12. Electronic polarizability of an atom is proportional to
(a) radius (b) $(radius)^2$ (c) $(radius)^3$ (d) $\sqrt{radius}$

13. Choose the correct relation for orientation polarization
(a) $\alpha_0 = \mu^2 kT$ (b) $\alpha_0 = \mu^2/kT$ (c) $\alpha_0 = \frac{3kT}{\mu^2}$ (d) $\alpha_0 = \frac{\mu^2}{3kT}$

14. Choose the correct relation for electronic polarization
(a) $\alpha_e = \frac{\varepsilon_0(\varepsilon_r+1)}{N}$ (b) $\alpha_e = \frac{\varepsilon_0(\varepsilon_r-1)}{N}$
(c) $\alpha_e = \frac{\varepsilon_r(\varepsilon_0-1)}{N}$ (d) $\alpha_e = \frac{\varepsilon_r(\varepsilon_0+1)}{N}$

15. The electric field which a dipole experiences in a medium is
(a) called the local field (b) called the internal field
(c) [illegible] (d) equal to external field

16. Expression for internal field (Lorentz field) is given by
(a) $E_i = E - \frac{P}{3\varepsilon_0}$ (b) $E_i = E + \frac{3P}{\varepsilon_0}$
(c) $E_i = E - \frac{3P}{\varepsilon_0}$ (d) $E_i = E + \frac{P}{3\varepsilon_0}$

17. Clausius-Mosotti equation with usual meaning of notations can be written as
(a) $\frac{\varepsilon_r+2}{\varepsilon_r-1} = N\alpha_e$ (b) $\frac{\varepsilon_r+1}{\varepsilon_r-2} = \frac{N\alpha_e}{3\varepsilon_0}$
(c) $\frac{\varepsilon_r-1}{\varepsilon_r+2} = \frac{N\alpha_e}{3\varepsilon_0}$ (d) $\frac{\varepsilon_r-1}{\varepsilon_r+2} = \frac{N\varepsilon_0}{3\alpha_e}$

18. The materials in which spontaneous polarization changes with temperature is known as
(a) piezo-electric (b) ferro-electric (c) pyro-electric (d) magneto-optic

19. The property of becoming electrically polarized when mechanical stress is applied is known as
(a) ferro-electric (b) piezo-electric (c) pyro-electric (d) none of these

20. Hysteresis in polarization-electric field relation is exhibited by:
(a) piezo-electric material (b) ferro-electric material
(c) pyro-electric material (d) electro-optic material

21. Piezo-electricity [illegible]
(a) increase in charge (b) decrease in charge
(c) disappearance of charge (d) distortion of charge symmetry

22. Dielectric medium should have the property
(a) high dielectric constant (b) high specific resistance
(c) high dielectric strength (d) all

ANSWERS

1. (b)	2. (d)	3. (a)	4. (a)	5. (c)	6. (d)	7. (d)	8. (a)
9. (b)	10. (a)	11. (d)	12. (c)	13. (d)	14. (b)	15. (c)	16. (d)
17. (c)	18. (c)	19. (b)	20. (b)	21. (d)	22. (d)		

13 DEVELOPMENT OF QUANTUM MECHANICS

INTRODUCTION

Towards the end of 17th century, Newton proposed [illegible] of light. According to this theory, light consists of minute fast moving [illegible] *corpuscles**. The phenomena of interference, diffraction, polarisation, [illegible] on the basis of corpuscular theory. To explain these phenomena [illegible] theory of light. According to wave theory, the light travels in the form of [illegible]. Wave theory of light successfully explained the phenomenon of [illegible], polarisation, etc. The wave theory was followed in 1864 by [illegible]. There are certain other experimentally observed phenomenon, [illegible], Compton effect, emission and absorption of light, etc. which could not be [illegible] theories. These phenomena gave birth to the [illegible].

In order to explain the shape of the black [illegible] Max Planck in 1900 proposed the quantum theory. According to this theory, matter is [illegible] of a large number of oscillating particles which vibrate with different [illegible] classical theory the particles can have any value of frequency (or vibrational energy). [illegible] quantum theory, the energy of oscillating particle is quantized. The oscillating particle [illegible] any energy but only those energies given by

[illegible]

where, h (6.625×10^{-34} J/sec) is the [illegible] and n [illegible] number which can take only integer values ([illegible], 2, 3, ... etc). In this [illegible] assumed that the vibrating particles does not radiate energy continuously but [illegible] discrete *quanta* or *photon*. So long as the oscillator remains in one the quantized [illegible] absorbs any energy. When the oscillator moves from one quantized [illegible] quantized state it emits energy. Thus, when n changes by one i.e., $\Delta n = 1$ the [illegible]

$$\Delta E = \Delta n \cdot h\nu = h\nu$$

This shows that although the oscillator [illegible] only in terms of quanta of magnitude $h\nu$.

*A Biological term, used for minute body or cell [illegible] constituent of matter or light.

While studying black body radiations, Max Planck in 1901 concluded that the emission and absorption of thermal energy is not a continuous process but it takes place in discrete amounts, *i.e.*, an integral multiple of a certain energy unit $h\nu$. According to this postulate, the exchange of energy between light and matter is not continuous, but it is small ***bundles*** or ***packets*** or ***quanta*** of definite energy proportional to frequency of light. These small packets of energy are called ***photons***. The photons propagate like particles with the speed of light.

13.2 BLACK BODY RADIATION AND DISTRIBUTION LAWS

Black Body

A ***black body*** is a body which absorbs radiations of all wavelengths incident upon it. It neither reflects nor transmits any of the incident radiation and therefore appears black whatever be the colour of incident radiation. When a black body is heated, it emits radiation which are known as black body radiations. No actual body is a perfect black body but lamp black may be regarded as black body as it absorbs nearly 99% of incident radiation. Here, we shall describe a black body devised by Ferry. It consists of a hollow thick-walled sphere painted lamp black internally and provided a small circular opening to enter the radiation. In front of the opening, there is a projection to prevent the direct reflection of radiation from inner surface. When any radiation enters the opening suffers multiple reflections inside the sphere and is finally absorbed. Similarly, when the walls of such a cavity are heated to temperature T, the radiation emitted fills the cavity and then leaves from the opening. These are called ***black body radiation*** and is a characteristic of its temperature. Here it should be remembered that thermal radiation and light radiation are identical. The only difference is of wavelength. The wavelength of visible light is smaller than that of thermal radiation.

Spectral Distribution

We know that a black body emits radiations of all possible wavelengths. It was a problem to the scientists that how the energy is distributed among different wavelengths. First of all Stefan gave the fourth power law. ***According to Stefan's law the total amount of radiant energy by a black body per unit area per second due to all wavelengths is directly proportional to the fourth power of absolute temperature, i.e.,***

$$E \propto T^4 \quad \text{or} \quad E = \sigma T^4$$

where σ = Stefan's constant $= 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

Actually this law gives the total energy radiated by a black body and does not throw any light on energy distribution, of course, this is experimentally verified. The distribution of energy in black body radiation for different wavelengths and at various temperatures was determined experimentally by Lummer and Pringsheim in 1899.

A graph was then drawn between the intensity E_λ and wavelength λ at different temperatures. The curves so obtained are shown in Fig. (1).

The experimental results are as follows:

1. The emission from a black body at any temperature is composed of radiation of all wavelengths.
2. At a given temperature, the energy is not uniformly distributed. As the temperature of the black body increases, the intensity of radiation for each wavelength increases. This shows that the total amount of energy radiated per unit area per unit time increases with rise of temperature.

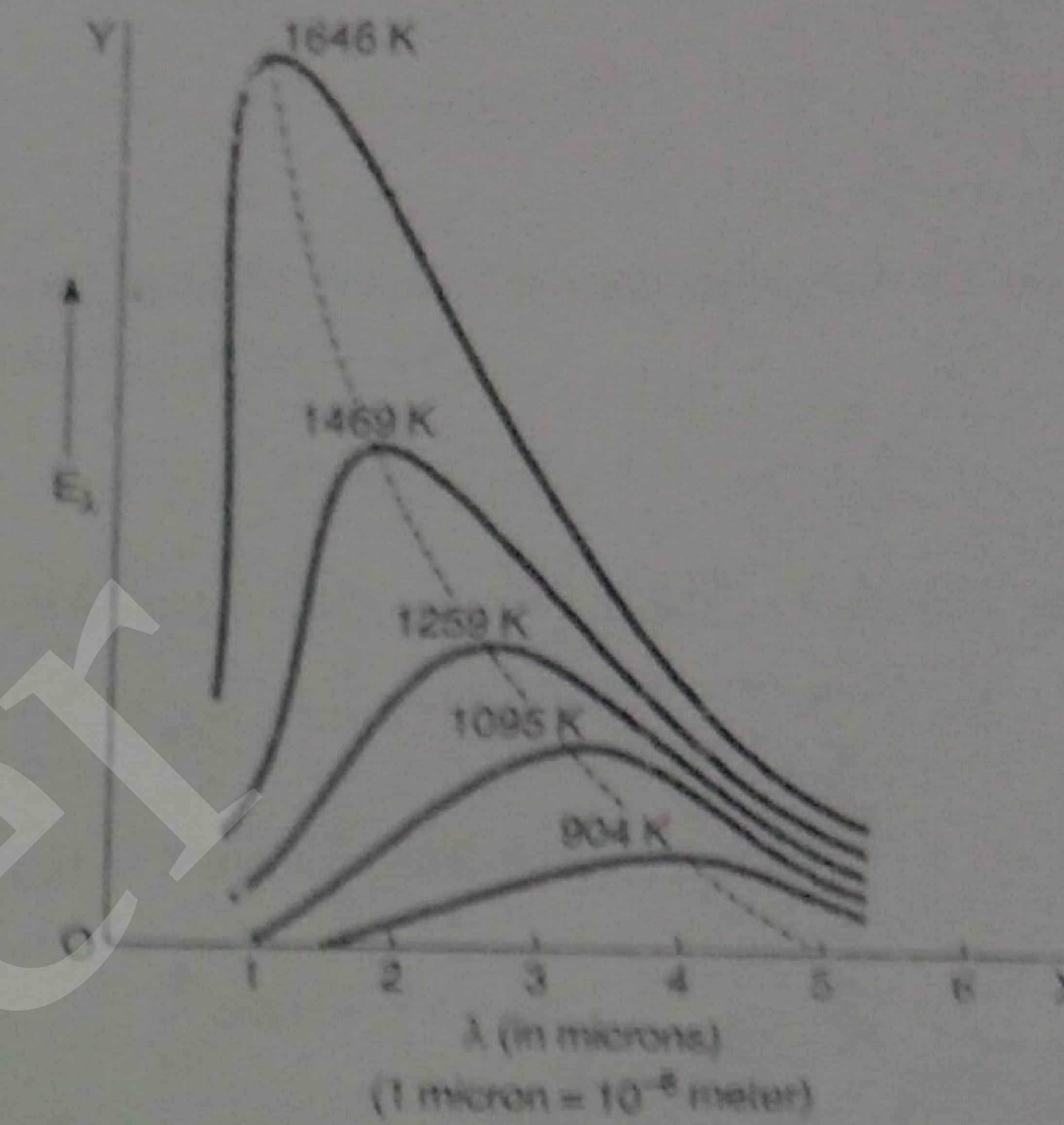


Fig. (1)

3. The total energy of radiation at any temperature is given by the area between the curve corresponding to that temperature and the horizontal axis. The increase in area is found in accordance with the Stefan's law.

$$E = \int_0^\infty E_\lambda \, d\lambda = \sigma T^4$$

4. The amount of radiant energy emitted is small at very short and very long wavelengths. At a particular temperature, the spectral radiancy E_λ is maximum at particular wavelength λ_m. Most of the energy is emitted at wavelengths not very different from λ_m.
5. The wavelength corresponding to the maximum energy represented by the peak of the curve shifts towards shorter wavelengths as the temperature increases. This is called Wien's displacement law. According to this law

$$\lambda_m \times T = \text{constant}$$

This shows that as the temperature is increased, the black body emits the radiation of shorter wavelengths such that the product of temperature T and maximum wavelength λ_m is a constant.

Different Laws

1. Wien's Law

In 1896, while investigating the energy distribution over different wavelengths, Wien showed that the maximum energy point shifts towards the shorter wavelengths side when the temperature of the body is raised. He showed that

$$\boxed{\lambda_m T = \text{constant}}$$

where λ_m is the wavelength corresponding to maximum energy emission from a black body at absolute temperature T. He also showed that the maximum energy emitted by a black body is proportional to the fifth power of its absolute temperature. Hence,

$$(E_\lambda)_{max} \propto T^5 \quad \text{or} \quad (E_\lambda)_{max} / T^5 = \text{constant}$$

This is called as Wien's displacement law.

Wien by applying Maxwell's law for distribution of velocities and the principle of equipartition of kinetic energy gave the expression for E_λ as

$$E_\lambda = C_1 \lambda^{-5} e^{-C_2/\lambda T}$$

where C_1 and C_2 are constants.

2. Rayleigh-Jean's Law

According to Rayleigh-Jean's law, the energy distribution in the thermal spectrum is given by

$$\boxed{E_\lambda = \frac{8\pi k T}{\lambda^4}}$$

where k is Boltzmann constant.

3. Planck's Law

On the basis of quantum theory, Planck's derived the following formula for the energy distribution in thermal spectrum

$$\boxed{E_\lambda = \frac{8\pi h c}{\lambda^5 (e^{hc/\lambda k T} - 1)}}$$

where h is Planck's constant, c, the velocity of light and k being the Boltzmann constant.

It should be remembered that Wien's formula agrees in short wavelengths region while Rayleigh-Jeans formula agrees for long wavelengths region. Planck's formula covers the entire range.

13.3 WIEN'S DISPLACEMENT LAW

We have studied that when radiation from a black body is passed through a prism, a continuous spectrum is obtained. The energy is distributed in various wavelengths varying from zero to infinity. The law that connects the intensity with wavelength is known as the law of distribution of intensity of black body radiation. Wien deduced thermodynamically that as the temperature of the body is raised, the maximum energy tends to be associated with shorter wavelengths. ***According to Wien's displacement law, the product of wavelength corresponding to maximum energy λ_m and the absolute temperature T is constant, i.e.,***

$$\boxed{\lambda_m T = \text{constant}}$$

This constant is called as Wien's displacement constant and has a value 0.2896×10^{-2} m K.

Derivation of formula

Consider a spherical enclosure with perfectly reflecting walls and capable of expansion radially outwards like a football bladder. Let this is filled with black body radiation of energy density u at a temperature T. If V be the volume of enclosure, then total internal energy U of radiation is given by

$$U = uV \qquad \ldots(1)$$

Let us imagine that the walls of enclosure move outward such that radiation inside it expands adiabatically. Let dV be the change in volume and p, the pressure of radiation on the walls of enclosure. Now, the workdone by the pressure of radiation on the walls of enclosure will be $p\,dV$. This is drawn from the internal energy of the radiation. Let dU be the decrease of internal energy of radiation. Then, from first law of thermodynamics,

$$dU + p\,dV = dQ = 0 \qquad \ldots(2)$$

($\because dQ = 0$ as the change is adiabatic)

By electromagnetic theory, the radiation exerts a pressure p on the wall of the enclosure which is equal to one-third of the energy density, *i.e.*,

$$P = \frac{1}{3} u \qquad \ldots(3)$$

Substituting the value of U and p from eqs. (1) and (3) in eq. (2), we get

$$d(uV) + \frac{1}{3} u\,dV = 0$$

or $$u\,dV + V\,du + \frac{1}{3} u\,dV = 0$$

or $$\frac{4}{3} u\,dV + V\,du = 0$$

Dividing by uV, we have

$$\frac{4}{3}\frac{dV}{V} + \frac{du}{u} = 0$$

Integrating the above expression, we have

$$\frac{4}{3}\log_e V + \log_e u = 0 \quad \text{or} \quad V^{4/3} u = \text{constant}$$

According to Stefan's law, $u = \sigma T^4$ where σ is Stefan's constant.

$\therefore$ $$V^{4/3} (\sigma T^4) = \text{constant}$$

or $$V^{1/3} T = \text{constant} \qquad \ldots(A)$$

The radiation filled in the enclosure will undergo continuous reflections from the moving walls of the enclosure. Due to Doppler's effect, the wavelength will be changed. The change in wavelength can be calculated as follows :

As shown in Fig. (2), let OA be a ray of wavelength λ incident at an angle θ on the wall in position S_1. A particular wave crest strikes the wall at A and is reflected along AC. Let $AC = \lambda$. As the reflected wave crest reaches at C, the next crest will reach at A in time T, T being the period of wave motion. When the next crest reaches A, the wall has moved a distance AM, equal to vT where v is the velocity of the expansion of the wall. Now, the crest is reflected from the point B of the new position S_2 of the wall. The crest is reflected in the direction BD. Let $AB + BD = \lambda_1$.

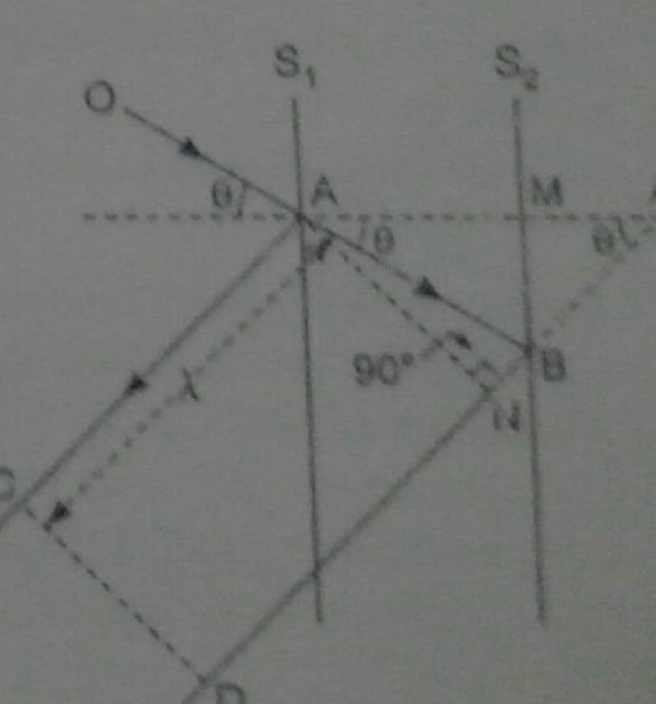


Fig. (2)

The change in wavelength,

$$d\lambda = \lambda_1 - \lambda = (AB + BD) - AC$$

$$= (AB + BN + ND) - AC$$

$$= AB + BN \qquad (\because ND = AC)$$

$$= A'B + BN \qquad (\because AB = A'B)$$

$$= A'N = AA' \cos\theta$$

$$= 2\,AM\cos\theta = 2\,vT\cos\theta$$

$$= 2\frac{v\lambda}{c}\cos\theta \qquad \left(\because T = \frac{\lambda}{c}\right) \qquad \text{...(B)}$$

where c is velocity of radiation.

Every ray inside the spherical enclosure undergoes repeated reflections. The path of a single ray is shown in Fig. (3). It is obvious from the figure that between two successive reflections, say at A and B, the wave travels a distance $2r\cos\theta$.

$\therefore$ number of reflections per sec:

$$= \frac{c}{2r\cos\theta}$$

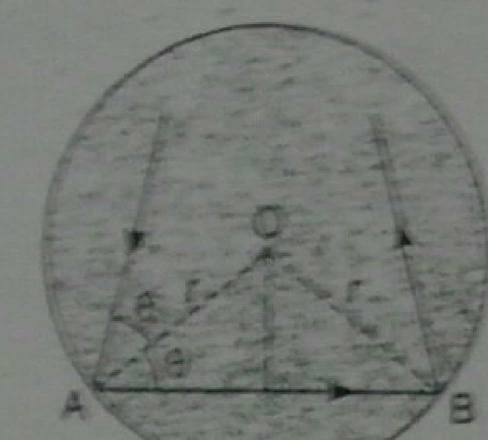


Fig. (3)

and Number of reflection in time dt

$$= \frac{c\,dt}{2r\cos\theta}$$

The change in wavelength in time dt is given by

$d\lambda$ = change in wavelength in one reflection × number of reflections in time dt

$$\therefore \quad d\lambda = \frac{2v\lambda\cos\theta}{c} \times \frac{c\,dt}{2r\cos\theta} = \frac{v\lambda}{r}\cdot dt$$

or
$$\frac{d\lambda}{\lambda} = \frac{v\,dt}{r} = \frac{dr}{r} \qquad \left(\because v = \frac{dr}{dt}\right)$$

Integrating this expression

$$\log_e \lambda = \log_e r + \log_e C \text{ (a constant)}$$

or
$$\lambda = rC \quad \text{or} \quad (\lambda/r) = \text{constant} \qquad \text{...(C)}$$

From eq. (A), $V^{1/3}\,T = \text{constant}$

or
$$\left(\frac{4}{3}\pi r^3\right)^{1/3} T = \text{constant} \qquad \left(\because V = \frac{4}{3}\pi r^3\right)$$

or
$$rT = \text{constant} \qquad \text{...(D)}$$

From eqs. (C) amd (D), we get

$$\frac{\lambda}{r} \times rT = \text{constant}$$

or
$$\boxed{\lambda T = \text{Constant} = \lambda' T'} \qquad \text{...(E)}$$

Thus, if radiation of a particular wavelength at a certain temperature is adiabatically altered to another wavelength, then the temperature changes in the inverse ratio. In the mathematical form, this relation indicates that as the temperature is raised, the maximum intensity of radiation emitted is displaced towards the shorter wavelength side. This is called Wien's displacement law.

