p and thus we see that the above equation is an identity between polynomials of two variables p, q which holds for all positive integral values of p, q. Thus this identity must be true identically for all real values of p, q and thus the evaluation of c_m from equation (3) is complete and we have

$$c_m = \binom{p}{0}\binom{q}{m} + \binom{p}{1}\binom{q}{m-1} + \cdots + \binom{p}{m-1}\binom{q}{1} + \binom{p}{m}\binom{q}{0} = \binom{p+q}{m} \tag{4}$$

for all real values of p, q. The series $h(x) = \sum c_m x^m$ thus turns out to be a binomial series $f(x, p + q)$. From Abel's theorem on multiplication of infinite series it follows that

$$f(x, p)f(x, q) = f(x, p + q)$$

when all the three binomial series involved are convergent. By ratio test it is easily proved that binomial series is convergent whenever $|x| < 1$. Thus we have the following result:

The binomial series

$$f(x, n) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots = \sum_{m=0}^{\infty}\binom{n}{m}x^m$$

is convergent for all values of x with $|x| < 1$ and if $|x| < 1$ then it satisfies the following functional equation

$$f(x, p)f(x, q) = f(x, p + q) \tag{5}$$

for all real values of p, q.

The above functional equation reminds us of the property satisfied by exponential function namely $F(x+y) = F(x)F(y)$ as far as the parameter p of $f(x, p)$ is concerned. By simple algebraic arguments we can thus establish that

$$f(x, n) = \{f(x, 1)\}^n = (1+x)^n$$

for all rational values of n. This completes the proof of general binomial theorem when index n is rational. The extension to irrational values of n is achieved by noting that $f(x, n)$ is a continuous function of n for a fixed value of x with $|x| < 1$.

Behavior of Binomial Series at $x = \pm 1$

For the sake of completeness it is best to discuss the behavior of the binomial series $f(x, n)$ in equation (1) for $x = \pm 1$. Let's first consider $x = 1$ and then we need to check the convergence (and find the sum if it is convergent) of the series

$$f(1, n) = 1 + \binom{n}{1} + \binom{n}{2} + \cdots = \sum_{m=0}^{\infty} \binom{n}{m}$$

The general term of this series

$$a_m = \binom{n}{m} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}$$

and clearly if $n \leq -1$ then

$$\left|\binom{n}{m}\right| \geq 1$$

and thus the general term of the series does not tend to 0. Therefore the binomial series does not converge if $n \leq -1$. If $n > -1$ then we can see that

$$\frac{a_{m+1}}{a_m} = \frac{n-m}{(m+1)} = -\left(1 - \frac{n+1}{m+1}\right)$$

so that a_m ultimately alternates in sign and since $n > -1$ the terms decrease in absolute value after a certain value of m. Next we note that

$$\log\left|\frac{a_{m+1}}{a_m}\right| = \log\left(\frac{m-n}{m+1}\right) = \log\left(1 - \frac{n+1}{m+1}\right) < -\frac{n+1}{m+1}$$

and hence

$$\log\left|\frac{a_{m+p}}{a_{m+1}}\right| < -(n+1)\sum_{i=1}^{p} \frac{1}{m+i}$$

and the expression on right tends to $-\infty$ if $p \to \infty$. It follows that $a_{m+p} \to 0$ as $p \to \infty$. Thus $a_m \to 0$ as $m \to \infty$. It follows from Leibniz test for alternating series that the series under discussion is convergent if $n > -1$. To calculate its sum we need to apply Taylor's series on the function $g(x) = (1+x)^n$ and we have

$$2^n = g(1) = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m-1} + R_m$$

where the remainder R_m in Lagrange's form is given by

$$R_m = \frac{g^{(m)}(\theta)}{m!} = \binom{n}{m}(1+\theta)^{n-m}$$

and hence for $m > n$ we have

$$|R_m| < \left|\binom{n}{m}\right| = |a_m|$$

and since $a_m \to 0$ as $m \to \infty$ for $n > -1$, it follows that $R_m \to 0$ as $m \to \infty$ for $n > -1$. Thus we have the following result:

The binomial series

$$f(x, n) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{m}x^m + \cdots$$

for $x = 1$ is convergent only when $n > -1$ and then its sum as expected is equal to 2^n.

Next we consider the behavior of series $f(x, n)$ for $x = -1$. If we write $n = -p$ then we can see that the general term is given by

$$(-1)^m\binom{n}{m} = (-1)^m\binom{-p}{m} = \frac{p(p+1)\cdots(p+m-1)}{m!}$$

and we can sum the series directly to get

$$\begin{aligned}1 + p + \frac{p(p+1)}{2!} + \cdots + \frac{p(p+1)\cdots(p+m-1)}{m!}\\ = \frac{(p+1)(p+2)\cdots(p+m)}{m!}\\ = (-1)^m\binom{n-1}{m}\end{aligned}$$

and we have seen earlier that the final expression converges if and only if $n - 1 > -1$ i.e. $n > 0$ and then it tends to 0 as $m \to \infty$. Hence the binomial series for $x = -1$ is convergent if and only if $n > 0$ and then its sum is 0.

To summarize, *the binomial series*

$$f(x, n) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

has the following behavior:

- *it is convergent for all real real values of x with $|x| < 1$ and all real values of n and its sum is $(1+x)^n$.*
- *for $x = 1$ it is convergent for all real values of n with $n > -1$ and its sum is 2^n.*
- *for $x = -1$ it is convergent for all real values of n with $n > 0$ and its sum is 0.*
- *for any other combination of real values of x, n the series does not converge.*

By Paramanand Singh

Saturday, July 23, 2016

Labels: Mathematical Analysis