بسم الله الرحمن الرحیم

Brownian motion solved problem

(Problems from all over the world)

By: darya (masoud) Khosrotash

شابک : ۹۷۸-۶۰۰-۹۶۲۵۷-۳-۴
شماره کتابشناسی ملی : ۴۲۵۰۷۰۴
عنوان و نام پدیدآور : Brownian motion solved problem (problems from all over the world)/ by Darya (Masoud) Khosrotash.
مشخصات نشر : شهر ری: کوله پشتی من، ۱۳۹۵= ۲۰۱۶م.
مشخصات ظاهری : ۱۵۰ص.
یادداشت : انگلیسی.
آوانویسی عنوان : براونین...
موضوع : ریاضیات -- پرسش‌ها و پاسخ‌ها (عالی)
موضوع : Mathematics -- Questions and answers (Higher)
رده بندی دیویی : ۵۱۵
رده بندی کنگره : QA۳۰۰/خ۵ب۴ ۱۳۹۵
سرشناسه : خسروتاش، دریا، ۱۳۵۷ - Khosrotash, Masoud
وضعیت فهرست نویسی : فیپا

عنوان	brownian motion solved problem
نویسنده	دریا (مسعود) خسروتاش
صفحه آرا	الهام جلالیان
طراح جلد	
ناشر	انتشارات کوله پشتی من
شابک	۹۷۸-۶۰۰-۹۶۲۵۷-۳-۴
نوبت چاپ	اول ۹۵
تیراژ	۱۰۰۰
قیمت	
لیتوگرافی	راپیدگرافیک
چاپ و صحافی	سپه

Contents

پیش گفتار 13

Introduction 14

Mathematics 14

Physics 15

Preliminaries 17

Basic Properties 17

Numerical Simulation 19

Questions : 20

Question 20

Standard Brownian Motion 20

Answer 20

Question 21

About Brownian motion 21

Answer 22

Question 23

Deriving heat equation from brownian motion 23

Answer 23

Question 24

Brownian motion Convergence 24

answer 25

Question 26

Quadratic Brownian Motion 26

Answer 26

Question 27

Stopped brownian motion 27

Answer 28

Question 28
Brownian motion - Hölder continuity 28
Answer 29
Question 29
Simple Brownian Motion Proof 29
Answer 30
Question 31
differentiability of brownian motion 31
Answer 32
Question 33
Is this a Brownian motion? 33
Answer 34
Question 34
Reflected process - Brownian motion 34
Answer 36
Question 37
Brownian motion and covariance 37
Answer 38
Question 39
Local Maximum of Brownian motion. 39
Answer 39
Question 40
Harmonic functions and Brownian motion 40
Answer 40
Question 42
Brownian motion process and brownian bridge process 42
Answer 42
Question 43

Brownian motion - characteristic function 43
Answer 45
Question 46
Considering Brownian bridge as conditioned Brownian motion 46
Answer 47
Question 47
Ito's formula and $2-D$ Brownian motion 47
Answer 48
Question 49
Probability Inequality for Brownian motion 49
Answer 49
Question 50
Brownian Motion and Progressive Process 50
Answer 51
Question 51
Brownian motion and stochastic integration 51
Answer 52
Question 52
Normal probability and Brownian motion 52
Answer 53
Question 53
Absolute value of Brownian motion 53
Answer 54
Question 55
Converge to Brownian Motion problem 55
Answer 55
Question 59

Ito's Isometry using Brownian Motion 59
Answer 59
Question 60
local maximum of brownian motion 60
Answer 61
Question 61
Conditioned Maximum of Brownian Motion 61
Answer 62
Question 63
beginner's question about Brownian motion 63
Answer 64
Question 65
Prove increment of Brownian motion is Brownian motion 65
Answer 67
Question 67
An Integral Involving Brownian Motion 67
Answer 68
Question 69
Computing expectation of brownian motion 69
Answer 70
Question 70
Joint moments of Brownian motion 70
Answer 71
Conjecture 71
Question 72
Asymptotic behaviour of brownian motion 72
Answer 73
Question 74

Brownian motion transition density question..... 74
Answer 74
Question..... 75
Conditional distribution in Brownian motion..... 75
Answer 76
Question..... 77
Scaling property for Brownian motion..... 77
Answer 77
Question..... 78
(Multidimensional) Standard Brownian Motion: Convergence 78
Answer 79
Question..... 80
Geometric Brownian motion - Volatility Interpretation80
Answer 80
Question..... 82
Convergence to Brownian motion integral 82
Answer 83
Question..... 83
Hermite Polynomials and Brownian motion 83
Answer 84
Question..... 87
Brownian motion and posterior distribution 87
Answer 88
Question..... 90
Brownian motion at exponential time 90
Answer 90
Question..... 91