Another form of Wien's displacement law

Let us now suppose that the waves of wavelengths lying between λ and $\lambda + d\lambda$ in the spherical chamber are isolated and are allowed to undergo adiabatic expansion. Its energy U will be $U_\lambda\ d\lambda$. Now, λ as well as $d\lambda$ will change. Again, from first law of thermodynamics,

$$d(U_\lambda\ d\lambda) + p\,dV = 0$$

or
$$d(u_\lambda V\,d\lambda) + \frac{1}{3}u_\lambda\,d\lambda\,dV = 0 \qquad \left(\because p = \frac{1}{3}u_\lambda\,d\lambda\right)$$

or
$$V\,d(u_\lambda\,d\lambda) + u_\lambda\,d\lambda\,dV + \frac{1}{3}u_\lambda\,d\lambda\,dV = 0$$

or
$$V\,d(u_\lambda\,d\lambda) + \frac{4}{3}u_\lambda\,d\lambda\,dV = 0$$

or
$$\frac{d(u_\lambda\,d\lambda)}{u_\lambda\,d\lambda} = -\frac{4}{3}\frac{dV}{V} = -4\frac{dr}{r}^*$$

Integrating this expression, we get

$$\log_e (u_\lambda\,d\lambda) + 4\log_e r = \log_e C \text{ (constant)}$$

or
$$u_\lambda\,d\lambda\,r^4 = \text{constant}$$

or
$$u_\lambda\,dr\,r^4 = \text{constant} \qquad (\because \lambda \propto r \text{ or } d\lambda \propto dr)$$

Integrating over r, we get [See eq. (C)]

$$u_\lambda\,r^5 = \text{constant}$$

or
$$u_\lambda\,T^{-5} = \text{constant} \qquad \text{(See eq. D)}$$

We know that spectral radiancy E_λ is proportional to energy density u_λ. Therefore,

$$\boxed{E_\lambda T^{-5} = \text{constant} = E_\lambda'(T')^{-5}} \qquad \text{...(E)}$$

$$\therefore \quad \frac{E_\lambda}{E_\lambda'} = \frac{T^5}{T'^5}$$

13.4 RAYLEIGH-JEANS FORMULA

Rayleigh and Jeans developed a theory for the spectral distribution of black body radiation by the application of electrodynamics and statistical mechanics. Consider a hollow cubic enclosure of side l with perfectly reflecting walls. Let us place a black particle inside the enclosure. The radiations emitted by the particle will be reflected by the walls.

According to electromagnetic theory, the radiation is supposed to consists of number of waves. The waves in the enclosure travel in all possible directions. They undergo multiple reflections from the various walls of the enclosure. In course of time, the enclosure will be filled with the stationary waves of all wavelengths. The reason is that reflected wave interferes with the corresponding incident wave to form stationary wave similar to sound waves. For example, when a wire of length l is fixed at both ends and plucked at its mid point, the stationary waves are formed with nodes at the

* $$V = \frac{4}{3}\pi r^3$$

$$dV = \frac{4}{3}\pi(3r^2)\,dr$$

or
$$\frac{dV}{V} = 3\frac{dr}{r}$$

fixed points, as shown in Fig. (4). If the wire vibrates in n loops due to formation of stationary waves, then

$$n\frac{\lambda}{2} = l \quad \text{or} \quad \lambda = (2\,l/n) \quad \text{where } n = 1, 2, 3 \ldots \infty \qquad \ldots(1)$$

Similar is the case with an enclosure filled with radiation [Fig. (5)]. If l be the distance between the walls, then the corresponding allowed frequencies (overtones) are

$$\nu = \frac{c}{\lambda} = \frac{c\,n}{2\,l} \quad \text{where } n = 1, 2, 3 \ldots \infty \qquad \ldots(2)$$

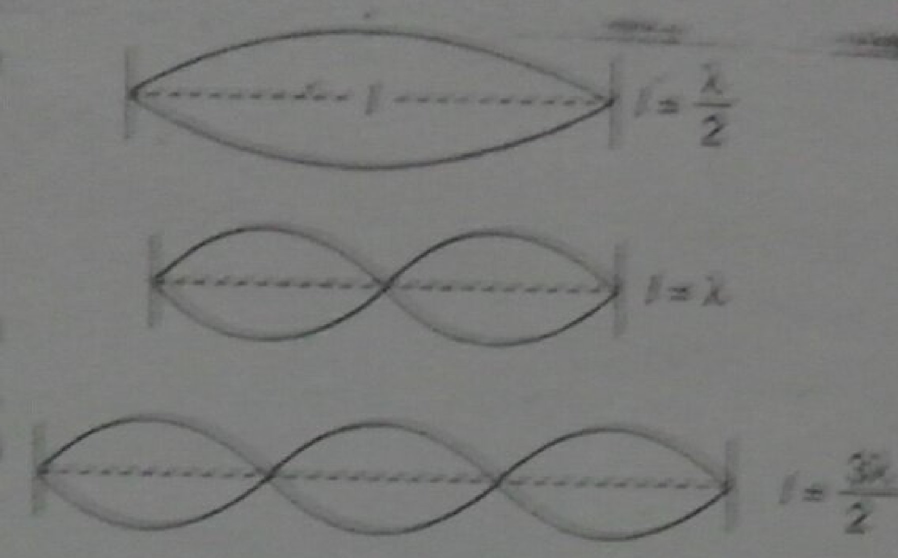


Fig. (4)

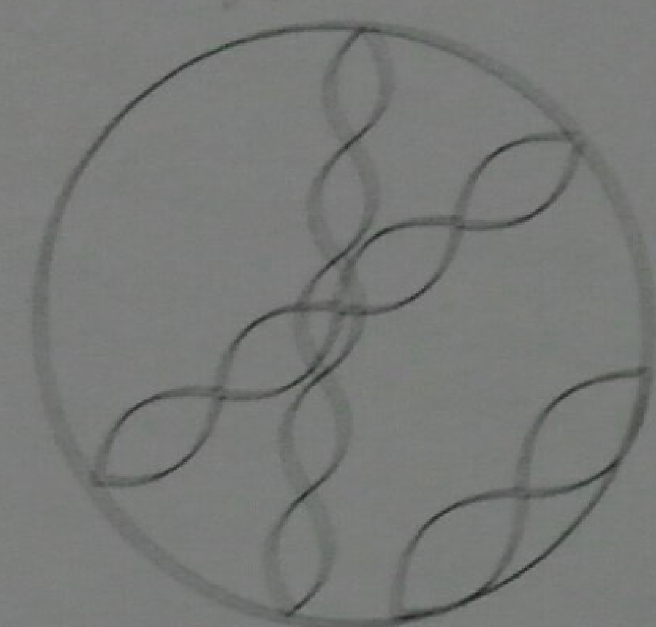


Fig. (5)

Here, c is velocity of light.

The number of overtones may be referred to as the several degrees of freedom with which the system is capable of vibrating. Moreover, every allowed frequency is called a *mode of vibration*.

If the three edges of the cube from the three axes in the space, the number of loops, in each direction n_x, n_y and n_z are given by

$$n_x = \frac{2\,l}{\lambda} \qquad \text{from eq. (1)}$$

$$\therefore \quad l = n_x\,(\lambda/2) = n_y\,(\lambda/2) = n_z\,(\lambda/2) \qquad \ldots(3)$$

As the radiation are diffuse, the waves can be inclined at any angle to the rectangular axes. For waves making an angle α, β and γ with three axes [see Fig. (6)], we have

$$\left.\begin{aligned} l\cos\alpha &= n_x\,(\lambda/2) \\ l\cos\beta &= n_y\,(\lambda/2) \\ l\cos\gamma &= n_z\,(\lambda/2) \end{aligned}\right\} \qquad \ldots(4)$$

Here, the direction cosines obey the relation*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad \ldots(5)$$

From eqs. (3) and (4), we can write

$$n_x^{\,2} + n_y^{\,2} + n_z^{\,2} = \left(\frac{2\,l}{\lambda}\right)^2 \qquad \ldots(6)$$

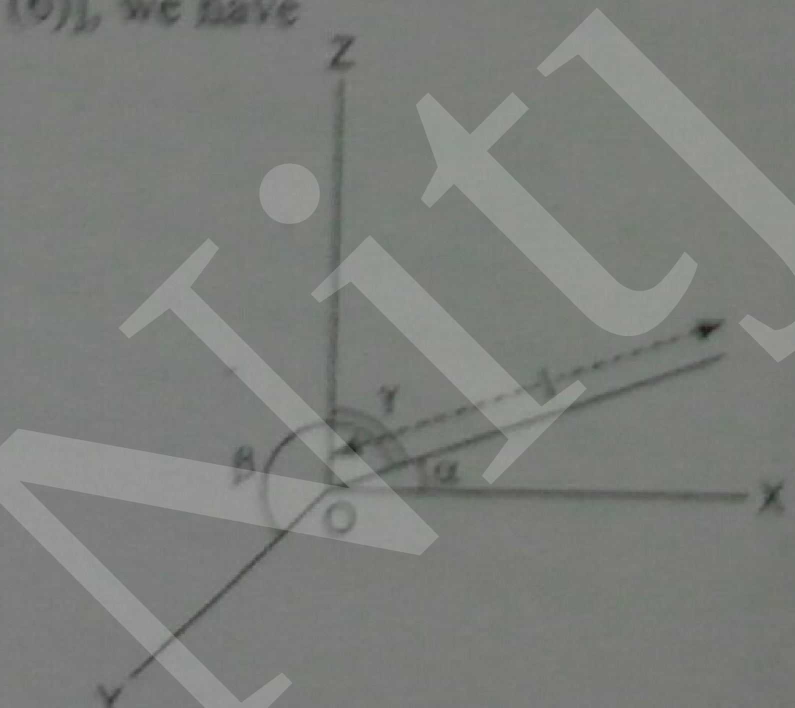


Fig. (6)

* $l\cos\alpha = l_x$, $l\cos\beta = l_y$ and $l\cos\gamma = l_z$
and $l^2 = l_x^{\,2} + l_y^{\,2} + l_z^{\,2}$

If $\quad n_x^{\,2} + n_y^{\,2} + n_z^{\,2} = r^2$, then $r^2 = \left(\frac{2\,l}{\lambda}\right)^2$ or $r = \frac{2l}{\lambda}$

Equation (6) represents an ellipsoid with n_x, n_y and n_z directions as coordinate axes. Each set of values of n_x, n_y and n_z satisfying the above equation corresponds to one mode of vibration. The total number of modes of vibration are the total number of possible set of (n_x, n_y, n_z). The number of modes of vibration in wavelength interval λ and $\lambda + d\lambda$ can be found using above equation.

To make the picture clear, let us consider the number of modes of vibration in two dimensional plane. For two dimensional plane eq. (6) has the form

$$n_x^{\,2} + n_y^{\,2} = \left(\frac{2\,l}{\lambda}\right)^2$$

This equation represents a circle when we plot n_x on X-axis and n_y on Y-axis as shown in Fig. (7). Every point of intersection gives a mode of vibration. Since, positive values of n_x and n_y are allowed, we have to consider intersections in positive quadrant only. The area of each square is unity. Hence, the number of squares, as it (which is equal to the area of the quadrant $\frac{1}{4}\pi \times (\text{radius})^2$. This is equal to

$$\frac{1}{4}\pi\left(\frac{2\,l}{\lambda}\right) = \frac{\pi\,l^2}{\lambda^2}$$

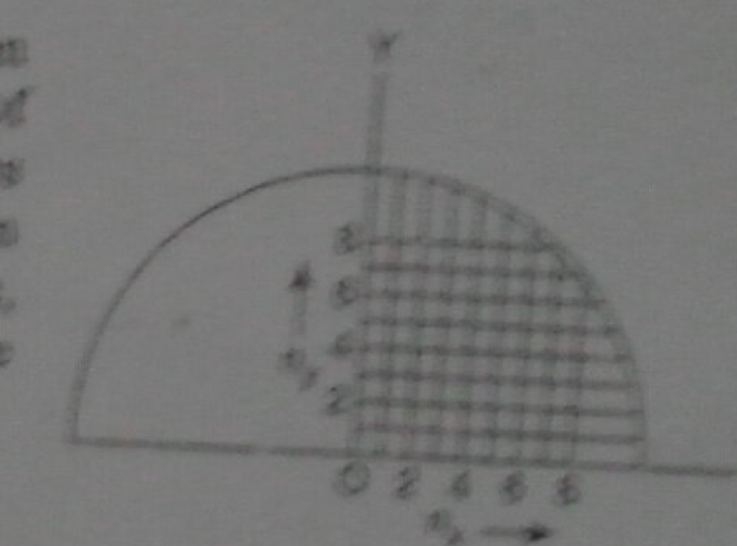


Fig. (7)

So the number of modes of vibration with wavelength lying between λ and $\lambda + d\lambda$ can be obtained by differentiating the above equation.

If we now extend the above idea to three dimensional space, the circle will be sphere and each unit square will be a unit cube. Now the total number of modes of vibrations f upto wavelength λ will be those in the octant of the spherical shell of radii $(2\,l/\lambda)$. So,

$$f = \frac{1}{8}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{8}\cdot\frac{4\pi}{3}\cdot\left(\frac{2\,l}{\lambda}\right)^3 = \frac{4\,\pi\,l^3}{3\lambda^3} \qquad \ldots(7)$$

The number of modes of vibration, *i.e.*, degree of freedom between λ and $\lambda + d\lambda$ is obtained by differentiating the above expression

$$\therefore \quad df = \frac{4\,\pi\,l^3}{3}\cdot\frac{d}{d\lambda}\left(\frac{1}{\lambda^3}\right) = \frac{4\pi\,l^3}{3}\left(-\frac{3}{\lambda^4}\,d\lambda\right) \qquad \ldots(8)$$

Neglecting negative sign as $d\lambda$ is positive, we have

$$df = \frac{4\,\pi V}{\lambda^4}\,d\lambda \qquad \ldots(9)$$

where V is the volume of enclosure.

The above number should be multiplied by 2 as transverse electromagnetic waves have two polarisation for each mode. Thus, the total number of modes of vibration is given by

$$df = \frac{8\,\pi V}{\lambda^4}\,d\lambda \qquad \ldots(10)$$

$\therefore$ Number of modes of vibration per unit volume

$$df = \frac{8\,\pi\,d\lambda}{\lambda^4} \qquad \ldots(11)$$

Equation (11) gives the number of oscillations per unit volume in wavelength λ and $\lambda + d\lambda$.

Rayleigh and Jeans assumed that the law of equipartition of energy holds good in case of radiation also. According to this law, the average energy per mode of vibration is kT, where k is Boltzmann's constant and T is absolute temperature.

$\therefore$ Energy density within wavelength λ and $\lambda + d\lambda$ is given by

$E_\lambda \, d\lambda$ = Total number of modes of vibration × average energy per mode

or
$$E_\lambda \, d\lambda = \frac{8\pi \, d\lambda}{\lambda^4} \times kT$$

or
$$\boxed{E_\lambda \, d\lambda = \frac{8\pi kT}{\lambda^4} \, d\lambda} \qquad ...(12)$$

This is Rayleigh-Jean's formula for energy distribution.

❑ **EXAMPLE** *Show that for an electromagnetic radiation enclosed, the number of modes of vibration per unit volume of the enclosure, lying in the frequency range ν and $\nu + d\nu$ is $8\pi\nu^2 \, d\nu/c^3$, where c is the velocity of light.*

Solution According to eq. (11), the number of modes of vibration per unit volume

$$f = \frac{8\pi \, d\lambda}{\lambda^4} \qquad ...(13)$$

Converting wavelength into frequency $\nu \, (= c/\lambda)$ we have

$$\lambda = c/\nu \quad \text{and} \quad d\lambda = c \, d\nu/\nu^2$$

$$\boxed{f = \frac{8\pi\nu^2 \, d\nu}{c^3}} \qquad ...(14)$$

13.5 PLANCK'S RADIATION LAW

Planck's Hypothesis. In 1900, Max Planck introduced the revolutionary concept of radiation known as quantum theory of radiation. He made the following assumptions:

1. A black body radiator contains simple harmonic oscillators of possible frequencies.
2. The oscillators can not emit or absorb energy continuously. This is contrary to electromagnetic theory which allows a continuous emission or absorption of energy.
3. Emission or absorption of energy takes place in discrete amounts, *i.e.*, energy of oscillator is quantised. The energy of an atomic oscillator of frequency ν can have only certain values like $0, h\nu, 2h\nu$... This is an integral multiple of a small unit of energy $h\nu$ called the quantum or photon. In general for an oscillator of frequency ν, the possible values of the energy are given by

$$\varepsilon = n h \nu$$

where n is any positive integer and h is Planck's constant.

Average Energy of an Oscillator

Let N be the total number of Planck's oscillators and E be their total energy. Then the average energy per Planck's oscillator $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \frac{E}{N} \qquad ...(1)$$

Let there be $N_0, N_1, N_2, N_3 \ldots N_r \ldots$ etc. oscillators having energy $0, \varepsilon, 2\varepsilon, 3\varepsilon \ldots r\varepsilon, \ldots$ etc. respectively. Now we have

$$N = N_0 + N_1 + N_2 + N_3 + \ldots + N_r + \ldots \qquad ...(2)$$

and
$$E = 0 + \varepsilon N_1 + 2\varepsilon N_2 + 3\varepsilon N_3 + \ldots + r\varepsilon N_r + \ldots \qquad ...(3)$$

We know that according to Maxwell's distribution formula, the number of oscillators having energy $r\varepsilon$ is given by

$$N_r = N_0 \, e^{-r\varepsilon/kT} = N_0 \exp\left(-\frac{r\varepsilon}{kT}\right) \qquad ...(4)$$

where k is Boltzmann constant.

Substituting the values of $N_1, N_2, N_3 \ldots$ from eq. (4) in eq. (2), we get

$$N = N_0 + N_0 \exp\left(-\frac{\varepsilon}{kT}\right) + N_0 \exp\left(-\frac{2\varepsilon}{kT}\right) + \ldots + N_0 \exp\left(-\frac{r\varepsilon}{kT}\right) + \ldots$$

$$= N_0\left[1 + \exp\left(-\frac{\varepsilon}{kT}\right) + \exp\left(-\frac{2\varepsilon}{kT}\right) + \exp\left(-\frac{3\varepsilon}{kT}\right) + \ldots\ldots + \exp\left(-\frac{r\varepsilon}{kT}\right) + \ldots\right]$$

or
$$N = \frac{N_0}{\left[1 - \exp\left(\frac{\varepsilon}{kT}\right)\right]} \qquad \left(\because 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}\right) \qquad ...(5)$$

Substituting the values of $N_1, N_2, N_3 \ldots$ from eq. (4) in eq. (3), we get

$$E = (N_0 \times 0) + \varepsilon N_0 \exp\left(-\frac{\varepsilon}{kT}\right) + 2\varepsilon N_0 \exp\left(-\frac{2\varepsilon}{kT}\right)$$

$$+ 3\varepsilon N_0 \exp\left(-\frac{3\varepsilon}{kT}\right) + \ldots + r\varepsilon N_0 \exp\left(-\frac{r\varepsilon}{kT}\right) + \ldots$$

$$= N_0 \varepsilon \exp\left(-\frac{\varepsilon}{kT}\right)\left[1 + 2\exp\left(-\frac{\varepsilon}{kT}\right)\right.$$

$$\left. + 3\exp\left(-\frac{2\varepsilon}{kT}\right) + \ldots + r\exp\left(-\frac{(r-1)\varepsilon}{kT}\right) + \ldots\right]$$

$$= N_0 \varepsilon \exp\left(-\frac{\varepsilon}{kT}\right) \frac{1}{\left\{1 - \exp\left(-\frac{\varepsilon}{kT}\right)\right\}^2} \qquad ...(6)$$

$$\left(\because 1 + 2x + 3x^2 + \ldots + r x^{r-1} = \frac{1}{(1-x)^2}\right)$$

Now, the average energy of oscillator is given by

$$\bar{\varepsilon} = \frac{E}{N} = \frac{E_0 \varepsilon \exp.(-\varepsilon/kT)/\{1 - \exp.(-\varepsilon/kT)\}^2}{N_0/\{1 - \exp.(-\varepsilon/kT)\}}$$

$$= \frac{\varepsilon \exp.(-\varepsilon/kT)}{\{1-\exp.(-\varepsilon/kT)\}}$$

$$= \frac{\varepsilon}{\{\exp.(\varepsilon/kT)-1\}}$$

$$= \frac{h\nu}{\{\exp.(h\nu/kT)-1\}} = \frac{h\nu}{\{e^{h\nu/kT}-1\}} \quad ...(7)$$

Thus, we see that the average energy of oscillator given by eq. (7) is different from the energy kT of a classical oscillator.

