Matt's Special Challenge #1

1 2.4.38

> ***Problem Statement:*** Prove that the function $H(t) = \begin{cases} 0, & t < 0 \\ 1, & t = 0 \end{cases}$ has no limit as $t \to 0$.

Proof. Suppose to show a contradiction that $\lim_{t\to 0} H(t) = L$ with L some real number. By definition this means for $\epsilon = \frac{1}{2}$ there exists a positive number $\delta > 0$ such that for all $|t-0| = |t| < \delta$ it holds that $|H(t) - L| < \frac{1}{2}$. Since $-\delta/2$ and $\delta/2$ both have absolute value less than δ it follows that

$$|H(-\delta/2) - L| < \frac{1}{2} \qquad \text{and} \qquad |H(\delta/2) - L| < \frac{1}{2}.$$

But since δ is positive we know that $H(-\delta/2) = 0$ and $H(\delta/2) = 1$. The above line then says that

$$|0 - L| < \frac{1}{2} \qquad \text{and} \qquad |1 - L| < \frac{1}{2}.$$

Rewriting the absolute value signs, this implies that

$$-\frac{1}{2} < L < \frac{1}{2} \qquad \text{and} \qquad -\frac{1}{2} < 1 - L < \frac{1}{2}.$$

The right inequality can be rewritten as $\frac{1}{2} < L < \frac{3}{2}$ by subtracting 1 from both sides and multiplying by (-1) (which flips the inequalities) we arrive at the contradiction

$$-\frac{1}{2} < L < \frac{1}{2} \qquad \text{and} \qquad \frac{1}{2} < L < \frac{3}{2}.$$

So, the limit cannot exist (since L would have to both be strictly smaller than $\frac{1}{2}$ and strictly larger than $\frac{1}{2}$.) □

2 2.4.44

> ***Problem Statement:*** Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$ with c a constant.
>
> (a) Prove that $\lim_{x\to a}[f(x) + g(x)] = \infty$.
>
> (b) Prove that if $c > 0$ then $\lim_{x\to a}[f(x)g(x)] = \infty$.

Part (a)

Proof. Since $\lim_{x\to a} g(x) = c$, let δ_1 be such that for all $|x-a| < \delta_1$ we have $|g(x)-c| < 1$ (notice we are letting $\epsilon = 1$ and choosing the corresponding δ_1). This give us a bound on $g(x)$; for $|x-a| < \delta_1$ it must be the case that

$$c - 1 < g(x) < c + 1.$$

Fix an arbitrary positive number M. Let $M_2 = M + c - 1$. Because $\lim_{x\to a} f(x) = \infty$, we know that there exists a number $\delta_2 > 0$ such that for all $|x - a| < \delta_2$ it holds that $f(x) > M_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ (that is let δ equal whichever is smallest of δ_1 and δ_2). We now have for $|x - a| < \delta$ it holds that

$$f(x) + g(x) > M_2 + g(x) > M_2 - (c - 1) = M + (c - 1) - (c - 1) = M.$$

We have shown that for all $|x - a| < \delta$ it holds that $|f(x) + g(x)| > M$, which is precisely the definition that $\lim_{x\to a}[f(x) + g(x)] = \infty$. □

Part (b)

Proof. Since $c > 0$ we let $\epsilon = \frac{c}{2}$ and let δ_1 be such that for $|x - a| < \delta_1$ it holds that $|g(x) - c| < \frac{c}{2}$. As in Part (a) this guarantees that

$$c - \frac{c}{2} < g(x) < c + \frac{c}{2}.$$

In particular, when $|x - a| < \delta_1$ we know that g is positive and therefore we can ignore the absolute value signs and write

$$0 < \frac{c}{2} < g(x) < \frac{3c}{2}.$$

Fix a number $M > 0$. Let $M_2 = M \cdot \left(\frac{2}{c}\right)$. Since $\lim_{x\to\infty} f(x) = \infty$ we let δ_2 be such that for all $|x - a| < \delta_2$ it holds that $f(x) > M_2$. Letting $\delta = \min\{\delta_1, \delta_2\}$ we now have for $|x - a| < \delta$

$$f(x)g(x) > M_2 \left(\frac{c}{2}\right) = M \left(\frac{2}{c}\right) \left(\frac{c}{2}\right) = M.$$

Hence $\lim_{x\to\infty} f(x)g(x) = \infty$. □