Definition of the Brownian motion 91
Answer 92
Question 93
Brownian bridge expression for a Brownian motion 93
Answer 94
Question 95
n times integrated Brownian motion 95
Answer 97
Question 98
Brownian motion and hitting frequency 98
Answer 99
Question 99
Mean of exponential Brownian motion 99
Answer 100
Question 102
Expectation of Brownian motion Integral 102
Answer 103
Question 104
Brownian Motion independent increment computation104
Answer 105
Question 105
Arcsine law for Brownian motion 105
Answer 106
Question 107
Distribution of Brownian Motion help 107
Answer 108
Question 109

Is $W_{2t} - W_t$ a brownian motion? 109
Answer 109
Question 110
quadratic variations of Brownian motion squared 110
Answer 110
Question 112
Correlated Brownian motion and Poisson process 112
Answer 112
Question 113
Discontinuous expression cannot be a Brownian motion? 113
Answer 113
Question 114
Martingale property of negative Brownian motion 114
Answer 114
Question 115
Probability of exponential of brownian motion 115
Answer 115
Question 116
Covariance of Brownian-motion-like processes 116
Answer 116
Question 118
Brownian motion in upper half-plane 118
Answer 119
Question 120
“Continuity” of stochastic integral wrt Brownian motion 120
Answer 120
Question 121

Law of large numbers for Brownian Motion 121
answer 121
Question 122
Integral with respect to brownian motion 122
Answer 123
Question 124
Scaled integrated Brownian motion has limit 124
Answer 124
Question 126
modulus of continuity of Brownian Motion 126
Answer 126
Question 127
Expectation of expression involving Brownian motion127
Answer 127
Question 128
change of sign for brownian motion 128
answer 128
Question 129
Laplace Transform of a Brownian motion 129
Answer 129
Question 130
Second order Cone and Brownian motion 130
Answer 131
question 132
Expected value of average of Brownian motion..... 132
Answer 132
Question 133
Ito Integral Properties with Brownian Motion 133

Answer 133
Question 134
Bound probability Brownian motion stays in $[-1,1]$. 134
Answer 135
Question 137
Conditional Integral of Square of Brownian Motion? . 137
Answer 137
Question 139
Is this true about Brownian Motion? 139
Answer 139
Question 140
Brownian motion recurrence theorems and Hausdorff Dimension 140
Answer 142
Question 143
A counter example of Brownian Motion 143
Answer 144
Reference 145

پیش گفتار

این کتاب شامل سوالاتی است که در سراسر جهان پرسیده و پاسخ داده شده است . در بزرگترین سیستم آموزشی (پرسش و پاسخ) جهان که آدرس ان در ذیل آمده است ، افراد سوال مورد نظر خود را به زبان انگلیسی مطرح می کنند و کسانی دیگر از گوشه و کنار جهان بدانها پاسخ می دهند . همچنین افرادی خبره پاسخ ها را رصد می کنند و پاسخ های درست را تایید می نمایند . این افراد اساتیدی از دانشگاه های برتر در کلیه کشور ها می باشند . بنابراین وقتی سوالی با پاسخ اش تایید شد به معنی این است که تا حد بسیار زیادی قابلیت اعتماد دارد . در کتابی که روبروی شماست ، سوالاتی مربوط به حرکت براونی و حواشی آن برگزیده و گردآوری شده است . امید است که این مسایل بتواند گره گشایی تمرینات و مفاهیم پایه دروسی همچون آنالیز تصادفی ، معادلات دیفرانسیل تصادفی و ریاضیات مالی گردد .

سعی بر آن بوده که مطالب خالی از اشکالات تایپی و گرامری باشند و باز بینی بسیار بر مطالب صورت گرفته ، اما به هر حال در کار های چنین گونه درصدی از خطا وجود دارد .بنابراین هر گونه انتقاد و یا پیشنهادی را به گوش جان شنیدارم .