Planck's Formula

We know that number of oscillators per unit volume in frequency range ν and $\nu + d\nu$ is given by

$$f = \frac{8\pi\nu^2}{c^3} d\nu \qquad \text{[See eq. (14) of previous article.]} \quad ...(8)$$

Multiplying eq. (8) with the average energy of oscillators, given by eq. (7) we get the total energy per unit volume belonging to the range $d\nu$ or the energy density belonging to range $d\nu$ is

$$E_\nu \, d\nu = \frac{8\pi\nu^2}{c^3} d\nu \times \frac{h\nu}{\{\exp.(h\nu/kT)-1\}}$$

or

$$E_\nu \, d\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{d\nu}{\{\exp.(h\nu/kT)-1\}} \quad ...(9)$$

This is known as Planck's radiation law.

This law can be expressed in terms of wavelengths as:

$$\nu = \frac{c}{\lambda} \quad \text{or} \quad |d\nu| = \left|\left(-\frac{c}{\lambda^2}\right)d\lambda\right| = \left(\frac{c}{\lambda^2}\, d\lambda\right)$$

$$\therefore \quad E_\lambda \, d\lambda = \frac{8\pi h}{c^3}\left(\frac{c^3}{\lambda^3}\right) \cdot \frac{1}{\{\exp.(hc/\lambda kT)-1\}}\left(\frac{c}{\lambda^2}\, d\lambda\right)$$

or

$$\boxed{E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{\{\exp.(hc/\lambda kT)-1\}} \cdot d\lambda} \quad ...(10)$$

This formula agrees well with the experimental curves throughout the whole range of wavelengths. This equation contains the constants c (velocity of light in vacuum), k (Boltzmann constant) and h (Planck's constant $= 6.6256 \times 10^{-34}$ Js). The validity of this equation can be checked by the following experimentally established laws:

(i) Wien's formula for shorter wavelengths,
(ii) Rayleigh-Jeans law at longer wavelengths,
(iii) Wien's displacement law, and
(iv) Stefan's-Boltzmann law for total radiation.

13.6 DIFFERENT LAWS FROM PLANCK'S RADIATION FORMULA

1. Wien's Formula

For short wavelengths, *i.e.*, when λ is very small

$e^{hc/\lambda kT} >> 1$, *i.e.*, one can be neglected in comparison of $e^{hc/\lambda kT}$.

According to Planck's formula

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{\{\exp.(hc/\lambda kT)-1\}} \, d\lambda \quad ...(1)$$

For shorter wavelengths, formula (1) can be written as

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{\exp.(hc/\lambda kT)} \cdot d\lambda$$

or

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot e^{-hc/\lambda kT} \cdot d\lambda \quad ...(2)$$

Putting $8\pi hc = A$ and $hc/k = B$, we have

$$E_\lambda \, d\lambda = \frac{A}{\lambda^5} \cdot e^{-B/\lambda T} \cdot d\lambda \quad ...(3)$$

This is Wien's formula which agrees with experiment at short wavelengths.

2. Rayleigh-Jeans Formula

For longer wavelengths, *i.e.*, when λ is very large $hc/\lambda kT$ is small and $e^{ch/\lambda kT}$ can be expanded as

$$e^{hc/\lambda kT} = 1 + \frac{hc}{\lambda kT} + \frac{h^2c^2}{\lambda^2k^2T^2} + \dots$$

$$= 1 + \frac{hc}{\lambda kT} \quad \text{neglecting higher powers.} \quad ...(4)$$

Now

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{\left(1+\frac{hc}{\lambda kT}\right)-1}$$

or

$$E_\lambda \, d\lambda = \frac{8\pi hc \times \lambda kT}{\lambda^5 \, hc} \times d\lambda = \frac{8\pi kT}{\lambda^4} \, d\lambda \quad ...(5)$$

This is Rayleigh-Jeans formula which agrees with experimental values at long wavelenghts.

3. Wien's Displacement Law

According to Planck's radiation formula

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{(e^{hc/\lambda kT}-1)}$$

or

$$E_\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{(e^{hc/\lambda kT}-1)}$$

$$E_\lambda = A\lambda^{-5}(e^{hc/\lambda kT}-1)^{-1} \quad ...(6)$$

where

$$A = 8\pi hc$$

To find out the wavelength at which the spectral radiancy is maximum, we differentiate eq. (6) w.r.t. λ and equate it to zero, *i.e.*, we put $(dE_\lambda/d\lambda) = 0$. Thus,

$$A\left[\lambda^{-5}(-1)(e^{hc/\lambda kT}-1)^{-2} e^{hc/\lambda kT}\left(-\frac{hc}{\lambda^2 kT}\right) - 5(\lambda)^{-6}(e^{hc/\lambda kT}-1)^{-1}\right] = 0$$

or $$(e^{hc/\lambda kT}-1)^{-1}\, e^{hc/\lambda kT}\left(\frac{hc}{\lambda^2 kT}\right)=\frac{5}{\lambda}$$

or $$\left(\frac{hc}{\lambda kT}\right)\frac{e^{hc/\lambda kT}}{(e^{hc/\lambda kT}-1)}=5$$

Putting $(hc/\lambda kT)=x$, we get

$$\frac{x e^x}{(e^x-1)}=5$$

or $$\frac{x}{1-e^{-x}}=5 \quad \text{or} \quad \frac{x}{5}+e^{-x}=1 \qquad \ldots(7)$$

The equation has two roots $x=0$ and $x=4.965$. Ignoring first value

$$x=\frac{hc}{\lambda kT}=4.955$$

Therefore, the wavelengths λ_m at which the spectral radiancy per unit range of wavelength has its maximum value is given by

$$\lambda_m T=\frac{hc}{4.965\,k}=b \text{ (say)}$$

∴ $$\lambda_m T=b \qquad \ldots(8)$$

This is Wien's displacement law.

SOLVED EXAMPLES

❑ **EXAMPLE 1** *Determine the temperature of sun with the help of Wien's law, given $b=2.92\times10^{-3}$ m K. Maximum wavelength = 4900 Å.*

Solution According to Wien's displacement law

$$\lambda_m T=b \quad \text{or} \quad T=b/\lambda_m$$

∴ $$T=\frac{2.92\times10^{-3}}{4900\times10^{-10}}$$

$$= \mathbf{5967\ K}$$

❑ **EXAMPLE 2** *A body at 1500 K emits maximum energy at a wavelength 20,000 Å. If the sun emits maximum energy at a wavelength 5500 Å what would be the temperature of the sun?*

Solution According to Wien's displacement law

$$\lambda_m T=\text{constant} \quad \text{or} \quad \lambda_m T=\lambda_m' T'$$

∴ $$T'=\frac{\lambda_m T}{\lambda_m'}=\frac{20000\times1500}{5500}$$

$$= \mathbf{5454\ K}$$

❑ **EXAMPLE 3** *Find the wavelength at which maximum energy is radiated by a black body having a temperature of 327°C. The Wien's constant is 2.897×10^{-3} m K.*

Solution According to Wien's displacement law

$$\lambda_m T=b \quad \text{or} \quad \lambda_m=b/T$$

∴ $$\lambda_m=\frac{2.897\times10^{-3}\text{ m K}}{(327+273)\text{ K}}=\frac{2.897\times10^{-3}}{600}\text{ m}$$

$$=482.83\times10^{-8}\text{ m}$$

$$= \mathbf{48283\ Å}$$

❑ **EXAMPLE 4** *In an atomic explosion, the maximum temperature produced was of the order of 10^7 K. Calculate the wavelength of maximum energy. Wien's constant is 0.292 cm K.*

Solution According to Wien's displacement law

$$\lambda_m T=b \quad \text{or} \quad \lambda_m=b/T$$

∴ $$\lambda_m=\frac{0.292}{10^7}=0.292\times10^{-7}\text{ cm}$$

$$=2.92\times10^{-8}\text{ cm}$$

$$= \mathbf{2.92\ Å}$$

❑ **EXAMPLE 5** *A black body at 1127°C radiates maximum wavelength of 2 microns. if the wavelength of maximum energy of moon is 14 microns, what is the temperature of the moon?*

Solution According to Wien's displacement law

$$\lambda_m T=\text{constant}=b$$

Here, $$T=1127°\text{C}=(1127+273)=1400\text{ K}$$

$$\lambda_m=2\text{ microns}=2\times10^{-6}\text{ m}$$

∴ $$b=(2\times10^{-6})\times(1400) \qquad \ldots(1)$$

For moon $$(14\times10^{-6})\,T_m=b \qquad \ldots(2)$$

From eqs. (1) and (2), we get

$$(14\times10^{-6})\,T_m=(2\times10^{-6})\times1400$$

$$T_m = \mathbf{200\ K}$$

❑ **EXAMPLE 6** *Calculate the surface temperature of the sun and moon given that $\lambda_m=4753$ Å and $14\,\mu$ respectively, λ_m being wavelength of maximum intensity of emission.*

Solution According to Wien's displacement law

$$\lambda_m T=\text{constant}=0.2898\times10^{-2}\text{ m K}$$

or $$T=\frac{0.2898\times10^{-2}}{\lambda_m}$$

(i) For sun, $$\lambda_m=4753\text{ Å}=4753\times10^{-10}\text{ m}$$

∴ $$T_s=\frac{0.2898\times10^{-2}}{4753\times10^{-10}}$$

$$= \mathbf{6097\ K}$$

(ii) For moon,
$$T_m = \frac{0.2898 \times 10^{-2}}{14 \times 10^{-6}} \qquad (\because 1\mu = 1 \text{ micron} = 10^{-6} \text{ m})$$
$$= 207 \text{ K}$$

13.7 WAVE AND PARTICLE DUALITY

To understand the wave and particle duality, it is necessary to know what is a particle and what is a wave.

The concept of a particle is easy to grasp. It has mass, it is located at some definite point, it can move from one place to another, it gives energy when slowed down or stopped. Thus, the particle is specified by (*i*) ***mass*** m, (*ii*) ***velocity*** v, (*iii*) ***momentum*** p, and (*iv*) ***energy*** E.

The concept of a wave is a bit more difficult than that of a particle. A wave is spread out over a relatively large region of space, it cannot be said to be located just here and there, it is hard to think of mass being associated with a wave. Actually a wave is nothing but rather a spread out disturbance. A wave is specified by its (*i*) ***frequency***, (*ii*) ***wavelength***, (*iii*) ***phase of wave velocity***, (*iv*) ***amplitude***, and (*v*) ***intensity***.

Considering the above facts, it appears difficult to accept the conflicting ideas that radiation has a dual nature, *i.e.*, radiation is a wave which is spread out over space and also a particle which is localised at a point in space. However, this acceptance is essential because radiation sometimes behaves as a wave and at other times as a particle as explained below :

(i) Radiations including visible light, infra-red, ultraviolet, X-rays, etc. behave as waves in experiments based on interference, diffraction, etc. This is due to the fact that these phenomena require the presence of two waves at the same position at the same time. Obviously, it is difficult for the two particles to occupy the same position at the same time. Thus, we conclude that radiations behave like wave.

(ii) Planck's quantum theory was successful in explaining black body radiation, the photoelectric effect, the Compton effect, etc. and had clearly established that the radiant energy, in its interaction with matter, behaves as though it consists of corpuscles. Here radiation interacts with matter in the form of photons or quanta. Thus, we conclude that radiations behave like particle.

Radiations, thus, sometimes behave as a wave and at some other time as a particle, *i.e.*, it has a ***wave particle dualism***. Here it should be remembered that radiation cannot exhibit its particle and wave properties simultaneously.

13.8 DE-BROGLIE'S HYPOTHESIS OF MATTER WAVES

In Newton time, matter and radiation, both were assumed to consist of particles. With the discovery of phenomena like interference, diffraction and polarisation it was established that light is a kind of a wave motion. In the beginning of 20th century, some new phenomena (photoelectric effect, Compton effect, etc.) were discovered which could not be explained on the basis of wave theory. These phenomenon were explained on the basis of quantum theory in which light quanta or photon are endowed with corpuscular properties-mass ($h\nu/c^2$), velocity c and momentum $h\nu/c$. But when the photon theory was applied to phenomena such as interference, diffraction, etc. (which had been fully explained on wave theory) it proved helpless to explain them. Thus, light has a dual nature, *i.e.*, it possesses both particle and wave properties. In some phenomena corpuscular nature predominate while in others wave nature predominate. The manifestation of properties depends upon the conditions under which the particular phenomena occurs. It is worthy to note that wave and particle never expected to appear together.

Louis de-Broglie in 1924 extended the wave particle parallelism of light radiations to all the fundamental entities of physics such as electrons, protons, neutrons, atoms and molecules, etc. Louis de-Broglie put a bold suggestion that the correspondence between wave and particle should not be confined only to electromagnetic radiation, but it should also be valid for material particles, *i.e.*, ***like radiation, matter also has a dual***, (*i.e.*, particle like and wave like) ***characteristic***.

In his thesis de-Broglie wrote that there was an intimate connection between waves and corpuscles not only in the case of radiation but also in the case of matter. A moving particle is always associated with a wave and the particle is controlled by the wave in a manner similar to that in which a photon is controlled by waves. His suggestion was based on the fact that nature loves symmetry, if radiation like light can act like wave some time and like a particle at other times, then the material particles (*e.g.*, electron, neutron, etc.) should act as wave at some other times.

Matter Waves

According to de-Broglie's hypothesis, a moving particle is associated with a wave which is known as de-Broglie wave. The wavelength of the matter wave is given by

$$\lambda = \frac{h}{mv} = \frac{h}{p},$$

where m is the mass of the material particle, v its velocity and p is its momentum.

13.9 EXPRESSION FOR DE-BROGLIE WAVELENGTH

The expression of the wavelength associated with a material particle be derived on the analogy of radiation as follows:

Considering the Planck's theory of radiation the energy of a photon (quantum) is given by

$$E = h\nu = \frac{hc}{\lambda} \qquad \ldots(1)$$

where c is the velocity of light in vacuum and λ is its wavelength.

According to Einstein energy-mass relation

$$E = mc^2 \qquad \ldots(2)$$

From eqs. (1) and (2), we get

$$mc^2 = \frac{hc}{\lambda}$$

or

$$\lambda = \frac{hc}{mc^2}$$

or

$$\lambda = \frac{h}{mc} \qquad \ldots(3)$$

where $mc = p$ (momentum associated with photon).

If we consider the case of a material particle of mass m and moving with a velocity v, *i.e.*, momentum mv, then the wavelength associated with this particle is given by

$$\lambda = \frac{h}{mv} = \frac{h}{p} \qquad \ldots(4)$$

Different expressions

(a) If E is the kinetic energy of the material particle, then

$$E = \frac{1}{2} m v^2 = \frac{1}{2} \frac{m^2 v^2}{m} = \frac{p^2}{2m}$$

or

$$p = \sqrt{(2mE)}$$

∴ de-Broglie wavelength

$$\boxed{\lambda = \frac{h}{\sqrt{(2mE)}}} \qquad \text{...(a)}$$

(b) When a charged particle carrying a charge q is accelerated by a potential difference V volts, then its K.E., E is given by

$$E = qV$$

Hence, the de-Broglie wavelength associated with this particle is given by

$$\boxed{\lambda = \frac{h}{\sqrt{(2mqV)}}} \qquad \text{...(b)}$$

(c) When a material particle is in thermal equilibrium at a temperature T, then

$$E = \frac{3}{2} kT$$

where k = Boltzmann's constant $= 1.38 \times 10^{-23}$ J/K.

So, the de-Broglie wavelength of a material particle at temperature T is given by

$$\lambda = \frac{h}{\sqrt{[2m(3/2 \cdot kT)]}}$$

or

$$\boxed{\lambda = \frac{h}{\sqrt{(3mkT)}}} \qquad \text{...(c)}$$

(d) If the velocity of the particle is comparable with the velocity of light, then the mass of the particle is given by

$$m = \frac{m_0}{\sqrt{\left[1 - \left(\frac{v^2}{c^2}\right)\right]}}$$

where m_0 is the rest mass of the particle.

The de-Broglie wavelength is given by

$$\lambda = \frac{h}{mv} = \frac{h\sqrt{\left\{1 - \left(\frac{v^2}{c^2}\right)\right\}}}{m_0 v} \qquad \text{...(d)}$$

(e) de-Broglie wavelength associated with electrons

Let us consider the case of an electron of rest mass m_0 and charge e which is accelerated by a potential V volt from rest to velocity v, then

$$\frac{1}{2} m_0 v^2 = eV \quad \text{or} \quad v = \sqrt{\left(\frac{2eV}{m_0}\right)}$$

Now,

$$\lambda = \frac{h}{m_0 v} = \frac{h\sqrt{(m_0)}}{m_0\sqrt{(2eV)}} = \frac{h}{\sqrt{(2eVm_0)}}$$

or

$$\lambda = \frac{6.625 \times 10^{-34}}{\sqrt{[(2 \times 1.632 \times 10^{-19})V \times 9.1 \times 10^{-31})]}}$$

$$= \frac{12.26}{\sqrt{V}} \text{ Å} \qquad \text{...(e)}$$

If $V = 100$ volt, then

$\lambda = 1.226$ Å

Therefore, the wavelength associated with an electron accelerated to 100 volt is 1.226 Å.

13.10 PROPERTIES OF MATTER WAVES

Following are the properties of matter waves:

1. Lighter is the particle, greater is the wavelength associated with it.
2. Smaller is the velocity of the particle, greater is the wavelength associated with it.
3. When $v = 0$ then $\lambda = \infty$, *i.e.*, wave becomes indeterminate and if $v = \infty$ then $\lambda = 0$. ***This shows that matter waves are generated only when material particles are in motion.***
   These waves are produced whether the particles are charged particles or they are uncharged ($\lambda = h/mv$ is independent of charge). This fact reveals that these waves are not electromagnetic waves but they are a new kind of waves (electromagnetic waves are produced only by motion of charged particles).
4. The velocity of matter wave depends on the velocity of matter particle, *i.e.*, it is not a constant while the velocity of electromagnetic wave is constant.
5. The velocity of matter wave is greater than the velocity of light.
   The velocity of matter wave** (de-Broglie wave velocity) **is greater than the velocity of light. This can be proved as under.
   A particle in motion with associated matter wave has two different velocities; one referring to the mechanical motion of the particle represented by v and second related to the propagation of the wave represented by w.
   We know that $\quad E = h\nu \quad$ and $\quad E = mc^2$

$$h\nu = mc^2 \quad \text{or} \quad \nu = \frac{mc^2}{h}$$

The wave velocity (w) is given by

$$w = \nu \times \lambda = \frac{mc^2}{h} \times \frac{h}{mv} \qquad \left(\because \lambda = \frac{h}{mv}\right)$$

or

$$w = \frac{c^2}{v}$$

As particle velocity v cannot exceed c (velocity of light), hence, w is greater than velocity of light. We can understand this unexpected result by considering the wave velocity (also known as phase velocity) and the group velocity.

6. The wave velocity or the phase velocity of the matter waves is given by

$$v_{phase} = \frac{c^2}{v_{group}}$$

Here, v_{group} is same as particle velocity. This velocity is always less than c, the velocity of light. Therefore, the velocity of matter waves is always greater than c. This shows that matter waves are not physical waves.

7. ***The wave and particle aspects of moving bodies can never appear together in the same experiment.*** What we can say is that waves have particle like properties and particles have wave like properties and the concepts are separately linked. Matter wave representation is only a symbolic representation.

8. The wave nature of matter introduces an uncertainty in the location of the position of the particle because a wave cannot be said exactly at this point or exactly at that point. However, where the wave is large (strong) there is a good chance of finding the particle while, where the wave is small (weak) there is very small chance of finding the particle.

13.11 EXPERIMENTAL VERIFICATION (DAVISSON AND GERMER'S ELECTRON DIFFRACTION EXPERIMENT)

Principle

The first experimental evidence of matter wave was given by two American physicists, Davisson and Germer in 1927. They also succeeded in measuring the de-Broglie wavelength associated with slow electrons. Davisson and Germer were studying the reflection of electrons from nickel target. Accidentally the nickel target was subjected to such a heat treatment that the reflection became anomalous. Now the reflected intensity showed striking maxima and minima. Thus, they suspected that electrons are diffracted like X-rays, *i.e.*, they behave like waves under certain conditions.

