#3: Calculating Limits, Continuity
Chapters 2.3, 2.5

Contents

1 Calculating Limits

1.1 Intro

We started by computing limits by looking at values of x that are close to the limiting value and inferring what value the function approaches. However, we saw with the example $g(x) = \sin(1/x)$ that this doesn't always work.

Goal: Be able to calculate $\lim_{x\to a} f(x)$ for a variety of functions.

1.2 Limit Laws

Suppose that f and g are functions with $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} g(x) = L_2$. With L_1, L_2 real numbers. The following equalities hold:

1. $\lim\limits_{x\to a}[f(x) + g(x)] = L_1 + L_2$
2. $\lim\limits_{x\to a} cf(x) = cL_1$
3. $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x) = L_1 \cdot L_2$
4. $\lim\limits_{x\to a} \frac{f(x)}{g(x} = \frac{L_1}{L_2}$

1.3 Direct Substitution Property

If f is a polynomial, rational, root, trig, exponential or logarithmic function and a is in the domain of f then

$$\lim_{x\to a} f(x) = f(a)$$

(Note: the domain of a function are the numbers x for which $f(x)$ is defined to have a value.
Example:

$$\lim_{x\to 1} \frac{\pi \cos(\pi x) \sin((\pi/2)x)}{e^x} + x = -\frac{\pi}{e} + 1$$

1.4 When Direct Sub Doesn't Work

Direct sub fails when the limiting value, a, isn't in the domain. As far as we are concerned there are three ways this can happen:

1. $f(a)$ involves dividing by zero. THIS NEVER MAKES SENSE and is the most common. We will shortly show how to get around this.
2. a simply is not included in the domain.
3. $\ln(0)$ is not defined.

Example:

Let $f(x) = \frac{x^2-1}{x-1}$ and take $\lim_{x\to 1} f(x)$. Divide by zero, so 1 is not in the domain of f. But, we can rewrite as

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1 = g(x)$$

Where $g(x)$ is a new function with 1 in its domain. Notice that $\lim_{x\to 1} g(x) = 2$.

WRABD: We can say that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$ because when we take a limit we are only interested in the values near but not equal to 1.

Diagram

This has a name and will be one of the most commonly used tools for computing limits.

Equal Almost Everywhere Property

If $f(x) = g(x)$ when $x \neq a$ then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$. Notice, however, that in our example above $f \neq g$, they are different functions since they have different domains.

1.5 Name of the Game (with limits)

What is going to happen (almost all of time) is you will be given a function that isn't defined at say $x = a$. Most often because we are dividing by zero in some way. Your task will be to use various algebra tricks and function know-how to be able to apply the equal almost everywhere property (EAEP).

1.5.1 Four Common Tricks

Example:

Conjugate:

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}.$$

We use the fact that $a^2 - b^2 = (a-b)(a+b)$

Computation

Divide and Conquer:

$$\lim_{x\to -2} \frac{x^4 + x^3 - 2x^2}{x^3 + 2x^2} = -3$$

Computation

Attack from All Sides (use left and right limits)

$$\lim_{x\to 0} |x|$$

Computation

Combine Fractions

$$\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

Computation

1.6 DNE

1.6.1 Disproving a Limit

Sometimes we want to show that a limit doesn't exist. *Attack From Both Sides* is the way to go.
Example:

$$\lim_{x\to 0} \frac{1}{x}$$

Computation

1.6.2 Patching Up Piecewise Functions

Sometimes you need to use left and right limits to decide if a piecewise function has a limit. Example:

$$g(t) = \begin{cases} \sqrt{t+c}, & t \geq -2 \\ \sin(\pi t), & t > -2 \end{cases}$$

Find c so that the $\lim_{t \to -2} g(t)$ exists.

Computation

1.7 Squeeze Theorem

Fact: If $f(x) \leq g(x)$ when x is near a then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$. We can use this fact in the squeeze theorem.

Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = L$.

Example:

$f(x) = x \sin(1/x)$, find $\lim_{x \to 0} f(x)$.
Notice that we can't break it up into products (since $\lim_{x \to 0} \sin(1/x) = DNE$). So we squeeze it!

Computation

2 Continuity

Continuity is formally all about limits. The intuitive notion of continuity is that the graph is a curve that can be drawn without picking up your pen. Try it out!

2.1 Definition of Continuity

Definition: A function f is **continuous** at a point a if

1. a is in the domain of f.
2. $\lim_{x \to a} f(x) = f(a)$.

Definition: If a function is not continuous at a point a we say that it is **discontinuous at** a.

Definition: There are three types of discontinuities:

(a) Removable

(b) Jump

(c) Infinity

Diagram

2.2 Left/Right Continuity

Oftentimes we look at continuity on intervals, i.e. $(-\infty, \infty)$, $[1, 3)$, $[0, 1]$ and we need to decide if a function is continuous at the endpoints of the interval.

Definition:

(a) f is **right continuous** at a if

- (a) a is in the domain of f.
- (b) $\lim_{x \to a^+} f(x) = f(a)$.

(b) f is **left continuous** at a if

- (a) a is in the domain of f.
- (b) $\lim_{x \to a^-} f(x) = f(a)$.

Definition: A function f is continuous on an interval $[a, b]$ if f is continuous at all points in (a, b) and right continuous at $x = a$ and left continuous at $x = b$.

Example:

$$f(x) = \begin{cases} 1, & x = 1 \\ x^2, & -1 < x < 1 \\ -1, & x = -1 \end{cases}$$

This function is continuous on $(-1, 1]$ but not continuous on $[-1, 1]$. This is true since

$$\lim_{x \to -1^-} f(x) = 1 \neq f(-1) = -1.$$

2.3 Nice Functions

Fact: The following types of functions are continuous at every point in their domain

1. Polynomials
2. Roots
3. Trig
4. Inverse Trig
5. Exponential
6. Logarithms

! Piecewise (or multipart) functions are not necessarily continuous.

Example:

$$f(x) = \begin{cases} \sin x, & x > 0 \\ \cos x, & x \leq 0 \end{cases}$$

Fact: Limits commute with composition of continuous functions. This means that if f and g are continuous at a then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Example:

$$\lim_{t \to 1} \arctan\left(\frac{x^2 - x}{x - 1}\right)$$

Computation

Fact: It follows from the previous fact that if f and g are continuous at a then the function $f(g(x))$ is continuous at a. (We check two things:)

Computation