من فقط می دانم ...کوشش بی ثمری نیست جهان

مسعود (دریا) خسروتاش

مدرس دانشگاه امام علی (ع) و دانشگاه ازاد اسلامی

amoodarya@yahoo.com

09121576134

https://telegram.me/riazidrKhosrotash

/khosrotash۱۰۴۱۷۱http://math.stackexchange.com/users/

Introduction

Mathematics

In mathematics, Brownian motion is described by the Wiener process; a continuous-time stochastic process named in honor of Norbert Wiener. It is one of the best known Lévy processes (càdlàg stochastic processes with stationary independent increments) and occurs frequently in pure and applied mathematics, economics and physics.

The Wiener process Wt is characterised by four facts:

1. $W_0 = 0$
2. W_t is almost surely continuous
3. W_t has independent increments

$$W_t - W_s \sim \quad (0, t-s)$$

(μ, σ^2) denotes the normal distribution with expected value μ and variance σ^2. The condition that it has independent increments means that if

$$0 \le s_1 < t_1 \le s_2 < t_2$$

then

$$W_{t_1} - W_{s_1}$$

and

$$W_{t_2} - W_{s_2}$$

are independent random variables.

An alternative characterisation of the Wiener process is the so-called *Lévy characterisation* that says that the Wiener process is an almost surely continuous martingale with $W_0 = 0$ and quadratic variation

$$[W_t, W_t] = t$$

A third characterisation is that the Wiener process has a spectral representation as a sine series whose coefficients are independent

$$(0,1)$$

random variables. This representation can be obtained using the Karhunen–Loève theorem.

Physics

The diffusion equation yields an approximation of the time evolution of the probability density function associated to the position of the particle going under a Brownian movement under the physical definition. The approximation is valid on short timescales.

The time evolution of the position of the Brownian particle itself is best described using Langevin equation, an equation which involves a random force field representing the effect of the thermal fluctuations of the solvent on the particle.

The displacement of a particle undergoing Brownian motion is obtained by solving the diffusion equation under appropriate boundary conditions and finding the rms of the solution. This shows that the displacement varies as the square root of the time (not linearly), which explains why previous experimental results concerning the velocity of Brownian particles gave nonsensical results. A linear time dependence was incorrectly assumed.

At very short time scales, however, the motion of a particle is dominated by its inertia and its displacement will be linearly dependent on time:

$$\Delta x = v \Delta t$$

. So the instantaneous velocity of the Brownian motion can be measured as

$$v = \frac{\Delta x}{\Delta t}$$

, when

$$\Delta t \ll \mathrm{T}$$

, where τ is the momentum relaxation time. In 2010, the instantaneous velocity of a Brownian particle (a glass microsphere trapped in air with an optical tweezer) was measured successfully.[1] The velocity data verified the Maxwell-Boltzmann velocity distribution, and the equipartition theorem for a Brownian particle.

The Brownian motion can be modeled by a random walk.[2] Random walks in porous media or fractals are anomalous.[3]

In the general case, Brownian motion is a non-Markov random process and described by stochastic integral equations.[4]

---

[1] Li, Tongcang; Kheifets, Simon; Medellin, David; Raizen, Mark (June 2010). "Measurement of the instantaneous velocity of a Brownian particle". Science 328(5986): 1673–1675. Bibcode:2010Sci...328.1673L

2 Weiss, G. H. (1994). Aspects and applications of the random walk. North Holland.

3 Ben-Avraham, D.; Havlin, S. (2000). Diffusion and reaction in disordered systems. Cambridge University Press.

4 Morozov, A. N.; Skripkin, A. V. (2011). "Spherical particle Brownian motion in viscous medium as non-Markovian random process". Physics Letters A 375 (46): 4113–4115. Bibcode:2011PhLA..375.4113M

Preliminaries

Basic Properties

Brownian motions constitute a particular class of continuous time stochastic processes. They basically are families of random variables parameterized by the non-negative reals.

Definition 1. A *brownian motion* W_t with *dispersion coefficient* $\sigma \geq 0$, also denoted W(t) or simply W, is a stochastic process such that,

1. It passes through the origin : $W_0 = 0$.
2. It has independent increments on non-overlapping intervals : The two random variables,

$$\Delta_1 = W_{t_2} - W_{t_1} \qquad \text{and} \qquad \Delta_2 = W_{t_4} - W_{t_3}$$

   are independent as soon as $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$.

3. Increments are normally distributed : For all t_1 and t_2 with $0 \leq t_1 \leq t_2$,the random variable $W_{t_2} - W_{t_1}$ has normal distribution, with mean 0 0 and standard deviation $\sigma\sqrt{t_2 - t_1}$.
4. With probability $1, W_t$ is continuous as a function of t

In other words, the variance of the increments are proportional to the length of the corresponding interval. The proportionality coefficient is the square σ^2 of the dispersion. A brownian motion is *qualified standard* when $\sigma = 1$.