Experimental arrangement

The experimental arrangement is shown in Fig. (8). The apparatus consists of an electron gun G where the electrons are produced and obtained in a fine pencil of electronic beam of known velocity. The electron gun consists of a tungsten filament F heated to dull red so that electrons are emitted due to thermionic emission. Now the electrons are accelerated in the electric field of known potential difference. After this the electrons are collimated by suitable slits to obtain a fine beam. The beam of electrons is directed to fall on a large single crystal of nickel, known as target T. The

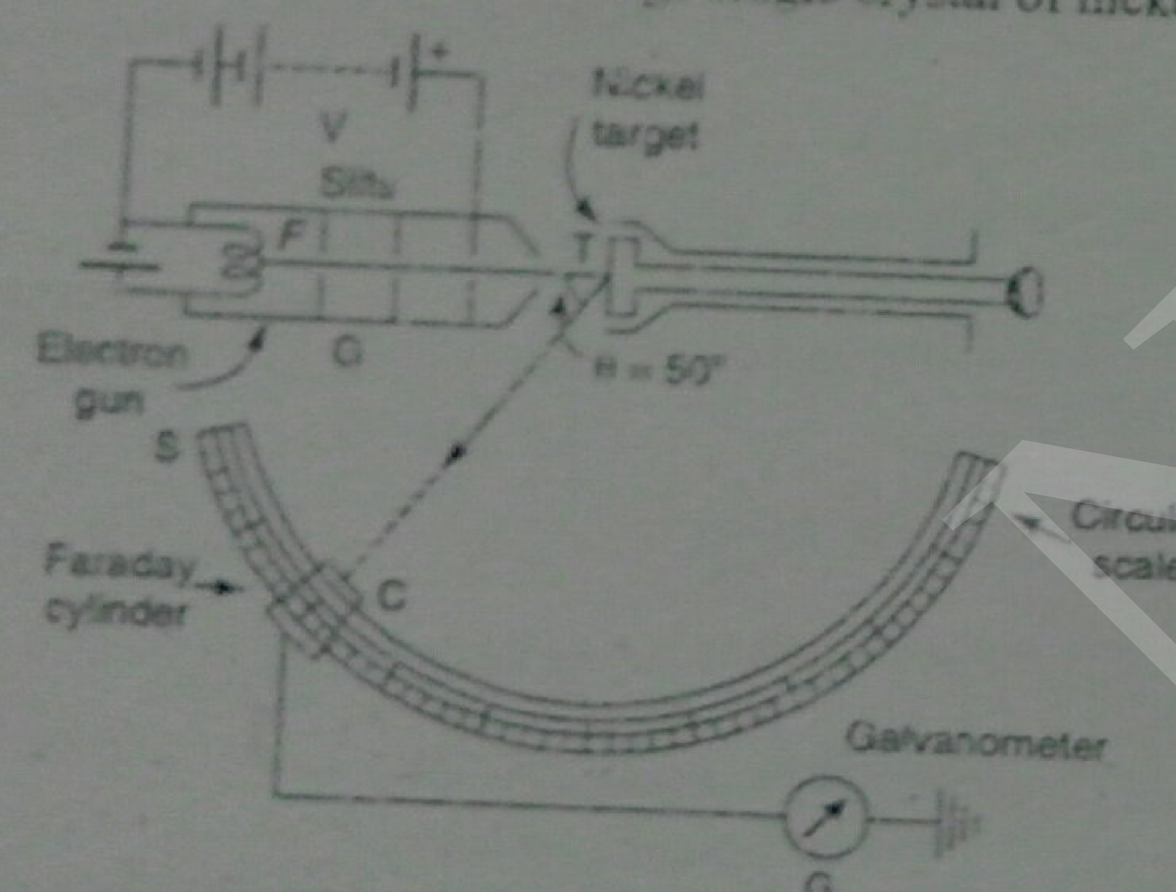


Fig. (8)

electrons, acting as the waves, are diffracted in different directions. The angular distribution is measured by an electron detector (Faraday cylinder C) which is connected to a galvanometer. The Faraday cylinder can move on a circular graduated scale S between 29° to 90° to receive the reflected electrons. The Faraday cylinder consists of two walls which are insulated from each other. A retarding potential is maintained between them so that only fast moving electrons coming from electron gun may enter inside it. The secondary electrons (slow electrons) produced by collisions with atoms from nickel target are reflected by Faraday cylinder. In this way the galvanometer deflection is only due to electrons coming from electron gun.

First of all, the accelerating potential V is given a low value and the crystal is set at any arbitrary azimuth (θ). Now, the Faraday cylinder is moved to various positions on the scale S and galvanometer current is noted for each position. Here, it should be remembered that galvanometer current is a measure of intensity of diffracted beam. A graph is then plotted between galvanometer current against angle θ between incident beam and beam entering the cylinder. The observations are repeated for different accelerating potentials. The corresponding curves are shown in Fig. (9).

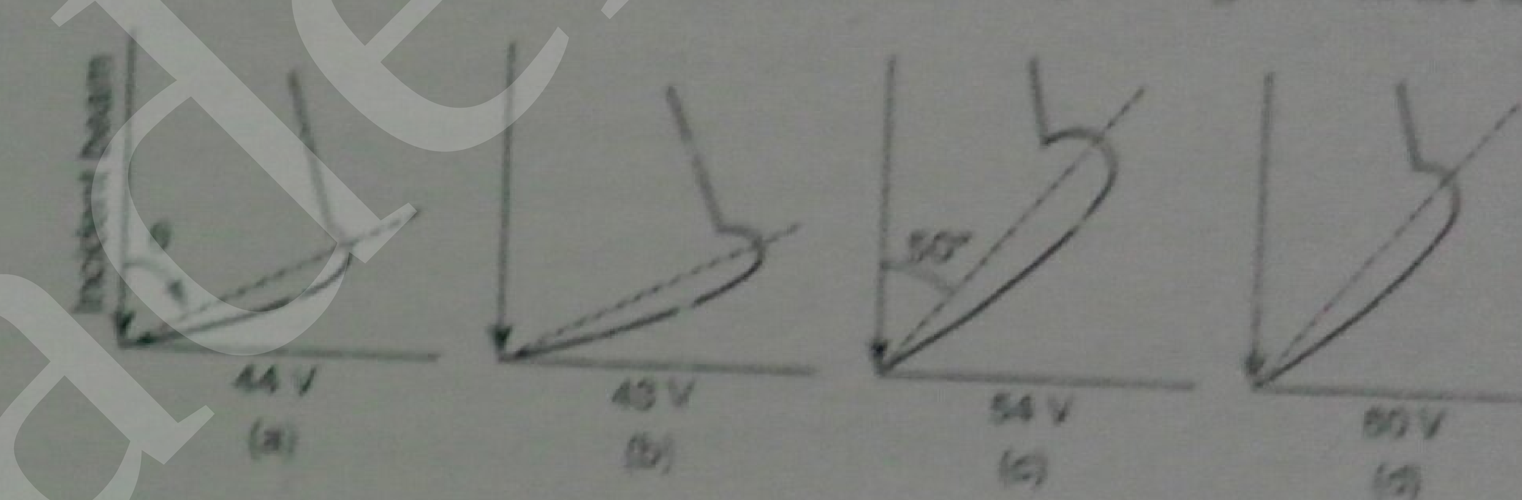


Fig. (9)

Evidence

It is observed that a 'bump' begins to appear in the curve for 44 volts electrons. Following points are also observed

(i) With increasing potential, the bump moves upwards.

(ii) The 'bump' becomes most prominent in the curve for 54 volt electrons at θ = 50°.

(iii) At higher potentials, the 'bumps' gradually disappear.

The bump in its most prominent state verifies the existence of electron waves. According to de-Broglie, the wavelength associated with electron accelerated through a potential V is given by

$$\lambda = \frac{12.26}{\sqrt{V}} \text{ Å}$$

Hence, the wavelength associated with 54 volt electron is

$$\lambda = \frac{12.26}{\sqrt{(54)}} = 1.67 \text{ Å}$$

From X-ray analysis, it is known that a nickel crystal acts as a plane diffraction grating with grating space $d = 0.91$ Å (see Fig. 10). According to experiment, we have diffracted electron beam at θ = 50°. The corresponding angle of incidence relative to the family of Bragg plane

$$\theta' = \frac{180 - 50}{2} = 65°$$

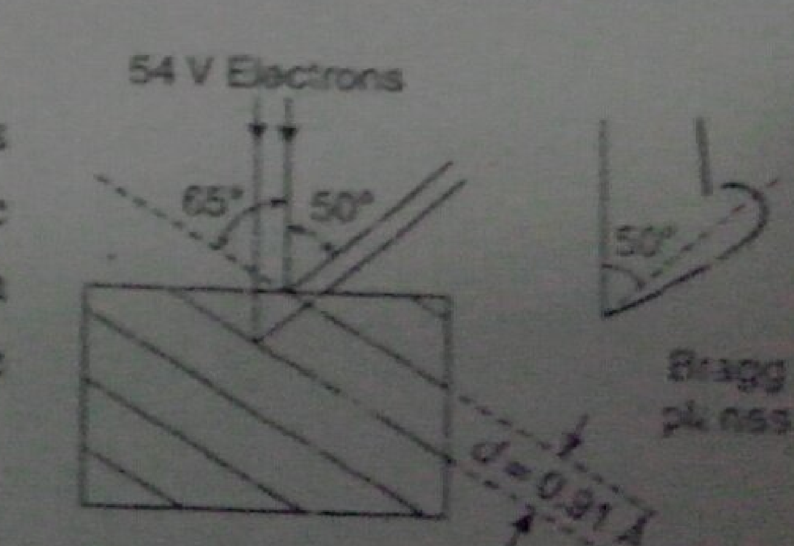


Fig. (10)

Using Bragg's equation (taking $n = 1$), we have

$$\lambda = 2\, d \sin\theta' = 2\,(0.91\ \text{Å}) \sin 65° = \mathbf{1.65\ Å}$$

This is in good agreement with the wavelength computed from de-Broglie hypothesis.

As the two values are in good agreement, hence, confirms the de-Broglie concept of matter waves.

Significance

Although the dual nature of matter is applicable to all the material objects but it is significant only for microscopic bodies such as electron, proton, atoms or molecules, etc. For large bodies such as a big ball, earth, etc., the associated waves are of very small wavelengths and cannot be measured. This will be clear by the following examples:

1. Consider an electron of mass 9.11×10^{-31} kg moving with the velocity 10^6 m/s. The de-Broglie wavelength associated with electron is given by

$$\lambda = \frac{h}{mv} = \frac{6.63\times10^{-34}}{(9.11\times10^{-31})\times10^{6}} = \mathbf{7.28\times10^{-10}}\ \text{m}$$

This wavelength is of the order of magnitude of X-rays. So, this wavelength can be measured.

2. Consider a ball of 10 gramme moving with the velocity 10^6 m/s. The de-Broglie wavelength associated with the ball is given by

$$\lambda = \frac{h}{mv} = \frac{6.63\times10^{-34}}{(10\times10^{-3})\times(10^{6})} = \mathbf{6.63\times10^{-38}}\ \text{m.}$$

This is shorter than any electromagnetic radiation. This cannot be measured by any method. So, the ball does not have waves associated with it.

13.12 WAVE VELOCITY AND GROUP VELOCITY

Wave Velocity or Phase Velocity

When a monochromatic wave, i.e., a wave of single frequency and wavelength, travels through a medium, its velocity of advancement in the medium is called as wave velocity. Consider a wave whose displacement y is expressed as

$$y = a \sin(\omega t - k x)$$

where, a is the amplitude, ω is angular frequency $(= 2\pi n)$ and $k\ (= 2\pi/\lambda)$ is the propagation constant of the wave.

The ratio of angular frequency ω to the propagation constant k is defined as wave velocity. This is expressed by v_p. Hence,

$$v_p = (\omega/k) \qquad \left(\because v_p = n\lambda = \frac{\omega}{2\pi}\times\frac{2\pi}{k} = \frac{\omega}{k}\right) \qquad \ldots(1)$$

For the wave $(\omega t - k x)$ is the phase of wave motion. For the planes of constant phase (wavefronts), we have

$$\omega t - k x = \text{constant} \qquad \ldots(2)$$

Differentiating eq. (2) with respect to time t

$$\omega - k\frac{dx}{dt} = 0$$

or

$$\frac{dx}{dt} = \frac{\omega}{k} = v_p \qquad \ldots(3)$$

Thus, the wave velocity is the velocity with which the planes of constant phase advance through the medium. Due to this reason, the wave velocity is also called as phase velocity.

Group velocity

Consider pulses rather than monochromatic wave. The pulse consists of a number of waves slightly differing in frequency from one another. The superposition of such waves is known as wave group (wave packet) [Fig. (11)]. When such a group travels in the medium, the phase velocities of different components are different. However, the observed velocity is the velocity with which the maximum amplitude of the group advances. This is called group velocity. *Thus, the group velocity is the velocity with which the group (wave packet) is transmitted.*

Wave Packet (Wave velocity and group velocity)

We have seen that the velocity of matter wave is greater than the velocity of light. According to Einstein's theory of relativity, no material particle can have a speed greater than the speed of light c. Therefore, the speed of matter waves cannot be greater than speed of light. Secondly, if the speed of de-Broglie wave associated with the particle is greater than the speed of particle, the particle will be left behind. So, only one wave cannot be associated with material particle. Hence, Schroedinger assumed that a moving material particle is equivalent to a wave packet (number of waves) instead of single wave.

A wave packet comprises a group of waves slightly differing in velocity and wavelength, with phases and amplitude such that they interfere constructively over a small region of space where the particle can be located and outside this space they interfere destructively so that the amplitude reduces to zero. Such a wave packet is shown in Fig. (11).

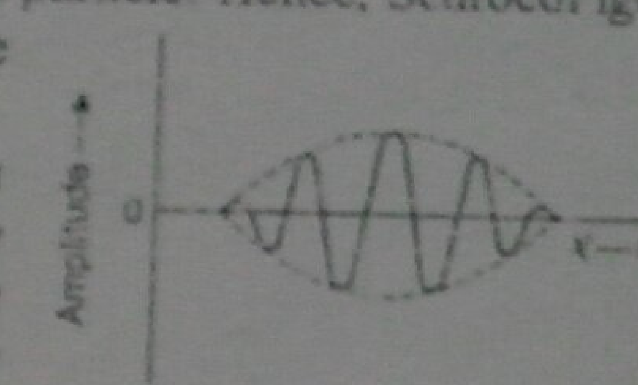


Fig. (11) Wave packet

The velocity of component waves of a wave packet is called as *phase velocity* of those waves. This is denoted by symbol v_p.

The velocity with which the wave packet obtained due to superposition of waves travelling in a group, is called ***group velocity***. This is denoted by v_g.

Expression for Group Velocity

Let us consider the case of two wave trains having same amplitude a but slightly different angular frequencies (ω and ω') and phase velocities (u and u'). The waves can be represented as

$$y_1 = a \sin(\omega t - k x) \qquad \ldots(1)$$

and

$$y_2 = a \sin(\omega' t - k' x) \qquad \ldots(2)$$

where k and k' are propagation constants defined as $(2\pi/\lambda)$. The resultant wave is given by

$$y = y_1 + y_2 = a \sin(\omega t - k x) + a \sin(\omega' t - k' x)$$

$$= 2a \cos\left[\left(\frac{\omega-\omega'}{2}\right)t - \left(\frac{k-k'}{2}\right)x\right] \times \sin\left[\left(\frac{\omega+\omega'}{2}\right)t - \left(\frac{k+k'}{2}\right)x\right]$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= 2a \cos\left[\left(\frac{d\omega}{2}\right)t - \left(\frac{dk}{2}\right)x\right] \sin(\omega t - k x) \qquad \ldots(3)$$

where $\quad \dfrac{\omega+\omega'}{2}=\omega \quad$ and $\quad \dfrac{k+k'}{2}=k$ (approx.)

Equation (3) represents a wave of angular frequency ω and propagation constant k. The phase velocity υ_p of the resultant wave is given by

$$\upsilon_p=\frac{\omega}{k}\ \text{(approx.)} = \text{same as that of each composing wave.}$$

The amplitude of the resultant wave is modified. The amplitude is given by

$$2a\cos\left[\left(\frac{d\omega}{2}\right)t-\left(\frac{dk}{2}\right)x\right]$$

or

$$2a\cos\frac{d\omega}{2}\left[t-\frac{dk}{d\omega}x\right]=2a\cos\frac{d\omega}{2}\left[t-\frac{x}{\upsilon_g}\right]$$

where υ_g is known as group velocity. This is given as

$$\boxed{\upsilon_g=\frac{d\omega}{dk}=\frac{\omega-\omega'}{k-k'}} \qquad \text{...(4)}$$

Relation between Group Velocity and Wave Velocity

$$\omega=\upsilon_p\,k \quad\text{or}\quad d\omega=d\upsilon_p\,k+\upsilon_p\,dk$$

$$\frac{d\omega}{dk}=\upsilon_p+k\frac{d\upsilon_p}{dk} \qquad \text{...(5)}$$

From eqs. (4) and (5), we get

$$\upsilon_g=\upsilon_p+k\frac{d\upsilon_p}{dk}$$

or

$$\upsilon_g=\upsilon_p+k\frac{d\upsilon_p}{d\lambda}\times\frac{d\lambda}{dk} \qquad \text{...(6)}$$

Since, $k=\dfrac{2\pi}{\lambda}$ hence, $\quad \dfrac{d\lambda}{dk}=-\dfrac{2\pi}{k^2}$

Now, eq. (6) becomes,

$$\boxed{\upsilon_g=\upsilon_p-\lambda\frac{d\upsilon_p}{d\lambda}} \qquad \text{...(7)}$$

Equation (7) represents a relationship between group velocity υ_g and phase velocity υ_p in a dispersive medium in which the wave velocity is frequency dependent.

Let us consider a medium in which wave velocity is independent of frequency. For such a medium υ_p is independent of frequency, *i.e.*, υ_p = constant. Therefore,

$$\frac{d\upsilon_p}{d\lambda}=0$$

From eq. (7), we get

$$\boxed{\upsilon_g=\upsilon_p}$$

So, *for a non-dispersive medium* (such as free space), *the group velocity of a wave packet is equal to wave velocity of phase velocity.*

Group Velocity and Particle Velocity

Equation (7) can be written as

$$\upsilon_g=\lambda^2\left[\frac{\upsilon_p}{\lambda^2}-\frac{1}{\lambda}\left(\frac{d\upsilon_p}{d\lambda}\right)\right]$$

$$=-\lambda^2\frac{d}{d\lambda}\left(\frac{\upsilon_p}{\lambda}\right)=-\lambda^2\frac{d\nu}{d\lambda}$$

or

$$\frac{1}{\upsilon_g}=-\frac{1}{\lambda^2}\frac{d\lambda}{d\nu}=\frac{d}{d\nu}\left(\frac{1}{\lambda}\right) \qquad \text{...(8)}$$

If E and V represent the total and potential energy of the particle respectively, then kinetic energy of particle is given by

$$\frac{1}{2}m\upsilon^2=E-V \quad \text{where } \upsilon \text{ is the particle velocity}$$

or

$$\upsilon=\left[\frac{2(E-V)}{m}\right]^{1/2}$$

According to de-Broglie formula $\lambda=h/m\upsilon$

$$\frac{1}{\lambda}=\frac{m\upsilon}{h}=\frac{m}{h}\left[\frac{2(E-V)}{m}\right]^{1/2} \qquad \text{...(9)}$$

Substituting the value of $(1/\lambda)$ from eq. (9) in eq. (8), we get

$$\frac{1}{\upsilon_g}=\frac{d}{d\nu}\left[\frac{m}{h}\left\{\frac{2(E-V)}{m}\right\}^{1/2}\right]$$

$$=\frac{d}{d\nu}\left[\frac{m}{h}\left\{\frac{2(h\nu-V)}{m}\right\}^{1/2}\right] \qquad (\because E=h\nu)$$

$$=\frac{1}{h}\frac{d}{d\nu}\left[\{2m(h\nu-V)\}^{1/2}\right]$$

$$=\frac{1}{h}\frac{1}{2}\{2m(h\nu-V)\}^{-1/2}\cdot 2mh$$

$$=\frac{m}{[2m(E-V)]^{1/2}}=\left[\frac{m}{2(E-V)}\right]^{1/2}=\frac{1}{\upsilon_p} \qquad \text{...(10)}$$

$$\boxed{\text{Group velocity } \upsilon_g = \text{particle velocity } \upsilon_p.}$$

Hence, we can say that a material particle in motion is equivalent to group of waves or a wave packet.

When the material particle is considered to be a wave packet, it is necessary to postulate the existence of a guiding wave for it. The equation of guiding wave was derived by Schroedinger. The physical significance of the Schroedinger wave equation is that it relates the amplitude of guiding wave to the probability of finding material particle at a point. If the amplitude of guiding wave is zero at a certain point in space, then the probability of finding the particle at the point is infinitesimal. The mechanical process is accompanied by wave process. On the other hand, if the amplitude of the guiding wave is large at any point in the space, then the probability of finding the particle at that point is maximum. This is equivalent to a particle in the packet which moves with group velocity. ***Thus, the wave packet plus guiding wave possess the properties of a particle moving with velocity*** v. With this idea the phenomena like interference, diffraction, etc. can be explained.