The dynamic of brownian motions is both rich and simple. Rich in that it models a lot of natural phenomenon of stochastic nature. Simple in that it is subject to some very nice symmetries. For instance, each of the following relation defines a brownian motion Y from a given one W, both of the same dispersion.

1. Symmetry :

$$Y_t = -W_t$$

2. Scaling :

$$Y_t = sW_{\frac{t}{s^2}}$$

3. Translation :

$$Y_t = W_{t+T} - W_T$$

4. Reflexion principle :

$$Y_t = \begin{cases} W_t & t \le T \\ 2W_T - W_t & t \ge T \end{cases}$$

5. Time inversion :

$$Y_t = \begin{cases} 0 & t = 0 \\ tW_{1/t} & t > 0 \end{cases}$$

Each of those symmetries is in fact a theorem worth proving. We invite the interested reader to try to prove them using only the properties of the definition.

One more thing, if W and W' are two independent brownian motions having the same dispersion σ, then one can piece them together at a particular point of time T to produce another brownian motion Y of dispersion σ.

$$Y_t = \begin{cases} W_t & t \le T \\ W_T + W'_{t-T} & t \ge T \end{cases}$$

Numerical Simulation

Suppose first that we want to simulate the values taken by a brownian motion WW at regularly spaced time intervals. We call Δt the distance between two successive time intervals. Then for $k = 0,1,...$ we have $t_k = k\Delta t$ and

$$W_{t_{k+1}} = W_{t_k} + \sigma\sqrt{\Delta t}\, X_k$$

where $X_k \sim \quad (0,1)$ are independent standard gaussian variables. Standard here means, of zero mean and unit variance.

We've just approximated a brownian motion by a discrete stochastic process (i.e. a sequence of random variables $(U_k)_{k\geq 0}$ with $U_k = U_{k_t}$. But the converse is also possible and useful.

Questions :

Question

Standard Brownian Motion

Let $\{X_t, t \geq 0\}$ be a standard Brownian motion. Compute the density of X_t conditioned by X_{t_1} and X_{t_2} assuming that $t_1 < t < t_2$.

Can anyone give me some hint to start the question?

Answer

Let $f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ be the $N(0,t)$ density, and let $f_{s,t}$ be the joint density of (X_s, X_t). If $s < t$, then since the bijection $(X_s, X_t - X_s) \mapsto (X_s, X_t)$ has inverse $(u,v) \mapsto (u, v-u)$ with jacobian determinant 1,

$$f_{s,t}(x,y) = f_{s,t-s}(x, y-x) = f_s(x) f_{t-s}(y-x).$$

Then

$$f_{t|s}(x \mid y) = \frac{f_{s,t}(x,y)}{f_s(x)} = f_{t-s}(y-x)$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)}$$

Apply this to (X_{t_1}, X_t, X_{t_2}): the bijective map

$(x, y, z) \mapsto (x, y-x, z-y)$ has inverse
$(u, v, w) \mapsto (u, u+v, u+v+w)$ with jacobian determinant 1, and so for $s < t < u$

$$f_{s,t,u}(x, y, z) = f_{s,t-s,u-t}(x, y-x, z-y)$$
$$= f_s(x) f_{t-s}(y-x) f_{u-t}(z-y)$$ Then

$$f_{t|s,u}(y \mid x, z) = \frac{f_{s,t,u}(x, y, z)}{f_{s,u}(x, y)}$$

$$= \frac{f_s(x) f_{t-s}(y-x) f_{u-t}(z-y)}{f_{s,u}(x, y)} = \frac{f_{t-s}(y-x) f_{u-t}(z-y)}{f_{u-s}(y-x)}$$

Question

About Brownian motion

Let T be the last time before 1 a Brownian motion visits 0. Explain why

$X(t) = B(t+T) - B(T) = B(t+T)$ is not a Brownian motion. This problem is from *Introduction to Stochastic Calculus with Applications*, by Klebaner,Exercise 3.15, and I can't understand the solution provided by the book.

Any hints, thanks.

Answer

Let $(B_t)_{t\geq 0}$ a Brownian Motion and $\xi_t := \sup\{s \leq t; B_s = 0\}$. Then we know (by arc-sine law, see René L. Schilling/Lothar Partzsch: "Brownian Motion - An Introduction to stochastic processes", Theorem 6.19)

$$[\xi_t < s] = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}}$$

for all $s \leq t$. Thus

$$[\exists t \in (0,\varepsilon) : B_t = 0] = 1 - [\forall t \in (0,\varepsilon) : B_t \neq 0] = 1 - [\xi_\varepsilon = 0] = 1$$

for all $\varepsilon > 0$ which means that you can (a.s.) find for every $w \in \Omega$ a sequence $(t_n)_n$ such that $t_n \to 0$, $B(t_n, w) = 0$.

And this means that $(X_t)_{t\geq 0}$ can't be a Brownian motion: Let $w \in \Omega$, then we have $X_t(w) \neq 0$ for all $0 < t \leq 1 - T(w)$ (by definition of T). So there can't exist a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to 0$, $X_{t_n}(w) = 0$.

Question

Deriving heat equation from brownian motion

Today my prof gave me an equation of random walk: $p(x_i, t+\Delta t) = \frac{1}{2}(p(x_i - \Delta t) + p(x_i + \Delta t)) - p(x_i, t)$ Using this he get $P_t = P_{xx}$ when $\Delta t << 1$

But how and what's actually the meaning of the probability equation? e.g. one half of particles goes left and another half goes right, while nothing left in the middle. But why divided by 2 an how does this actually make sense?

Answer

The expression in the OP is incorrect. It should be

$$p(x_i, t+\Delta t) = \frac{1}{2}(p(x_i + \Delta x, t) + p(x_i - \Delta x, t)) \qquad (1)$$

Expanding the left-hand side of (1) in a Taylor series in Δt, we obtain

$$p(x_i, t+\Delta t) = p(x_i, t) + p_t(x_i, t)\Delta t + O(\Delta t)^2 \quad (2)$$

Expanding the left-hand side of (1) in a Taylor series in Δt, we obtain

$$p(x_i + \Delta x, t) + p(x_i - \Delta x, t) =$$
$$p(x_i, t) + p_x(x_i, t)\Delta x + \frac{1}{2} p_{xx}(x_i, t)(\Delta x)^2 + O(\Delta x)^3$$

$$+p(x_i,t)-p_x(x_i,t)\Delta x+\frac{1}{2}p_{xx}(x_i,t)(\Delta x)^2+O(\Delta x)^3$$

$$=p(x_i,t)+p_{xx}(x_i,t)(\Delta x)^2+O(\Delta x)^3 \qquad (3)$$

Equating (2) and (3), dividing both sides by Δt,

$$p_t(x_i,t)+O(\Delta t)=\frac{1}{2}p_{xx}(x_i,t)\frac{(\Delta x)^2}{\Delta t}+\frac{O(\Delta x)^3}{\Delta t}$$

If $\lim_{\Delta t\to 0}\frac{(\Delta x)^2}{2\Delta t}=1$, then (4) becomes in the limit as $\Delta t\to 0$,

$$p_t(x_i,t)=p_{xx}(x_i,t)$$

as was to be shown!

Question

Brownian motion Convergence

If X is a standard $1d$ brownian motion and

$M_t=\max\{X_S:0\le s\le t\}$, what can we say about M_t/t as $t\to\infty$?

mainly, what can we say about the behavior of this martingale?

My attempt: P(Msubt > a) = 2P(B(t) >a), but what integral does this fit in with?

answer

Since B_t is symmetric, the result recalled in the post reads $\mathrm{P}(M_t > at) = \mathrm{P}(|B_t| > at)$ $(*)$.

Since B_t equals $\sqrt{t}B_1$ in distribution, $(*)$ is $\mathrm{P}(|B_1| > a\sqrt{t})$. Since $|B_1|$ is almost surely finite and $a\sqrt{t} \to +\infty$ for every $a > 0$, the last probability goes to zero, hence $\mathrm{P}(M_t / t > a) \to 0$ for every $a > 0$. That is, $M_t / t \to 0$ in probability.

One can strengthen this convergence, noting that the series $\sum_n \mathrm{P}(|B_1| > a\sqrt{n})$ converges since $\mathrm{P}(|B_1| > a\sqrt{n})$ is $\mathrm{e}^{-\frac{1}{2}a^2 n + o(n)}$. By the first Borel-Cantell[5]i lemma, $M_n / n \to 0$ almost surely when the integer n goes to infinity. For every t 1 there exists an integer n 1 such that n $t < n+1$, and $(M_t)_t$ is non decreasing hence M_t / t M_{n+1} / n $2M_{n+1} / (n+1)$. This proves that $M_t / t \to 0$ almost surely when the real number t goes to infinity.