13.13 PHASE VELOCITY (OR WAVE VELOCITY) OF de-BROGLIE WAVES

In this article we shall show that phase velocity of de-Broglie waves is greater than velocity of light.

According to de-Broglie, a particle of mass m moving with velocity v is associated with a wave whose wavelength λ is given by

$$\lambda = \frac{h}{mv}$$

The propagation constant k of the wave is given by

$$k = \frac{2\pi}{\lambda} = \frac{2\pi(mv)}{h} \qquad ...(1)$$

Let E be the energy of particle. Then according to quantum condition $E = h\nu$, the frequency ν of associated wave is given by

$$\nu = \frac{E}{h}$$

$\therefore$ Angular frequency, $\omega = 2\pi\nu = (2\pi E/h)$

Further $$\omega = \frac{2\pi mc^2}{h} \qquad (\because E = mc^2) \quad ...(2)$$

Now de-Broglie wave velocity v_p is given by

$$v_p = \frac{\omega}{k} = \left(\frac{2\pi mc^2}{h}\right) \times \frac{h}{2\pi(mv)} = \frac{c^2}{v}$$

or $$\boxed{v_p \times v_g = c^2} \qquad (\because v_g = v)$$

Thus, the product of phase velocity (v_p) and group velocity (v_g) or particle velocity (v) is equal to the square of velocity of light (c^2).

As v is less than c and hence, the de-Broglie wave velocity must be greater than c (an unexpected result). So, the de-Broglie wave train associated with the particle would travel much faster than the particle itself and would leave the particle far behind. This statement is nothing but the collapse of the wave description of the particle.

13.14 GROUP VELOCITY OF de-BROGLIE WAVES

The angular frequency and propagation constant of de-Broglie waves associated with a particle of rest mass m_0 can be calculated as follows:

According to theory of relativity,

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \qquad ...(a)$$

We have derived

$$\omega = \frac{2\pi mc^2}{h} \quad \text{and} \quad k = \frac{2\pi mv}{h} \qquad \text{(See article 13.9)}$$

Substituting the value of m from eq. (a), we get

$$\omega = \frac{2\pi m_0 c^2}{h\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \qquad ...(1)$$

and $$k = \frac{2\pi m_0 v}{h\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \qquad ...(2)$$

Differentiating eqs. (1) and (2), we have

$$\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h\left[1 - \frac{v^2}{c^2}\right]^{3/2}} \qquad ...(3)$$

and $$\frac{dk}{dv} = \frac{2\pi m_0}{h\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \qquad ...(4)$$

The group velocity v_g of de-Broglie waves associated with the particle is given by

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv} \qquad ...(5)$$

Substituting the values of $(d\omega/dv)$ and (dk/dv) from eqs. (3) and (4) respectively in eq. (5), we get

$$\boxed{v_g = v}$$

Thus, de-Broglie wave group associated with a moving particle travels with same velocity as the particle.

SOLVED EXAMPLES

EXAMPLE 1 *Calculate the de-Broglie wavelength associated with the following:*

(i) A golf ball of 50 g moving with a velocity of 20 m/sec.

(ii) A proton moving with a velocity of 2200 m/sec.

(iii) An electron moving with a kinetic energy of 10 eV.

Solution (i) We know that $\lambda = h/mv$

$$\therefore \quad \lambda = \frac{6.625 \times 10^{-34} \text{ joule-sec}}{0.05 \text{ kg} \times 20 \text{ m/sec}} = 6.625 \times 10^{-34} \text{ m}$$

This is too small to be detected.

(ii) In case of a proton having mass 1.67×10^{-27} kg, we have

$$\lambda = \frac{6.625 \times 10^{-34} \text{ joule-sec}}{1.67 \times 10^{-27} \text{ kg} \times 2200 \text{ m/sec}} = 1.80 \times 10^{-10} \text{ m} = 1.80 \text{ Å}$$

This wavelength is of the order of X-ray wavelength.

(iii) In case of an electron with kinetic energy 10 eV, we have

$$\lambda = \frac{h}{\sqrt{(2mE)}} \quad \text{where } E = \text{Kinetic energy}$$

$$= \frac{6.625 \times 10^{-34} \text{ J-s}}{\sqrt{[2\,(9.11 \times 10^{-31} \text{ kg})\,(10 \times 1.602 \times 10^{-19})]}}$$

$$= 3.9 \times 10^{-10} \text{ m} = \mathbf{3.9\ Å}$$

Here it should be remembered that if the kinetic energy of the electron becomes comparable to its rest mass, we must use the relativistic expression for momentum mv.

❑ **EXAMPLE 2** *Calculate de-Broglie wavelength associated with a proton moving with a velocity equal to 1/20th of the velocity of light.*

Solution Velocity of proton

$$v = \frac{1}{20} \times \text{velocity of light} = \frac{1}{20} \times 3 \times 10^8 = 1.5 \times 10^7 \text{ m/sec}$$

Mass of the proton $= 1.67 \times 10^{-27}$ kg

$$\lambda = \frac{h}{mv} = \frac{6.62 \times 10^{-34}}{1.67 \times 10^{-27} \times 1.5 \times 10^7} = \mathbf{2.64 \times 10^{-14}} \text{ m}$$

❑ **EXAMPLE 3** *Find the energy of the neutron in units of electron volt whose de-Broglie wavelength is* 1 Å.

Given mass of the neutron $= 1.674 \times 10^{-27}$ kg

Planck's constant $h = 6.60 \times 10^{-34}$ J-sec

Solution We know that

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{(2mE)}}$$

$$E = \frac{h^2}{2m\lambda^2}$$

In the given problem $m = 1.674 \times 10^{-27}$ kg, $\lambda = 1\ Å = 10^{-10}$ m

and $h = 6.60 \times 10^{-34}$ J-sec.

$\therefore$

$$E = \frac{(6.60 \times 10^{-34})^2}{2 \times 1.674 \times 10^{-27} \times (10^{-10})^2} = 13.01 \times 10^{-21} \text{ J}$$

$$= \frac{13.01 \times 10^{-21}}{1.6 \times 10^{-19}} = \mathbf{8.13 \times 10^{-2}} \text{ eV}$$

❑ **EXAMPLE 4** *Calculate the energy in electron volt of an electron wave* $\lambda = 3 \times 10^{-12}$ m. *Given* $h = 6.62 \times 10^{-34}$ J-s.

Solution We know that energy is given by

$$E = \frac{h^2}{2m\lambda^2}$$

Here $h = 6.62 \times 10^{-34}$ J-s, $m = 9.1 \times 10^{-31}$ kg and $\lambda = 3 \times 10^{-12}$ m

$$E = \frac{(6.62 \times 10^{-34})^2}{2 \times 9.1 \times 10^{-31} \times (3 \times 10^{-12})^2} = 3.370 \times 10^{-19} \text{ J}$$

$$= \frac{3.370 \times 10^{-19}}{1.6 \times 10^{-19}} \text{ eV} = \mathbf{2.107\ eV}$$

❑ **EXAMPLE 5** *What will be the kinetic energy of an electron if its de-Broglie wavelength equals the wavelength of the yellow line of sodium (5896 Å)? The rest mass of electron is* $m = 9.1 \times 10^{-31}$ kg *and* $h = 6.63 \times 10^{-34}$ J-s.

Solution de-Broglie wavelength λ is given by

$$\lambda = \frac{h}{mv} \quad \text{or} \quad v = \frac{h}{m\lambda} \qquad \ldots(1)$$

If K be the kinetic energy, then

$$K = \frac{1}{2} m v^2 \qquad \ldots(2)$$

From eqs. (1) and (2), we get

$$K = \frac{1}{2} m \left(\frac{h^2}{m^2 \lambda^2} \right) = \frac{h^2}{2m\lambda^2}$$

$$= \frac{(6.63 \times 10^{-34})^2}{2 \times (9.1 \times 10^{-31}) \times (5896 \times 10^{-10})^2} = 6.95 \times 10^{-25} \text{ J}$$

$$= \frac{6.95 \times 10^{-25} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = \mathbf{4.34 \times 10^{-6}} \text{ eV}$$

❑ **EXAMPLE 6** *What voltage must be applied to an electron microscope to produce electrons of wavelength* 0.40 Å?

Solution de-Broglie wavelength λ is given by

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{(2mE)}} = \frac{h}{\sqrt{(2meV)}}$$

$E = eV$ where V is in volt

Here $h = 6.6 \times 10^{-34}$ J-s, $m = 9.1 \times 10^{-31}$ kg and $\lambda = 0.4 \times 10^{-10}$ m

$\therefore$

$$0.4 \times 10^{-10} = \frac{6.6 \times 10^{-34}}{\sqrt{(2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19}\ V)}}$$

Solving, we get $V = 960$ V.

❑ **EXAMPLE 7** *Determine the velocity and kinetic energy of a neutron having de-Broglie wavelength* 1 Å. *Mass of neutron is* 1.67×10^{-27} kg *and* $h = 6.63 \times 10^{-34}$ J-s.

Solution de-Broglie wavelength λ is given by

$$\lambda = \frac{h}{mv} \quad \text{or} \quad v = \frac{h}{m\lambda}$$

Substituting the given values, we have

$$v = \frac{6.63\times10^{-34}}{(1.67\times10^{-27})(10^{-10})} = 3.97\times10^{3}\ \text{m/sec.}$$

The K.E. of particle is given by

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\times(1.67\times10^{-27})(3.97\times10^{3})^2$$

$$= 1.32\times10^{-20}\ \text{J} = \frac{1.32\times10^{-20}}{1.6\times10^{-19}}\ \text{eV} = \mathbf{0.0825\ eV}$$

❑ **EXAMPLE 8** *Each of a photon and an electron has an energy of 1 keV. Calculate the corresponding wavelength. $h = 6.6\times10^{-34}$ J-s, $m = 9.1\times10^{-31}$ kg, 1 eV $= 1.6\times10^{-19}$ J.*

Solution The energy of a photon is given by

$$E = h\nu = \frac{hc}{\lambda}$$

$$\lambda = \frac{hc}{E}$$

Substituting the given values, we get

$$\lambda = \frac{(6.6\times10^{-34})(3\times10^{8})}{(1000\times1.6\times10^{-19}\ \text{J})}$$

$$(\because 1\ \text{keV} = 1000\ \text{eV} = 1000\times1.6\times10^{-19}\ \text{J})$$

$$= 12.4\times10^{-10}\ \text{m} = \mathbf{12.4\ Å}$$

If E is the kinetic energy, then

$$\frac{1}{2}mv^2 = E \quad \text{or} \quad mv^2 = 2E \qquad \therefore v = \sqrt{(2E/m)}$$

Now,

$$\lambda = \frac{h}{mv} = \frac{h}{m\sqrt{(2E/m)}} = \frac{h}{\sqrt{(2mE)}}$$

Substituting the given values, we have

$$\lambda = \frac{6.6\times10^{-34}\ \text{J s}}{\sqrt{(2\times(9.1\times10^{-31}\ \text{kg})\times(1000\times1.6\times10^{-19}\ \text{J}))}}$$

$$= \frac{6.6\times10^{-34}\ \text{J s}}{17\times10^{-24}\ \text{kg m/sec}} = 0.39\times10^{-10}\ \text{m} = \mathbf{0.39\ Å}$$

❑ **EXAMPLE 9** *What would be the wavelength of quantum of radiant energy emitted, if an electron is transmitted into radiation and converted into one quantum?*

Solution According to Planck, the energy E associated with one quantum is $E = h\nu$, where ν is the frequency of radiation.

When the energy of an electron is transmitted into radiation, we have

$$E = mc^2\ \text{(mass energy relation)}$$

$$h\nu = mc^2 \quad \text{or} \quad \frac{hc}{\lambda} = mc^2$$

$$\lambda = \frac{h}{mc} = \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}$$

$$= 0.0241\times10^{-10}\ \text{m} = \mathbf{0.0241\ Å}$$

❑ **EXAMPLE 10** *Compute the de-Broglie wavelength of 10^{11} keV neutron. Mass of neutron may be taken as 1.675×10^{-27} kg.*

Solution Here, kinetic energy of neutron

$$= 10^{11}\ \text{keV} = 10^{14}\ \text{eV} = 10^{14}\times1.6\times10^{-19}\ \text{J} = 1.6\times10^{-5}\ \text{J}$$

Now

$$\frac{1}{2}mv^2 = 1.6\times10^{-5}$$

or

$$v = \left(\frac{2\times1.6\times10^{-5}}{1.675\times10^{-27}}\right)^{1/2}\ \text{m/s}$$

Again

$$\lambda = \frac{h}{mv} = \frac{6.625\times10^{-34}}{1.675\times10^{-27}}\times\left(\frac{1.675\times10^{-27}}{2\times1.6\times10^{-5}}\right)^{1/2} = \mathbf{2.16\times10^{-18}\ m}$$

❑ **EXAMPLE 11** *Find the de-Broglie wavelength of a neutron of energy 12.8 MeV. Given mass of neutron $= 1.675\times10^{-27}$ kg.*

Solution Given that, K.E. $= 12.8$ MeV $= 12.8\times10^{6}$ eV $= 12.8\times10^{6}\times(1.6\times10^{-19})$ J

Now,

$$\frac{1}{2}mv^2 = 12.8\times10^{6}\times(1.6\times10^{-19})\ \text{J}$$

$$v = \left[\frac{2\times12.8\times1.6\times10^{-13}}{1.675\times10^{-27}}\right]^{1/2} = 1.554\times10^{7}\ \text{m/sec}$$

$$\lambda = \frac{h}{mv} = \frac{6.625\times10^{-34}}{(1.675\times10^{-27})\times(1.554\times10^{7})} = 2.529\times10^{-14}\ \text{m}$$

❑ **EXAMPLE 12** *Compute the de-Broglie wavelength of a proton whose kinetic energy is equal to the rest energy of an electron. Mass of a proton is 1836 times that of the electron.*

Solution According to Einstein's energy-mass relation, we have

$$E = mc^2 = 9.1\times10^{-31}\times(3\times10^{8})^2 = 81.9\times10^{-15}\ \text{J}$$

Mass of the proton $= 1836\times9.1\times10^{-31}$

Here

$$\frac{1}{2}mv^2 = 81.9\times10^{-15}$$

or

$$v = \left(\frac{2\times81.9\times10^{-15}}{1836\times9.1\times10^{-31}}\right)^{1/2}$$

Again

$$\lambda = \frac{h}{mv} = \frac{6.62\times10^{-34}}{1836\times9.1\times10^{-31}}\times\left(\frac{1836\times9.1\times10^{-31}}{2\times81.9\times10^{-15}}\right)^{1/2}$$

$$= 4\times10^{-14}\ \text{m} = 4\times10^{-4}\ Å = 0.0004\ Å$$

EXAMPLE 13 *Energy of a particle at absolute temperature T is of the order of kT. Calculate the wavelength of thermal neutrons at 27°C. Given mass of the neutron $= 1.67 \times 10^{-27}$ kg, Planck's constant $h = 6.60 \times 10^{-34}$ J-s, Boltzmann's constant $k = 8.6 \times 10^{-5}$ eV deg^{-1}.*

Solution We know that

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{(2mE)}}$$

According to the problem $E = kT$

$$\lambda = \frac{h}{\sqrt{(2mkT)}}$$

Here $h = 6.60 \times 10^{-34}$, $m = 1.67 \times 10^{-27}$ kg, $T = 27°C = 300° K$

$k = 8.6 \times 10^{-5} \times 1.6 \times 10^{-19}$ J deg^{-1} $= 1.376 \times 10^{-23}$ J deg^{-1}

$$\lambda = \frac{6.60 \times 10^{-34}}{\sqrt{[(2 \times 1.67 \times 10^{-27} \times 1.376 \times 10^{-23} \times 300)]}}$$

$$= 1.777 \times 10^{-10} \text{ m} = 1.777 \text{ Å}$$

EXAMPLE 14 *The phase velocity of sea waves is given by $\sqrt{\left(\frac{g\lambda}{2\pi}\right)}$. Calculate the group velocity of these waves. The wavelength is 580 m. Take $g = 9.8$ m/s^2.*

Solution The relation between group velocity v_g and phase velocity v_p is

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda} \quad \ldots(1)$$

Given

$$v_p = \sqrt{\left(\frac{g\lambda}{2\pi}\right)} = \sqrt{\left(\frac{g}{2\pi}\right)} (\lambda)^{1/2}$$

$$\frac{dv_p}{d\lambda} = \sqrt{\left(\frac{g}{2\pi}\right)} \frac{1}{2} (\lambda)^{-1/2} = \frac{1}{2}\sqrt{\left(\frac{g}{2\pi\lambda}\right)}$$

$$= \frac{1}{2\lambda}\sqrt{\left(\frac{g\lambda}{2\pi}\right)} = \frac{v_p}{2\lambda} \quad \ldots(2)$$

Substituting the value of $\left(\frac{dv_p}{d\lambda}\right)$ from eq. (2) in eq. (1), we get

$$v_g = v_p - \lambda \frac{v_p}{2\lambda} = \frac{v_p}{2} = \frac{1}{2}\sqrt{\left(\frac{g\lambda}{2\pi}\right)} \quad \ldots(3)$$

Substituting the given values, we get

$$v_g = \frac{1}{2}\sqrt{\left(\frac{9.8 \times 580}{2 \times 3.14}\right)} = 15 \text{ m/s}$$

EXAMPLE 15 *Calculate the group velocity of light waves ($\lambda = 5893$ Å) through carbon-di-sulphide ($\mu = 1.635$). Given $\frac{d\mu}{d\lambda} = -1.89 \times 10^{-5}$ Å^{-1}.*

Solution The relation between group velocity v_g and phase velocity v_p is

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda}$$

The phase velocity of light in carbon-di-sulphide is given by

$$v_p = \frac{c}{\mu} \qquad (c = \text{velocity of light})$$

$$\frac{dv_p}{d\lambda} = -\frac{c}{\mu^2}\frac{d\mu}{d\lambda}$$

$$\therefore \quad v_g = \frac{c}{\mu} + \frac{c}{\mu^2}\frac{d\mu}{d\lambda} = \frac{c}{\mu}\left[1 + \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right]$$

Substituting the given values, we get

$$v_g = \frac{3 \times 10^8}{1.635}\left[1 + \frac{5893 \text{ Å}}{1.635} \times (-1.89 \times 10^{-5} \text{ Å}^{-1})\right] = 1.835 \times 10^8 [1 - 0.0681]$$

$$= 1.835 \times 10^8 \times 0.932 = 1.71 \times 10^8 \text{ m/s}$$

13.15 HEISENBERG'S UNCERTAINTY PRINCIPLE

In 1927, Heisenberg proposed a very interesting principle, which is a direct consequence of the dual nature of matter, known as *uncertainty principle*. In classical mechanics, moving particle at any instant has a fixed position in space and a definite momentum which can be determined if the initial values are known. However, in wave mechanics the particle is described in terms of a wavepacket.

When we represent a particle as a wave packet, its dimension and position loose their precise meanings. The particle may be any where inside the wave packet. The position of the particle becomes more definite as the wavepacket becomes smaller and smaller. But the average value of wavelength in a smaller wavepacket becomes less defined because smaller wavepacket contains less number of waves. In this case the momentum ($p = h/\lambda$) is not well defined. So, when the position of particle is well defined then its momentum becomes less defined and *vice-versa*.

As shown in Fig. 12 (a), the wave packet is narrow. In this case, the position of the particle can be determined with accuracy but wavelength (and hence momentum) of the particle cannot be measured accurately. As shown in Fig. 12 (b), the wavepacket is large. In this case, the position of particle becomes uncertain while the wavelength (and hence momentum) can be determined fairly accurately. The reason is that the wavepacket contains enough waves

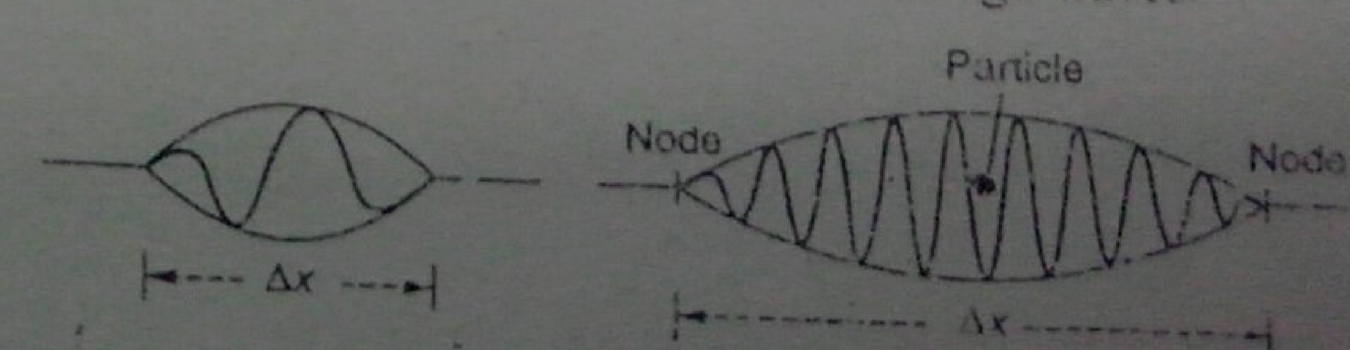


Fig. (12)

According to Heisenberg uncertainty principle, ***it is impossible to measure both the position and momentum of a particle simultaneously to any desired degree of accuracy.*** In general, this principle states that ***"It is impossible to specify precisely and simultaneously the values of both members of particular pairs of physical variables that describe the behaviour of an atomic system."***

Heisenberg showed that even if we design an ideal experiment to measure simultaneously the position x and corresponding component of momentum p_x of the particle, there is always an uncertainty of Δx in position and an uncertainty of Δp_x in momentum such that

$$\boxed{\Delta x \, \Delta p_x \approx h} \quad \ldots(1)$$

Similarly,

$$\boxed{\Delta E \, \Delta t \approx h} \quad \ldots(2)$$

and

$$\boxed{\Delta J \, \Delta\theta \approx h} \quad \ldots(3)$$

where ΔE and Δt are uncertainties in determining the energy and time while ΔJ and $\Delta\theta$ are uncertainties in determining the angular momentum and angle.

Elementary Proof

Consider two waves of angular frequencies ω_1 and ω_2 and propagation constants k_1 and k_2 travelling along X-axis. The waves are represented as

$$y_1 = a \sin(\omega_1 t - k_1 x) \quad \ldots(1)$$

and

$$y_2 = a \sin(\omega_2 t - k_2 x) \quad \ldots(2)$$

According to the superposition theorem, the resultant wave is

$$y = y_1 + y_2$$

or

$$y = a \sin(\omega_1 t - k_1 x) + a \sin(\omega_2 t - k_2 x)$$

or

$$y = 2a \sin(\omega t - kx) \cos\left(\frac{\delta\omega}{2} t - \frac{\delta k}{2} x\right)$$

where, $\omega = \frac{\omega_1 + \omega_2}{2}$, $k = \frac{k_1 + k_2}{2}$, $\delta\omega = \omega_1 - \omega_2$ and $\delta k = k_1 - k_2$.

The resultant wave is shown in Fig. 12 (b). The wave packet travels with group velocity, v_g. We know that group velocity is equal to particle velocity. Therefore, the loop is equivalent to the position of particle. According to uncertainty principle, the position of particle cannot be given by certainty but it may be somewhere between the nodes. So, its uncertainty is equal to the distance between the nodes.

A node is formed when $\cos\left(\frac{\delta\omega}{2} t - \frac{\delta k}{2} x\right) = 0$

i.e.,

$$\left(\frac{\delta\omega}{2} t - \frac{\delta k}{2} x\right) = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = (2n+1)\frac{\pi}{2} \quad \ldots(4)$$

where $n = 0, 1, 2, 3, \ldots$

If x_1 and x_2 denote the positions of two successive nodes, then at any instant t, we get

$$\frac{\delta\omega}{2} t - \frac{\delta k}{2} x_1 = (2n+1)\frac{\pi}{2} \quad \ldots(5)$$

and

$$\frac{\delta\omega}{2} t - \frac{\delta k}{2} x_2 = (2n+3)\frac{\pi}{2} \quad \ldots(6)$$

Subtracting eq. (6) from eq. (5), we get

$$\frac{\delta k}{2}(x_1 - x_2) = \pi$$

or

$$(x_1 - x_2) = \frac{2\pi}{\delta k}$$

or

$$\Delta x = \frac{2\pi}{\Delta\left(\frac{2\pi}{\lambda}\right)} = \frac{1}{\Delta(p/h)} = \frac{h}{\Delta p}$$

$\therefore$

$$\Delta x \cdot \Delta p \approx h$$

Note: The exact statement of uncertainty principle is as follows:

The product of uncertainties in determining the position and momentum of the particle is equal to or greater than $\frac{h}{4\pi}$ or $\frac{\hbar}{2}$. So, we have

$$\Delta x \, \Delta p \ge \frac{h}{4\pi}, \quad \Delta E \cdot \Delta t \ge \frac{h}{4\pi}, \quad \Delta J \cdot \Delta\theta \ge \frac{h}{4\pi}.$$

Physical Significance

Following are the physical significance of the Heisenberg uncertainty principle:

(i) This principle explains why it is possible for radiation and matter to have a dual (wave-particle) nature.

(ii) This principle helps in understanding many phenomena like, absence of electrons within nucleus, existence of protons and neutrons in nucleus, binding energy of an electron in atom, etc.

(iii) This principle also states that we can only predict the probable behaviour of quantum mechanical systems and not the *exact* behaviour.

13.16 TIME-ENERGY UNCERTAINTY PRINCIPLE

Here, we shall consider the time energy uncertainty with the help of position momentum uncertainty. Consider the case of a free particle with rest mass m_0 moving along X-direction with velocity v_x. The kinetic energy is given by

$$E = \frac{1}{2} m_0 v_x^2 = \frac{p_x^2}{2m_0} \quad \ldots(1)$$

If Δp_x and ΔE be the uncertainties in momentum and energy respectively, then differentiating eq. (1), we have

$$\Delta E = \frac{2 p_x \Delta p_x}{2 m_0}$$

or

$$p_x \Delta p_x = m_0 \Delta E$$

$\therefore$

$$\Delta p_x = \frac{m_0}{p_x} \Delta E = \frac{1}{v_x} \Delta E \quad \ldots(2)$$

Further, let the uncertainty in the time interval for measurement at point x is Δt, then uncertainty Δx in position is

$$\Delta x = v_x \cdot \Delta t \quad ...(3)$$

From eqs. (2) and (3), we get

$$\Delta x \, \Delta p_x = \Delta t \cdot \Delta E \quad ...(4)$$

We know that

$$\Delta x \, \Delta p_x \geq \frac{h}{4\pi}$$

$$\boxed{\Delta t \cdot \Delta E \geq \frac{h}{4\pi}} \quad ...(5)$$

13.17. EXPERIMENTAL ILLUSTRATION OF UNCERTAINTY PRINCIPLE

To illustrate uncertainty principle, we discuss the following two experiments :

1. Determination of the position of a particle by microscope

Consider the case of the measurement of the position of a particle (say electron) by high resolving power microscope as shown in Fig. (13). The resolving power, *i.e.*, the smallest distance between the two points that can be resolved by the microscope is given by

$$\Delta x = \frac{\lambda}{2 \sin \theta} \quad ...(1)$$

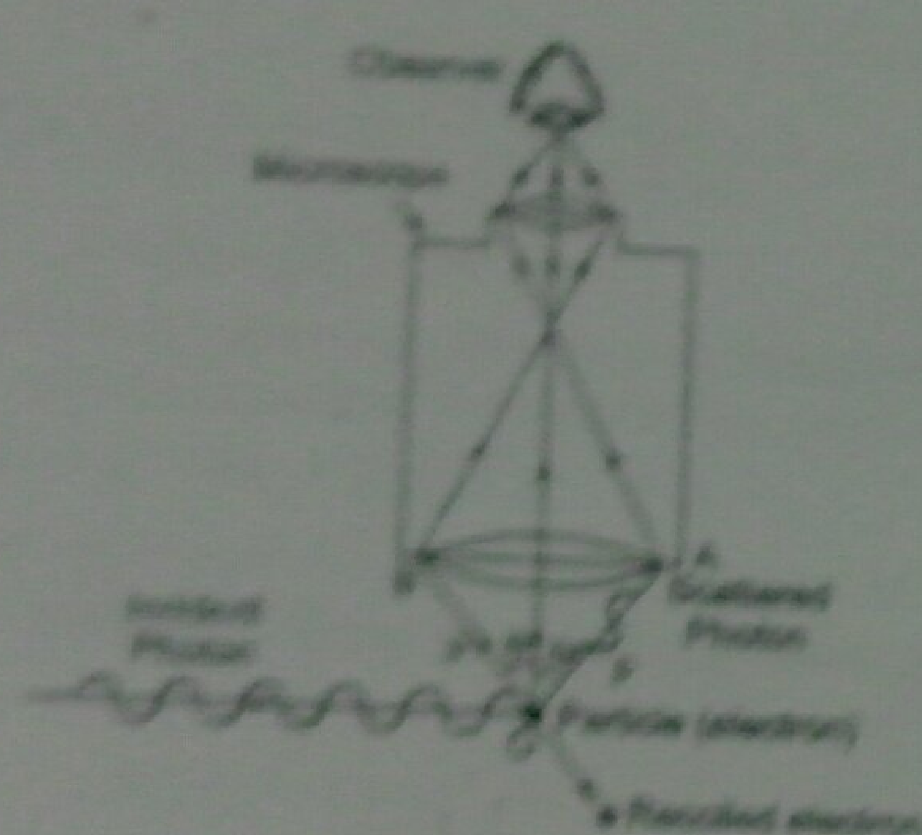


Fig. (13)

where λ is the wavelength of light used. θ is the semivertical angle of the cone of light rays entering the objective of microscope. Clearly, Δx represents the uncertainty in determining the position of the particle.

The electron is seen by the scattered photon into the microscope. In order to observe the electron, it is necessary that atleast one photon must strike the electron and scattered inside the microscope. The scattered photon can enter in the field of view of microscope between the angular range $+\theta$ to $-\theta$ as shown in the figure. The momentum (p) of the scattered photon is h/λ. If the photon enters the objective of microscope along OA, its momentum along X-axis is $p \sin \theta$ and if the photon enters in the direction of OB, then its momentum along x-axis is $-p \sin \theta$ as shown in Fig. (14).

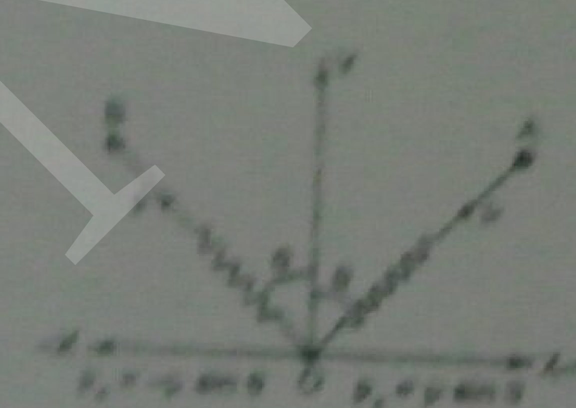


Fig (14)

Therefore, the uncertainty in the x-component of momentum Δp_x is given by

$$\Delta p_x = p \sin \theta - (-p \sin \theta)$$

or

$$\Delta p_x = 2p \sin \theta = 2\left(\frac{h}{\lambda}\right) \sin \theta$$

Due to conservation of momentum, the uncertainty in the measurement of x-component of electron (*i.e.*, Δp_x) will be the same as for electron. Therefore,

$$\Delta p_x = \frac{2h}{\lambda} \sin \theta \quad ...(2)$$

Thus, the product of uncertainties between Δx and Δp_x is given by

$$\Delta x \cdot \Delta p_x = \frac{\lambda}{2 \sin \theta} \times \frac{2h}{\lambda} \sin \theta = h \quad ...(3)$$

This is in accordance with the Heisenberg's uncertainty principle.

2. Diffraction of electron beam by a single slit

Suppose a narrow beam of electrons passes through a narrow single slit and produces a diffraction pattern on the screen as shown in Fig. (15).

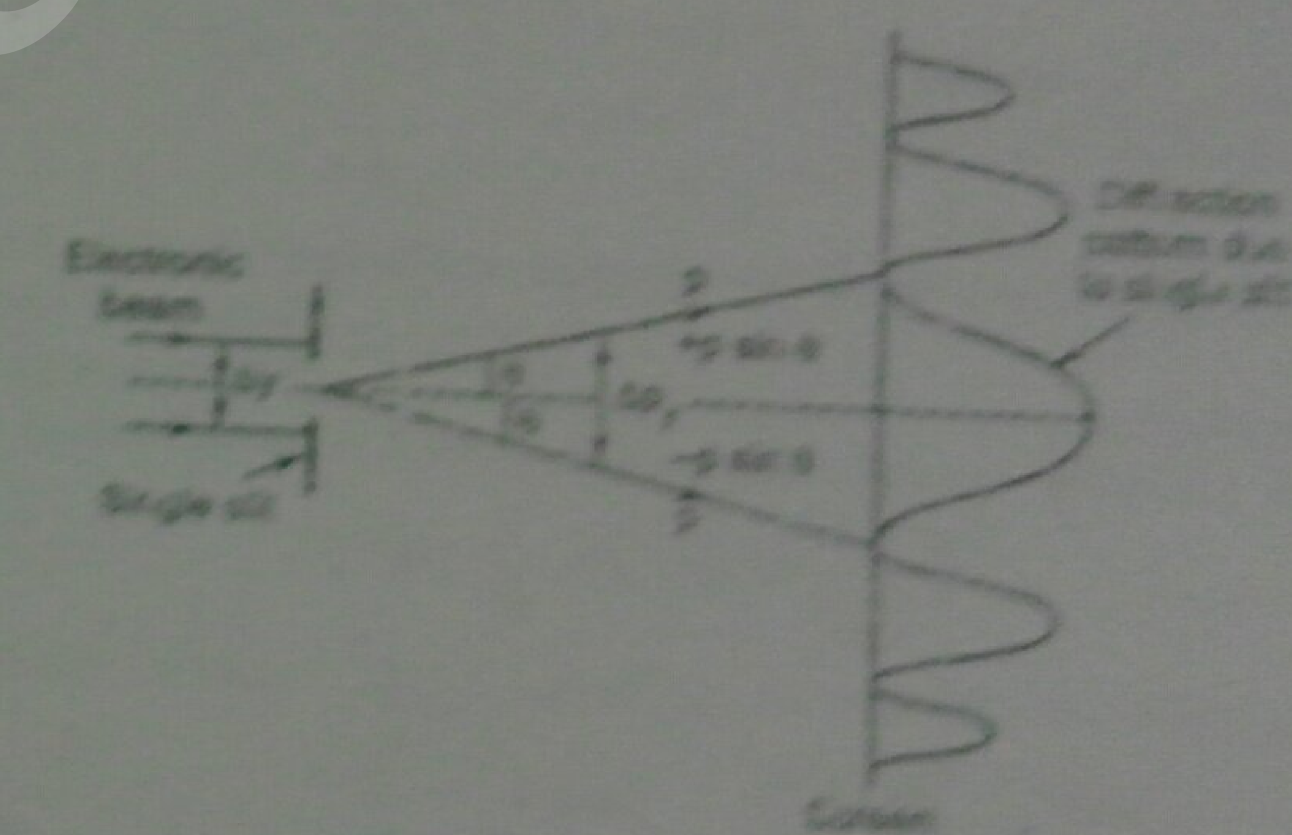


Fig. (15)

The diffraction pattern on the screen consists of a central bright band surrounding by alternate bright and dark bands on both sides. From the theory of Fraunhofer's diffraction at a single slit, the first minimum is given by

$$\Delta y \sin \theta = \lambda \qquad (\because 2d \sin \theta = \pm n\lambda) \quad ...(4)$$

where θ is the angle of diffraction corresponding to first minimum and λ is the wavelength of electron beam.

In producing the diffraction pattern on the screen, all the electrons have passed through the slit but we cannot say definitely at what place of the slit. Hence, the uncertainty in determining the position of electron is equal to the width Δy of the slit. From eq. (4), we have

$$\Delta y = \frac{\lambda}{\sin \theta} \quad ...(5)$$

Initially, the electrons are moving along the X-axis and hence they have no component of momentum along Y-axis. After diffraction on the slit, they are deviated from their initial path to form the pattern. Now, they have a component $p \sin\theta$. As y-component of momentum may be anywhere between $p\sin\theta$ and $-p\sin\theta$, hence the uncertainty in y-component of momentum is

$$\Delta p_y = p\sin\theta - (-p\sin\theta) = 2p\sin\theta$$

or

$$\Delta p_y = 2\left(\frac{h}{\lambda}\right)\sin\theta \qquad \left(\because p = \frac{h}{\lambda}\right) \quad ...(6)$$

From eqs. (5) and (6), we get

$$\Delta y\, \Delta p_y \geq \frac{\lambda}{\sin\theta} \times \frac{2h}{\lambda}\sin\theta = 2h \approx h$$

This relation shows that the product of uncertainties in position and momentum is of the order of Planck's constant.

13.18 APPLICATIONS OF UNCERTAINTY PRINCIPLE

1. Non-existence of electrons and existence of protons and neutrons in nucleus: The diameter of the nucleus of any atom is of the order of 10^{-14} m. If an electron is confined inside the nucleus, then uncertainty in the position Δx of the electron is equal to the diameter of the nucleus, *i.e.*, $\Delta x \approx 10^{-14}$ m. Using the Heisenberg's uncertainty relation, the uncertainty in momentum of electron is given by

$$\Delta p_x \geq \frac{h}{4\pi\, \Delta x} \geq \frac{6.63\times10^{-34}\ \text{J-s}}{4\times3.14\times(10^{-14}\ \text{m})} \qquad (\because \Delta x = 10^{-14}\ \text{m})$$

$$\geq 0.527\times10^{-20}\ \text{N-s}$$

It means that the momentum component p_x and hence the magnitude of total momentum $|\vec{p}|$ of the electron in the nucleus must be at least of the order of magnitude, *i.e.*,

$$|\vec{p}| \approx p_x \approx \Delta p_x \approx 0.527\times10^{-20}\ \text{N-s}.$$

Since the mass of the electron is 9.1×10^{-31} kg, the order of magnitude of momentum (0.527×10^{-20} kg m s^{-1}) is relativistic. Using the relstivistic formula for the energy E of the electron, we have

$$E^2 = p^2c^2 + m_0^2c^4$$

As the rest energy m_0c^2 of the electron is of the order of 0.511 MeV, which is much smaller than the value of first term, hence it can be neglected. Thus,

$$E^2 = p^2c^2 \quad \text{or} \quad E = pc$$

$\therefore$

$$E = (0.527\times10^{-20})\times(3\times10^8)\ \text{joule}$$

$$= \frac{(0.527\times10^{-20})(3\times10^8)}{1.6\times10^{-19}}\ \text{eV}$$

$$\approx 10\ \text{MeV}$$

This means that if the electrons exist inside the nucleus their energy must be of the order of 10 MeV. However, we know that the electrons emitted by radioactive nuclei during beta decay have energies only 3 to 4 MeV. Hence, in general electrons cannot exist in the nucleus.

For protons and neutrons, $m_0 = 1.67\times10^{-27}$ kg. This is a non-relativistic problems as $|\vec{v}| = |\vec{p}|/m_0 = 3\times10^6$ m s^{-1}. The kinetic energy E in this case is given by

$$E = \frac{p^2}{2m_0} = \frac{(0.527\times10^{-20})^2}{2\times1.67\times10^{-27}}\ \text{joule}$$

$$= \frac{(0.527\times10^{-20})^2}{2\times1.67\times10^{-27}\times1.6\times10^{-19}}\ \text{eV} \approx 52\ \text{keV}.$$

Since this theory is smaller than the energies carried by these particles emitted by nuclei, both these particles can exist inside the nuclei.

2. Radiation of light from an excited atom: We know that the average time period that an atom takes to come to its unexcited state from the excited state is of the order of 10^{-8} second. Thus, the uncertainty in the photon energy is given by

$$\Delta E \geq \frac{h}{4\pi \cdot \Delta t} \geq \frac{6.6\times10^{-34}}{4\times3.14\times10^{-8}} \approx 10^{-26}\ \text{joule}$$

Thus, the uncertainty in the frequency of the light is

$$\Delta\nu = \frac{\Delta E}{\frac{h}{4\pi}} \geq \frac{10^{-26}\times4\times3.14}{6.6\times10^{-34}} \approx 10^7\ \text{hertz}$$

This is the maximum limit to the accuracy with which one can determine the frequency of the radiation emitted by an atom.