---

[5] Let $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_n$ be a sequence of events in some probability space. The Borel–Cantelli lemma states:

> If the sum of the probabilities of the E_n is finite $\sum_{n=1}^{\infty} \Pr(\mathrm{E}_n) < \infty$, then the probability that infinitely many of them occur is 0, that is, $\Pr(\limsup_{n\to\infty} \mathrm{E}_n) = 0$.

The same identity $(*)$ shows the convergence in L^p for every finite $p \quad 1$.

Question

Quadratic Brownian Motion

If $B(t)$ is standard Brownian Motion then can we say that $B^2(t)-t$ is a martingale?? Given the following theorem:

If $\max_{1<k<n}(t_k - t_{k-1}) \to 0$ as $n \to \infty$ then

$$\sum_{1}^{n} |B(t_k) - B(t_{k-1})|^2 \to T \quad \text{a.s}$$

Many thanks

Answer

The theorem you mention shows that the quadratic variation of a Brownian motion on [0,T] equals T. This implies in particular that $(B_t^2 - t)_{t \geq 0}$ is a martingale. But actually, this argumentation is overkill since the martingale property follows readily from the properties of a Brownian motion:

Let $s \leq t$. Obviously,

$$B_t^2 = ((B_t - B_s) + B_s)^2 = (B_t - B_s)^2 + 2B_s \cdot (B_t - B_s) + B_s^2$$

Using that $B_t - B_s$ is independent of $_s$, B_s is $_s$-measurable and $(B_r) = 0$ for any

$$(B_t^2 - t \mid \ _s) =$$
$$((B_t - B_s)^2 \mid \ _s) + 2 \ (B_s \cdot (B_t - B_s) \mid \ _s) + \ (B_s^2 \mid \ _s) - t =$$
$$((B_t - B_s)^2) + 2B_s \cdot \ (B_t - B_s) + B_s^2 - t$$

$r \geq 0$, we find

By the stationarity of the increments, $B_t - B_s \sim B_{t-s}$. From

$(B_{t-s}^2) = t - s$ and $(B_{t-s}) = 0$ we conclude

$$(B_t^2 - t \mid \ _s) = (t - s) + B_s^2 - t = B_s^2 - s$$

This shows that $(B_t^2 - t)_{t \geq 0}$ is a martingale.

Question

Stopped brownian motion

Assume B_t is a standard complex (or 2D if you wish) brownian motion and τ is a stopping time relative to B_t. I want to know if it is possible to construct another brownian motion W_t such that for $t \leq \tau$, we have

$$W_t = B_{\tau - t} - B_\tau$$

I am not sure if this can be done? I know about strong markov property and splicing brownian motion. I tried to use them to show this but didn't work. I thought at least similar proof to those theorems should apply here, but I couldn't make it work.

Answer

I think not. Let $B_t = (B_t^1, B_t^2)$ be your Brownian motion, and let $\tau = \inf\{t \geq 0 : B_t^1 = 1\}$. By recurrence of 1-D Brownian motion, $\tau < \infty$ almost surely. If $W_t = B_{\tau - t} - B_\tau$ for $t \leq \tau$ as in your question, then W has the property that almost surely, there exists a random time $\tau > 0$ such that $W_s^1 \leq 0$ for all $0 \leq s \leq \tau$. The probability of a Brownian motion having this property is 0, so W is not a Brownian motion.

Question

Brownian motion - Hölder continuity

Let B stand for a Brownian motion on a finite interval [0,1]. If I am not wrong, I think that there exists a positive constant c, such that almost surely, for h small enough , for all $0 < t < 1 - h$

$$| B(t+h) - B(t) | < c\sqrt{h \log(1/h)}$$

or something like this. As a result

$$\left| \frac{B(t+h) - B(t)}{h} \right| < K(h)$$

Am I correct ?

Answer

Yes, the result is known as Lévy's modulus of continuity:

Let $(B_t)_{t\geq 0}$ a (one-dimensional) Brownian motion. Then

$$\left(\limsup_{h\to 0} \frac{\sup_{0\leq t\leq 1-h} |B(t+h)-B(t)|}{\sqrt{2h\,|\log h|}} = 1 \right) = 1.$$

See e.g. *René Schilling/Lothar Partzsch: Brownian motion - An introduction to stochastic processes* for a proof $(Section 10.3)$.

Question

Simple Brownian Motion Proof

I've been given the following question and solution:

Let W_t be a standard Brownian Motion w.r.t. $(\mathbf{P}, \quad _t)$.