3. Radius of Bohr's first orbit: If Δx and Δp be the uncertainties in position and momentum of the electron in the first orbit, then according to uncertainty principle, we have

$$\Delta x\, \Delta p \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta p \geq \frac{h}{4\pi\, \Delta x} \quad ...(1)$$

The uncertainty in kinetic energy ΔT of the electron may be written as

$$\Delta T = \frac{1}{2}m(\Delta v)^2 \geq \frac{m^2(\Delta v)^2}{2m} \geq \frac{(\Delta p)^2}{2m}$$

$\therefore$

$$\Delta T \geq \frac{h^2}{16\pi^2\, 2m\,(\Delta x)^2} \quad ...(2)$$

The uncertainty in potential energy ΔV of the same electron is

$$\Delta V \geq -\frac{Ze^2}{\Delta x} \quad ...(3)$$

The uncertainty in the total energy ΔE is given by

$$\Delta E \geq \Delta T + \Delta V \geq \frac{h^2}{16\pi^2\, 2m\,(\Delta x)^2} + \frac{-Ze^2}{\Delta x}$$

The uncertainty in energy will be minimum, if

$$\frac{d(\Delta E)}{d(\Delta x)} = 0 \quad ...(4)$$

Now,

$$\frac{d(\Delta E)}{d(\Delta x)} \geq \frac{-2h^2}{16\pi^2\, 2m\,(\Delta x)^3} + \frac{Z e^2}{(\Delta x)^2} = 0$$

or

$$\frac{h^2}{16\pi^2\, m\,(\Delta x)^3} = \frac{Z e^2}{\Delta x^2}$$

or

$$\Delta x = \frac{h^2}{16\pi^2\, m Z e^2} \qquad \text{...(5)}$$

This shows that in order to have minimum energy, the electron must be at least Δx away from the nucleus. We know that the energy of the electron is minimum in the first orbit. Therefore, the radius of the first orbit is given by

$$r = \Delta x = \frac{h^2}{16\pi^2\, m Z e^2} \qquad \text{...(6)}$$

SOLVED EXAMPLES

EXAMPLE 1 *If the uncertainty in position of an electron is 4×10^{-10} m, calculate the uncertainty in its momentum.*

Solution We know that

$$\Delta x \cdot \Delta p_x = h$$

$$\Delta p_x = \frac{h}{\Delta x} = \frac{6.6\times10^{-34}}{4\times10^{-10}} = 1.65\times10^{-24} \text{ kg m/sec}$$

EXAMPLE 2 *An electron has a speed of 600 m/s with an accuracy of 0.005%. Calculate the certainty with which we can locate the position of the electron. Given that*

$$h = 6.6\times10^{-34} \text{ joule-sec}, \quad m = 9.1\times10^{-31} \text{ kg}$$

Solution Uncertainty in speed,

$$\Delta v = 600 \times \frac{0.005}{100} \text{ m/s}$$

Uncertainty in momentum, $\Delta p_x = m\,\Delta v$

$$\therefore \quad \Delta p_x = (9.1\times10^{-31}) \times 600 \left(\frac{0.005}{100}\right)$$

$$= 9.1\times10^{-31} \times 5\times10^{-5} \times 600 \text{ kg m/s}$$

$$\therefore \quad \Delta x = \frac{h}{\Delta p_x} = \frac{6.6\times10^{-34}}{5\times10^{-5} \times 9.1\times10^{-31} \times 600} = 0.02254 \text{ m}$$

EXAMPLE 3 *Show that if the uncertainty in the location of a particle is equal to its de Broglie wavelength, the uncertainty in its velocity is equal to its velocity.*

Solution According to uncertainty principle

$$\Delta p \cdot \Delta x = h$$

or

$$\Delta(m v)\, \Delta x = h \quad \text{or} \quad \Delta v \cdot \Delta x = h/m$$

Given that

$$\Delta x = \lambda$$

$$\therefore \quad \Delta v = \frac{h}{m\lambda} = \frac{h}{m(h/mv)} \qquad (\because\ \lambda = h/mv)$$

or

$$\Delta v = v = \text{velocity of the particle.}$$

EXAMPLE 4 *A microscope using photons is employed to locate an electron in an atom to within a distance of 0.1 Å. What is the uncertainty in the momentum of the electron located in this way? What is the uncertainty in velocity? Rest mass of electron = 9.1×10^{-31} kg.*

Solution According to uncertainty principle,

$$\Delta p = \frac{h}{\Delta x} = \frac{6.63\times10^{-34} \text{ J-s}}{(0.1\times10^{-10} \text{ m})}$$

$$= 6.63\times10^{-23} \text{ kg-m/sec}$$

The uncertainty in the velocity of the electron is

$$\Delta v = \frac{\Delta p}{m_e} = \frac{6.63\times10^{-23} \text{ kg-m/sec}}{9.1\times10^{-31} \text{ kg}} = 7.28\times10^{7} \text{ m/sec}$$

EXAMPLE 5 *Compare the uncertainties in the velocities of an electron and a proton confined to 1 nm box. Their masses are 9.10×10^{-31} kg and 1.67×10^{-27} kg respectively.*

Solution Since electron and proton both are confined to the same box, the uncertainty in position Δx is same for both. Hence, uncertainty in Δp is also same.

Let $(\Delta v)_e$ and $(\Delta v)_p$ be the uncertainties in velocities of electron and proton respectively. Then

$$(\Delta v)_e = \frac{\Delta p}{m_e} \quad \text{and} \quad (\Delta v)_p = \frac{\Delta p}{m_p}$$

$$\therefore \quad \frac{(\Delta v)_e}{(\Delta v)_p} = \frac{m_p}{m_e} = \frac{1.67\times10^{-27} \text{ kg}}{9.1\times10^{-31} \text{ kg}} = 1835$$

EXAMPLE 6 *Using the uncertainty relation $\Delta E \cdot \Delta t = \frac{h}{2\pi}$, calculate the time required for the atomic system to retain the excitation energy for a line of wavelength 6000 Å and width 10^{-4} Å.*

Solution Here, the width of spectral line

$$d\lambda = 10^{-4} \text{ Å}$$

$$= 10^{-4}\times10^{-10} \text{ m} = 10^{-14} \text{ m}$$

and

$$\lambda = 6000 \text{ Å} = 6\times10^{-7} \text{ m}$$

We know that

$$E = h\nu = \frac{hc}{\lambda}$$

or

$$\Delta E = -\frac{hc}{\lambda^2}\Delta\lambda$$

Using uncertainty relation $\Delta E \cdot \Delta t = (h/2\pi)$, we obtain

$$\Delta t = \frac{h}{2\pi \cdot \Delta E} = \frac{h}{2\pi\left(\frac{hc}{\lambda^2}\right)\Delta\lambda} = \frac{\lambda^2}{2\pi c\,\Delta\lambda}$$

$$= \frac{(6\times10^{-7})^2}{2\times3.14\times(3\times10^8)\times10^{-14}} \text{ second} \qquad [\because d\lambda = \Delta\lambda = 10^{-14} \text{ m}]$$

$$= 1.9\times10^{-8} \text{ second}$$

❑ **EXAMPLE 7** *An electron has a speed of* 3.5×10^7 *cm/sec accurated to 0.0098%. With what fundamental accuracy can we locate the position of electron? (Mass of electron* $=9.11\times10^{-31}$ *kg).*

Solution Uncertainty in speed, $\Delta v = \left(\frac{0.0098}{100}\right)\times(3.5\times10^7)$

Uncertainty in momentum, $\Delta p = m\,\Delta v$

$$\therefore \quad \Delta p = (9.1\times10^{-31})\times\left[\left(\frac{0.0098}{100}\right)\times(3.5\times10^7)\right]$$

$$= (98\times10^{-6})\times(3.192\times10^{-23})$$

We know that

$$\Delta p\,\Delta x = \frac{h}{4\pi} \quad \text{or} \quad \Delta x = \frac{h}{4\pi\times\Delta p}$$

$$\therefore \quad \Delta x = \frac{6.63\times10^{-34}}{4\times3.14\times(98\times10^{-6})(3.192\times10^{-25})} = \mathbf{1.689\times10^{-6}} \text{ m}$$

❑ **EXAMPLE 8** *Assume that an electron is inside a nucleus of diameter* 10^{-15} *m. Using uncertainty principle, estimate the K.E. of the electron in eV.*

Solution The maximum uncertainty in the position

$$\Delta x = 10^{-15} \text{ m}$$

The corresponding maximum uncertainty in its momentum is given by

$$\Delta p = \frac{h}{4\pi\times\Delta x} = \frac{6.63\times10^{-34}}{4\times3.14\times(10^{-15})} = 5.276\times10^{-20} \text{ kg-m/sec.}$$

Now the momentum itself must be at least comparable in magnitude. Hence,

$$p = 5.276\times10^{-20} \text{ kg-m/sec.}$$

So

$$K = \frac{p^2}{2m_0} = \frac{(5.276\times10^{-20})^2}{2\times(9.1\times10^{-31})} = 1.530\times10^{-9} \text{ joule}$$

$$= \frac{1.530\times10^{-9}}{10^6\times(1.6\times10^{-19})} \text{ MeV} = \mathbf{9563} \text{ MeV}$$

❑ **EXAMPLE 9** *A microscope using photons is employed to locate an electron in an atom within a distance of 0.2 Å. What is the uncertainty in the momentum of electron located in this way? What is the uncertainty in velocity?* $h=6.63\times10^{-34}$ *J-s,* $m=9.1\times10^{-31}$ *kg.*

Solution $\Delta x = 0.2 \text{ Å} = 0.2\times10^{-10}$ m

The uncertainty in momentum is given by

$$\Delta p = \frac{h}{4\pi\,\Delta x} = \frac{6.63\times10^{-34}}{4\times3.14\times(0.2\times10^{-10})} = 2.64\times10^{-24} \text{ kg m/s}$$

The momentum of electron p is related to v by

$$p = mv \quad \text{or} \quad \Delta p = m\,\Delta v$$

$$\therefore \quad \Delta v = \frac{\Delta p}{m} = \frac{2.64\times10^{-24}}{9.1\times10^{-31}} = 2.90\times10^6 \text{ m/s}$$

❑ **EXAMPLE 10** *Calculate the uncertainty in the position of a dust particle with mass equal to 1 mg if uncertainty in its velocity is* 5.5×10^{-20} *m/s.*

Solution We know that $\Delta x\cdot\Delta p = \frac{h}{4\pi}$

Here, $\Delta p = m\,\Delta v = 10^{-6}\times(5.5\times10^{-20})$ $\qquad (\because m = 1 \text{ mg} = 10^{-6} \text{ kg})$

Now,

$$\Delta x = \frac{h}{4\pi\times\Delta p} = \frac{6.62\times10^{-34}}{4\times3.14\times[10^{-6}\times(5.5\times10^{-20})]}$$

$$= \frac{6.62\times10^{-34}}{69.08\times10^{-26}} = \mathbf{9.58\times10^{-13}} \text{ m}$$

❑ **EXAMPLE 11** *The life time of an excited state of nucleus is* 10^{-12} *s. What is the uncertainty in energy of γ-rays photon emitted?* $(h/2\pi = 1.054\times10^{-34}$ *J-s and* 1 eV $= 1.6\times10^{-19}$ J)

Solution The time interval Δt in which a photon of energy uncertainty ΔE is emitted is given by

$$\Delta E\,\Delta t \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta E \geq \frac{h/2\pi}{2\,\Delta t}$$

Therefore, the minimum uncertainty is

$$\Delta E = \frac{(h/2\pi)}{2\,\Delta t} = \frac{1.054\times10^{-34} \text{ J-s}}{2\times(10^{-12} \text{ s})} = 0.527\times10^{-22} \text{ J}$$

$$= \frac{0.527\times10^{-22} \text{ J}}{1.6\times10^{-19} \text{ J/eV}} = 3.3\times10^{-4} \text{ eV}$$

13.19 COMPTON EFFECT

When a beam of monochromatic X-rays strikes a target, the X-rays are dispersed in all possible directions. The phenomenon is called as ***scattering.*** The angle between the directions of incident and scattered rays is called as ***scattering angle.***

In 1921 Prof. A.H. Compton discovered that when a monochromatic beam of high frequency radiation (X-rays, γ-rays, etc.) is scattered by a substance, the scattered radiation contain two componentes–one having a lower frequency or greater wavelength and the other having the same frequency or wavelength. The radiation of unchanged frequency in the scattered beam is known as ***unmodified radiation*** while the radiation of lower frequency or slightly higher wavelength is called as ***modified radiation***. This phenomenon is known as Compton effect. This Compton scattering results in giving, (i) modified frequencies, (ii) unmodified frequency, and (iii) recoil electrons [See Fig. (16)].

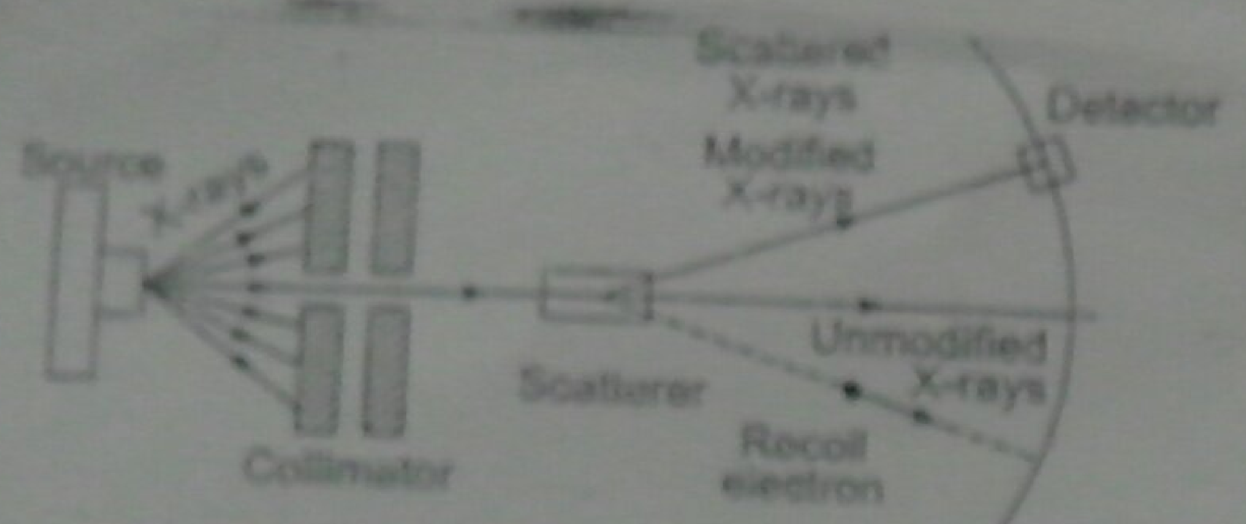


Fig. (16) Experimental arrangement for studying Compton effect

Explanation of Compton Effect

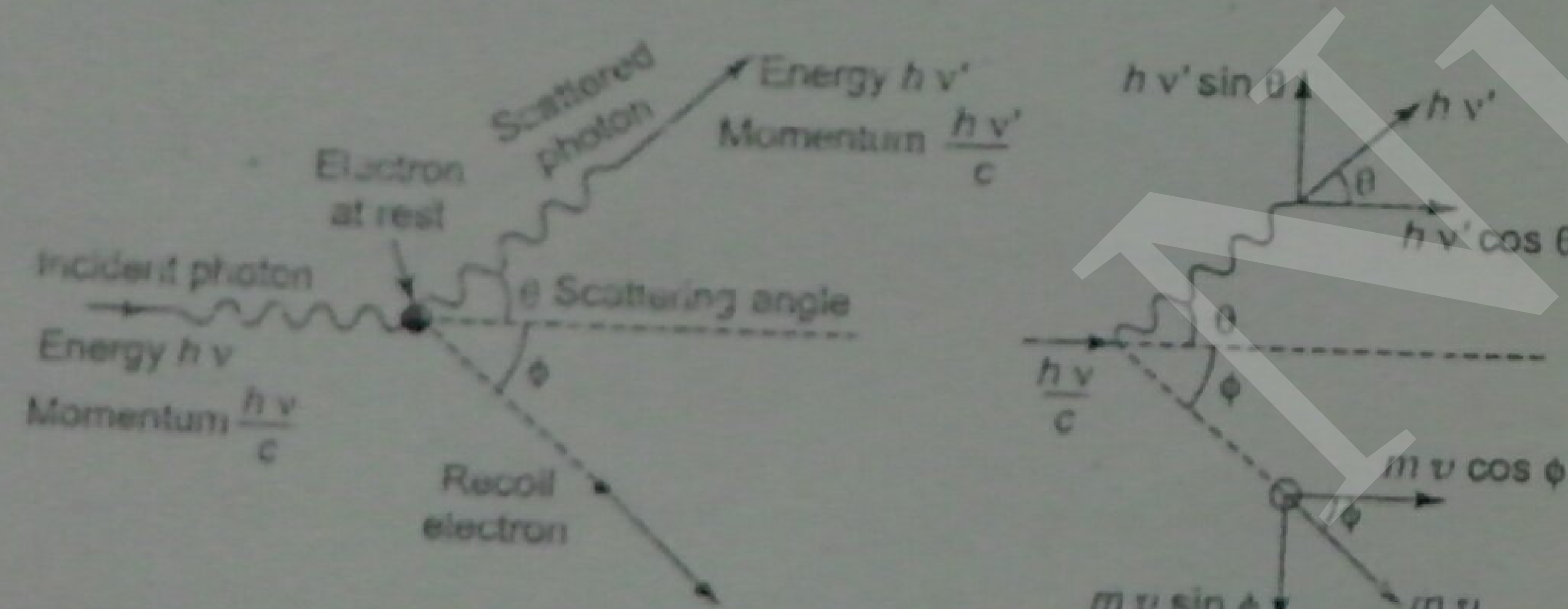
An adequate explanation of this effect was provided by Compton in 1922 on the basis of quantum theory of radiation. According to quantum concept of radiation, the radiation is constituted by energy packets called photons. The energy of photon is $h\nu$, where h is the Planck's constant and ν is the frequency of radiation. The photons move with the velocity of light c, possess momentum $h\nu/c$ and obey all the laws of conservation of energy and momentum. ***According to Compton, the phenomenon of scattering is due to an elastic collision between two particles, the photon of incident radiation and the electron of scatterer.*** When the photon of energy $h\nu$ collides with the electron of the scatterer at rest, it transfers some energy to the electron, *i.e.*, it loses energy. The scattered photon will therefore have a smaller energy $h\nu$ and consequently a lower frequency or greater wavelength than of the incident photon. ***The observed change in frequency or wavelength of the scattered radiation is known as Compton effect.*** In the scattering process, the electron gains kinetic energy and thus recoils with velocity v.

Theory of Compton effect

Considering the phenomenon of scattering as a collision between the photon and the electron and applying the laws of conservation of energy and momentum, Compton derived an expression for the change of wavelength in the following manner:

Compton assumed that the electron is free and is at rest before collision with the photon. After collision the relativistic mass of the electron is considered. The assumption is justified for the collision by high energy photons because only a small fraction of the photon energy imparts sufficient energy to the recoiled electron.

Let a photon of energy $h\nu$ collides with an electron at rest. Even if the electron is bound to the nucleus, a small fraction of $h\nu$ is used to free the electron, *i.e.*, work function is negligibly small as compared to the energy of photon and hence the electron is treated practically free. During the collision it gives a fraction of energy to the free electron. The electron gains kinetic energy and recoils. The process of recoiling of electron and scattering of photon is shown in Fig. (17). In figure, θ is the scattering angle while ϕ is the recoil angle.



(a) Geometry of Compton scattering

(b) Components of momentum before and after collision

Fig. (17)

Before Collision

(i) Energy of incident photon $= h\nu$

(ii) Momentum of incident photon $= \frac{h\nu}{c}$

(iii) Rest energy of the electron $= m_0 c^2$, where m_0 is rest mass of the electron.

(iv) Momentum of rest electron = 0.

After Collision

(i) Energy of scattered photon $= h\nu'$

(ii) Momentum of scattered photon $= \frac{h\nu'}{c}$

(iii) Energy of the electron $= mc^2$, where m is the mass of the electron moving with velocity v.

(iv) Momentum of recoil electron $= mv$, where v is the velocity of electron after collision

and
$$m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Energy of the system (photon-electron) before collision
$$= h\nu + m_0 c^2$$

Energy of the system after collision
$$= h\nu' + mc^2$$

According to the principle of conservation of energy
$$h\nu + m_0 c^2 = h\nu' + mc^2 \qquad ...(1)$$

Again using the principle of conservation of momentum along and perpendicular to the direction of incident, we get

Momentum before collision = Momentum after collision

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\theta + mv\cos\phi \qquad ...(2)$$

$$0 + 0 = \frac{h\nu'}{c}\sin\theta - mv\sin\phi \qquad ...(3)$$

From equation (2),
$$mvc\cos\phi = h\nu - h\nu'\cos\theta \qquad ...(4)$$

From equation (3),
$$mvc\sin\phi = h\nu'\sin\theta \qquad ...(5)$$

Squaring equations (4) and (5) and then adding, we get
$$m^2v^2c^2 = (h\nu - h\nu'\cos\theta)^2 + (h\nu'\sin\theta)^2$$
$$= h^2\nu^2 - 2h^2\nu\nu'\cos\theta + h^2\nu'^2\cos^2\theta + h^2\nu'^2\sin^2\theta$$
$$= h^2[\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta] \qquad ...(6)$$

From equation (1), we get
$$mc^2 = h(\nu - \nu') + m_0 c^2$$

Squaring $m^2 c^4 = h^2(\nu^2 - 2\nu\nu' + \nu'^2) + 2h(\nu - \nu')m_0 c^2 + m_0^2 c^4$ ...(7)

Subtracting equation (6) from equation (7), we have

$$m^2 c^4 - m^2 v^2 c^2 = -2h^2 \nu\nu'(1 - \cos\theta) + 2h(\nu - \nu')m_0 c^2 + m_0^2 c^4$$

or $$m^2 c^2 (c^2 - v^2) = -2h^2 \nu\nu'(1 - \cos\theta) + 2h(\nu - \nu')m_0 c^2 + m_0^2 c^4$$

or $$\frac{m_0^2 c^2}{1 - \frac{v^2}{c^2}}(c^2 - v^2) = -2h^2 \nu\nu'(1 - \cos\theta) + 2h(\nu - \nu')m_0 c^2 + m_0^2 c^4$$

or $$m_0^2 c^4 = -2h^2 \nu\nu'(1 - \cos\theta) + 2h(\nu - \nu')m_0 c^2 + m_0^2 c^4$$

∴ $$2h(\nu - \nu')m_0 c^2 = 2h^2 \nu\nu'(1 - \cos\theta)$$ ...[7(a)]

or $$\frac{\nu - \nu'}{\nu\nu'} = \frac{h}{m_0 c^2}(1 - \cos\theta)$$

or $$\boxed{\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0 c^2}(1 - \cos\theta)}$$ ...(8)

Equation (8) shows that $\nu' < \nu$ as h, m_0, c are the constants with positive values and the maximum value of $\cos\theta = 1$. This shows that the scattered frequency is always smaller than the incident frequency.

From equation (8), we have

$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_0 c}(1 - \cos\theta)$$

or $$\boxed{\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta)}$$ ...[8(a)]

or $$\Delta\lambda = \frac{2h}{m_0 c}\sin^2\frac{\theta}{2}$$ ...(9)

where $\Delta\lambda$ is change in wavelength.

The change in wavelength due to scattering is called as Compton effect.

Equation (9) shows that

(i) When $\theta = 0$, $\Delta\lambda = 0$, *i.e.*, there is no scattering along the direction of incidence.

(ii) When $\theta = \frac{\pi}{2}$, $\Delta\lambda = \frac{h}{m_0 c}$

$$= \frac{6.6 \times 10^{-34}}{9 \times 10^{-31} \times 3 \times 10^8}\text{ m} = 0.02426\text{ Å}$$

This difference in wavelength is known as Compton wavelength. Evidently, it is a constant quantity.

(iii) When $\theta = \pi$, $\Delta\lambda = \frac{2h}{m_0 c} = 0.4852\text{ Å}$

Hence, as θ varies from 0 to 180°, the wavelength of the scattered photon varies from λ to $\lambda + \frac{2h}{m_0 c}$, provided the wavelength of incident radiation is sufficiently small.

Figure (18) shows the Compton shift versus scattering angle θ.

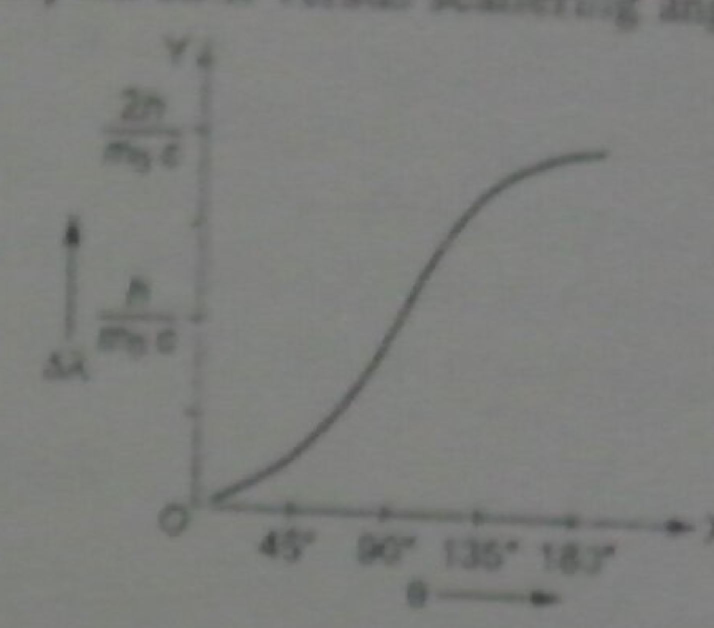


Fig. (18)

Importance of Compton Effect

Following are the importance of Compton effect :

(i) It provides the evidence of particle nature of electromagnetic radiation.

(ii) This verifies the Planck's quantum hypothesis.

(iii) This provides an indirect verification of the following relations

$$m = \frac{m_0}{\sqrt{\left[1 - \left(\frac{v^2}{c^2}\right)\right]}} \quad \text{and} \quad E = mc^2$$

because these expressions are used in Compton effect.

13.20 DIRECTION OF RECOIL ELECTRON

Dividing equation (5) by equation (4), of article 13.19 we get

$$\tan\phi = \frac{h\nu'\sin\theta}{h\nu - h\nu'\cos\theta} = \frac{\nu'\sin\theta}{\nu - \nu'\cos\theta}$$ ...(1)

$$= \frac{\frac{c}{\lambda'}\sin\theta}{\left(\frac{c}{\lambda}\right) - \left(\frac{c}{\lambda'}\right)\cos\theta}$$

$$= \frac{\lambda\sin\theta}{\lambda' - \lambda\cos\theta}$$ ...(2)

Using equation (8) of previous article, we get

$$\frac{1}{\nu'} = \frac{1}{\nu} + \frac{h}{m_0 c^2}(1 - \cos\theta) = \frac{1}{\nu} + \frac{h}{m_0 c^2}2\sin^2\frac{\theta}{2}$$

or $$\frac{1}{\nu'} = \frac{1 + \left(\frac{h\nu}{m_0 c^2}\right)2\sin^2\frac{\theta}{2}}{\nu}$$

or $$\nu' = \frac{\nu}{1 + \left(\frac{h\nu}{m_0 c^2}\right)2\sin^2\frac{\theta}{2}}$$ ...(3)

From equations (1) and (3), we have

$$\tan\theta = \frac{\dfrac{v\sin\theta}{1+\alpha\cdot 2\sin^2\dfrac{\theta}{2}}}{v-\left[\dfrac{v}{1+\alpha\cdot 2\sin^2\dfrac{\theta}{2}}\right]\cos\theta} \qquad \text{where } \frac{h\nu}{m_0c^2}=\alpha$$

or
$$\tan\phi = \frac{\sin\theta}{1+2\alpha\sin^2\dfrac{\theta}{2}-\cos\theta}$$

or
$$\tan\phi = \frac{\sin\theta}{(1-\cos\theta)+2\alpha\sin^2\dfrac{\theta}{2}} = \frac{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\sin^2\dfrac{\theta}{2}+2\alpha\sin^2\dfrac{\theta}{2}}$$

or
$$\tan\phi = \frac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}[1+\alpha]} = \frac{\cot\dfrac{\theta}{2}}{(1+\alpha)} = \frac{\cot\dfrac{\theta}{2}}{\left[1+\left(\dfrac{h\nu}{m_0c^2}\right)\right]} \qquad \text{...(4)}$$

13.21 EXPERIMENTAL VERIFICATION OF COMPTON EFFECT

Experimental arrangement. Fig. (19) shows the experimental set up for studying the Compton effect. A monochromatic beam of high energy *X*-rays are allowed to fall on a carbon block *C* (carbon acts as a good scatterer). The carbon block *C* scatters the *X*-rays in various directions. The intensity and wavelength of *X*-rays scattered through different angles θ were measured with Bragg spectrometer.

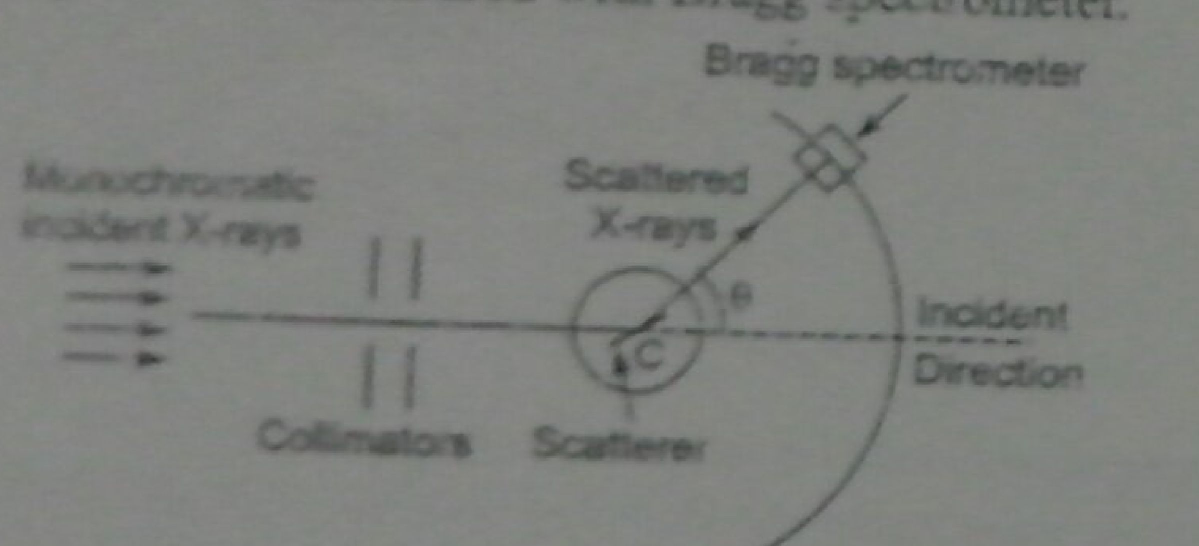


Fig. (19) Experimental arrangement for studying Compton effect

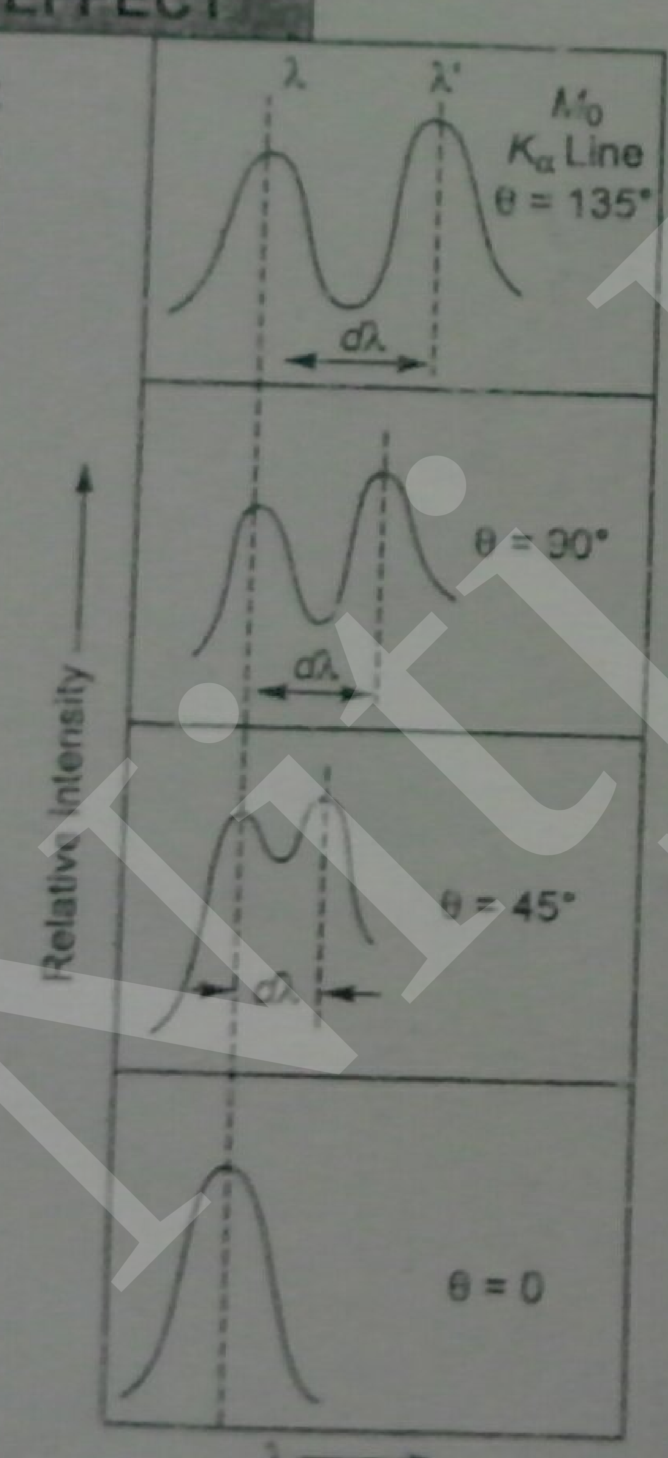


Fig. (20) Dependence of intensity of X-rays at different angle θ

In a typical experiment, the K_α line of molybdenum was scattered by graphite and measurements were made at scattering angle of 45°, 90° and 135°. The distribution of intensity of the scattered *X*-rays as function of their wavlengths was plotted for different scattering angles. Figure (20) shows the experimental results for different scattering angles. Figure (20) shows that for each value of θ, there are distinct intensity peaks for two wavelengths; one of which is the same as that of the incident radiation λ and the other

has a higher value λ'. The peak at λ' is called the modified radiation or modified peak. It is also obvious that $\lambda'-\lambda = d\lambda$ increases with scattering angle. Similar results were obtained by using different scatterers. In all the cases, λ' was found to depend on the scattering substance.

It is found that the shift in wavelength Δλ increases with θ in accordance with the result obtained from Compton theory. This gives the experimental verification of Compton theory.

■ SOLVED EXAMPLES

❑ **EXAMPLE 1** *In Compton scattering the incident photons have wavelength* 3.0×10^{-10} m. *Calculate the wavelength of scattered radiation if they are viewed at angle of 60° to the direction of incidence.*

Solution In Compton scattering, we have

$$\lambda'-\lambda = \frac{2h}{m_0c}\sin^2\left(\frac{\theta}{2}\right)$$

Here, $\lambda = 3.0\times10^{-10}$ m, $\theta = 60°$

$$\therefore \quad \lambda' = 3.0\times10^{-10} + \frac{2\times6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^8}\sin^2 30°$$

$$= 3.0\times10^{-10} + \frac{2\times6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^8}\times\frac{1}{4}$$

$$= 3.0\times10^{-10} + 0.012\times10^{-10}$$

$$= 3.012\times10^{-10}\ \text{m} = \mathbf{3.012\ Å}$$

❑ **EXAMPLE 2** *In a Compton experiment the wavelength of X-ray radiation scattered at an angle of 45° is 0.022 Å. Calculate the wavelength of incident X-rays.*

Solution In Compton scattering, we have

$$\lambda'-\lambda = \frac{2h}{m_0c}\sin^2\left(\frac{\theta}{2}\right) = \frac{h}{m_0c}(1-\cos\theta)$$

Here, $\lambda' = 0.022\ Å = 0.022\times10^{-10}$ m,

$h = 6.6\times10^{-34}$ J-s, $m_0 = 9.0\times10^{-31}$ kg

$c = 3\times10^8$ metre/sec and $\theta = 45°$

Now,
$$\lambda = \lambda' - \frac{h}{m_0c}(1-\cos\theta)$$

$$= 0.022\times10^{-10} - \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^8}\times(1-0.7071)$$

$$= 0.022\times10^{-10} - 0.7106\times10^{-12}$$

$$= 0.022\ Å - 0.007\ Å = \mathbf{0.015\ Å}$$

❑ **EXAMPLE 3** *A photon of energy 1.02 MeV is scattered through 90° by a free electron calculate the energy of photon and electron after interaction.*

Solution Change in wavelength of photon is given by

$$\Delta\lambda = \lambda' - \lambda = \frac{2h}{m_0 c}\sin^2\left(\frac{\theta}{2}\right)$$

$$= \frac{2\times 6.62\times 10^{-34}}{9.1\times 10^{-31}\times 3\times 10^{8}}\times\frac{1}{2} = 2.42\times 10^{-12}$$

$\therefore$ Change in frequency of photon

$$\Delta\nu = \frac{c}{\Delta\lambda} = \frac{3\times 10^{8}}{2.42\times 10^{-12}}$$

Change in energy of photon

$$\Delta E = h\,\Delta\nu = \frac{6.62\times 10^{-34}\times 3\times 10^{8}}{2.42\times 10^{-12}} = 0.51\ \text{MeV}$$

This energy is transferred to free electron.

Thus, the kinetic energy of electron after interaction is 0.51 MeV and the remaining energy of the photon after interaction

$$= 1.02\ \text{MeV} - 0.51\ \text{MeV} = \mathbf{0.51}\ \text{MeV}$$

❑ **EXAMPLE 4** *A photon recoils back after striking an electron at rest. What is the change in the wavelength of the photon?*

Solution The Compton shift $\Delta\lambda$ is given by

$$\Delta\lambda = \lambda' - \lambda = \frac{2h}{m_0 c}\sin^2\left(\frac{\theta}{2}\right)$$

Substituting the values, we get

$$\Delta\lambda = \frac{2\times 6.624\times 10^{-34}}{(9.1\times 10^{-31})(3\times 10^{8})}\sin^2\left(\frac{180^\circ}{2}\right)$$

$$= \frac{2\times 6.624\times 10^{-34}}{(9.1\times 10^{-31})(3\times 10^{8})}\sin^2 90^\circ$$

$$= 0.048\times 10^{-10}\ \text{metre} = \mathbf{0.048\ Å}$$

❑ **EXAMPLE 5** *An X-ray photon collides with an electron at rest. It is scattered through 90°. What is its frequency after collision if its initial frequency is 3×10^{19} Hz?*

Solution $\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\theta)$

Here, $\theta = 90^\circ$,

$\therefore$

$$\Delta\lambda = \frac{h}{m_0 c} = \frac{6.624\times 10^{-34}}{(9.1\times 10^{-31})(3\times 10^{8})}$$

$$= 0.024\times 10^{-10}\ \text{metre}$$

Now,

$$\Delta\lambda = \lambda' - \lambda = \frac{c}{\nu'} - \frac{c}{\nu} = \frac{0.024}{10^{10}}$$

or

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{0.024}{10^{10}\times(3\times 10^{8})} = \frac{0.024}{3\times 10^{18}}$$

$$\frac{1}{\nu'} = \frac{1}{\nu} + \frac{0.024}{3\times 10^{18}} = \frac{1}{3\times 10^{19}} + \frac{0.008}{10^{18}}$$

$$= \frac{1}{3\times 10^{19}} + \frac{0.08}{10^{19}} = \frac{1}{10^{19}}\left[\frac{1}{3} + 0.08\right] = \frac{1}{10^{19}}\times\frac{1.24}{3}$$

or

$$\nu' = \frac{3\times 10^{19}}{1.24} = \mathbf{2.419\times 10^{19}}\ \text{Hz}$$

❑ **EXAMPLE 6** *X-rays of wavelength 1 Å are scattered at such an angle that the recoil electron has the maximum K.E. Calculate the wavelength of the scattered ray.*

Solution We know that $\Delta\lambda = \frac{2h}{m_0 c}\sin^2\left(\frac{\theta}{2}\right)$

The Compton shift is maximum when $\theta = 180^\circ$

$\therefore$

$$(\Delta\lambda)_{max} = \frac{2h}{m_0 c} = 0.048\ Å$$

The incident wavelength $\lambda = 1$ Å

Hence, $\lambda' = \lambda + \Delta\lambda = 1\ Å + 0.048\ Å = 1.048\ Å$

❑ **EXAMPLE 7** *An X-ray photon is found to have its wavelength doubled on being scattered through 90°. Find the wavelength and energy of incident photon.*

Solution Given that $\lambda' = 2\lambda$ and $\theta = 90^\circ$

$\therefore$

$$\Delta\lambda = \lambda = \frac{h}{m_0 c}(1-\cos 90^\circ) = \frac{h}{m_0 c} = 0.024\ Å$$

This gives the wavelength of incident photon. The energy E is given by

$$E = h\nu = \frac{hc}{\lambda} = \frac{hc}{\left(\frac{h}{m_0 c}\right)} = m_0 c^2$$

$$= (9.1\times 10^{-31})(3\times 10^{8})^2 = 81\times 10^{-15}\ \text{joule}$$

❑ **EXAMPLE 8** *A beam of γ-radiation having photon energy of 510 keV is incident on a foil of aluminium. Calculate (a) the wavelength of radiation of 90°, (b) the energy and (c) direction of the corresponding electron.*

Solution (a) Energy of each photon

$$E = h\nu = \frac{hc}{\lambda}$$

Here, $E = 510\ \text{keV} = 510\times 10^{3}\times 1.6\times 10^{-19}$ joule

$\therefore$

$$\lambda = \frac{hc}{E} = \frac{6.6\times 10^{-34}\times 3\times 10^{8}}{510\times 10^{3}\times 1.6\times 10^{-19}} = 2.426\times 10^{-12}\ \text{m}$$