Prove that

$$E[|W_t|] < \infty, \forall\ t$$

Solution:

$$E[|W_t|] < E[1+W_t^2] < 1 + E[W_t^2] < 1 + t < \infty$$

My question is, what allows us to state the following?

$$E[|W_t|] < E[1+W_t^2]$$

Answer

Note that [6]

$$|W_t| = |W_t| \cdot 1_{\{|W_t| \le 1\}} + |W_t| \cdot 1_{\{|W_t| > 1\}}$$

$$\le 1 + |W_t|^2$$

This implies $(|W_t|) \le 1 + (W_t^2)$.

Remarks

- Please note that any random variable $X \in L^2$ is automatically integrable, i.e. $X \in L^1$. This follows from Jensen's inequality or the Cauchy Schwarz inequality. So if you know that $(W_t^2) < \infty$, this proves $W_t \in L^1$.

- As W_t is a Gaussian random variable, the (absolute) moments can be calculated explicitely, see [here][1][7]

[6] $1_{\{|W_t| \le 1\}} = \begin{cases} 1 & |W_t| \le 1 \\ 0 & \text{otherwise} \end{cases}$.

[7] http://en.wikipedia.org/wiki/Normal_distribution#Moments

Question

differentiability of brownian motion

for a fixed $t \in [0,\infty)$, I have to show that

$(D^{+}W_t = +\infty$ and $D_{+}W_t = -\infty)$, where D^{+} (and D_{+}) denotes the upper right-hand derivative (and respectively the lower right-hand derivative).

I construct a further Brownian motion X by time-inversion of a Brownian motion B, then:

$$D^{+}X(0) \geq \limsup_{n\to\infty} \frac{X(1/n) - X(0)}{1/n} \geq$$

$$\limsup_{n\to\infty} \sqrt{n}X(1/n) = \limsup_{n\to\infty} \frac{1}{\sqrt{n}} B(n)$$

Now all I need to prove is that $\limsup_{n\to\infty} \frac{1}{\sqrt{n}} B(n) = +\infty$ because then I define $X(s) = B(t+s) - B(t)$

which is a Brownian motion and differentiability of X at 0 is equivalent to that of B at t.

I thought about using Hewitt-Savage 0 - 1 law to prove the last claim but I can't conclude.

I think we could also use Borel-cantelli lemma but I don't know how.

If you see another more direct approach I will be glad to take it.

Answer

> Tools: Kolmogorov zero-one law, independence of Brownian increments, scaling property of the Brownian paths.

Note that the event $A = [\limsup_{n\to\infty} B_n / \sqrt{n} = +\infty]$ is also

$$A = [\limsup_{n\to\infty} (B_{n+k} - B_k)\sqrt{n} = +\infty],$$

for every k, hence A is in the tail sigma-algebra of the i.i.d. sequence $(B_n - B_{n-1})$.

Kolmogorov (independent) zero-one law shows that P[A] is 0 or 1. By symmetry, $P[A] = P[A']$ where $A' = [\liminf_{n\to\infty} B_n / \sqrt{n} = -\infty]$ hence A is almost sure as soon as $[\limsup_{n\to\infty} | B_n | / \sqrt{n} = +\infty]$ is almost sure.

Let $c > 0$. Note that, if $| x |$ $c\sqrt{2^n}$ and $| x + y |$ $c\sqrt{2^{n+1}}$ then $| y |$ $3c\sqrt{2^n}$. For every n, define $A_n = [| B_{2^n} |$ $c\sqrt{2^n}]$ then the preceding remark and the independence of the increments of the Brownian motion show that

$$P[A_{n+1} | A_n] \quad P[| B_{2^{n+1}} - B_{2^n} | \quad 3c\sqrt{2^n}] = P[| B_1 | \quad 3c].$$

The last equal sign uses the Brownian scaling property. Let p_c denote the RHS, thus $p_c < 1$. For every k ℓ,

$$P\left[\bigcap_{n=k}^{\ell+1} A_n\right] \quad p_c P\left[\bigcap_{n=k}^{\ell} A_n\right],$$

hence, for every k ,

$$P\left[\bigcap_{n\ \ k} A_n\right]=0,$$

thus, the event $\liminf_{n\to\infty} A_n$ has probability zero, that is, almost surely, $|B_{2^n}|>c\sqrt{2^n}$ for infinitely many n , thus,

$$P[\limsup_{n\to\infty}|B_n|/\sqrt{n}\quad c]=1.$$

This holds for every c hence the claim is proved.

Question

Is this a Brownian motion?

I am learning SDE, and here is some basic things I have trouble with,

Let $B(t)$ be a Brownian motion, and $F\in\ ^2$ is any stochastic process and I know $\int_0^t F(s)dB(t)$ is Ito process so $\int_0^t F(s)dB(t)$ is a martingale, but is it a Brownian motion?

Answer

Compute variance of increment from time t to time s using Ito's isometry

$$E[(\int_t^s F(u)dB(u))^2] = E(\int_t^s (F(u))^2 du) = \int_t^s E(F(u))^2 du$$

In general, it's not compatible with the variance of Brownian increment, which is $s-t$.

Question

Reflected process - Brownian motion

I am still new to stochastic processes and I tried to do this exercise. I don't know how to go on.

Define the maximum process

$M_t = \max_{0 \quad s \quad t} W_s$, and hitting times

$T_a = \inf\{s : W_s = a\}$. Give an expression for $(T_a \quad t)$ in terms of the maximum process .

For any fixed T and a standard Brownian motion W_t, the reflected process defined by

$\tilde{W}_t^T = \begin{cases} W_t, & t \quad T \\ 2W_T - W_t, & t > T, \end{cases}$ is itself a standard Brownian motion.

Prove that for any $x \quad a$,

Using symmetry, conclude that

For the first part, we have that

Assume that .

If , it is .

Suppose that . Then

which implies . So we must have .

But how can I prove in this case, that ?

If , we have

Moreover,

$$W_t = 2W_{T_a} + W_t - 2W_{T_a} =$$
$$2a - (2W_{T_a} - W_t) =$$
$$2a - \tilde{W}_t^{T_a} \quad 2a - (2a - x) = x.$$

Conversely, assume that $M_t \quad a \quad x$ and $W_t \quad x \quad a$.

If $t \quad T_a$, I do not know how to show that $W_t \quad 2a - x$.

If $t > T_a$, we get that

$$\tilde{W}_t^{T_a} \quad = 2W_{T_a} - W_t = 2a - W_t \quad 2a - x.$$

We get the second equality as follows:

$(\tilde{W}_t^{T_a} \quad 2a - x) \quad = 1 - \quad (\tilde{W}_t^{T_a} \quad 2a - x)$. Since $\tilde{W}_t^{T_a}$ is itself a standard Brownian motion with mean 0 and variance t, we get

$(\tilde{W}_t^{T_a} \quad 2a-x) = 1-\Phi\left(\frac{2a-x}{\sqrt{t}}\right)$. Now, we see

$(T_a \quad t) = \quad (M_t \quad a) =$

that $(M_t \quad a, W_t \quad x) + \quad (M_t \quad a, W_t > x) =$

$1-\Phi\left(\frac{2a-x}{\sqrt{t}}\right) + \quad (M_t \quad a, W_t > x)$

Here, I do not see how to obtain $2(1-\Phi(\frac{a}{\sqrt{t}}))$.

Answer

Let us start with your first problem. You want to calculate

$(M_t \quad a, W_t \quad x)$

As you noted,

$(M_t \quad a, W_t \quad x) = \quad (T_a \quad t, W_t \quad x) =$
$(T_a \quad t, W_t - 2W_{T_a} \quad x-2a) =$

$(T_a \quad t, \tilde{W}_t^{T_a} \quad 2a-x),$

where the last line follows because $t \geq T_a$. Lastly,

$(T_a \quad t, \tilde{W}_t^{T_a} \quad 2a-x) = \quad (\tilde{W}_t^{T_a} \quad 2a-x) - \quad (T_a > t, \tilde{W}_t^{T_a} \quad 2a-x),$

but if $T_a > t$,

$\{\tilde{W}_t^{T_a} \geq 2a-x\} = \{W_t \geq 2a-x\} \subset \{W_t \geq a\} \subset \{T_a \leq t\}$, which

means that $(T_a > t, \tilde{W}_t^{T_a} \quad 2a-x) = 0$.

For your second question, take $x = a$. The second term will simplify since $\{W_t > a\} \subset \{M_t \geq a\}$.

Note for the first part that we need the reflected Brownian motion to be a Brownian motion for any T hitting time (you wrote for any fixed T).

Question

Brownian motion and covariance

Show that for $B = (B_t)$ Brownian motion, its covariance is

$cov(B_s, B_t) = min(s,t)$.

The solution I was given was:

For s ≤ t ,

$B_t = B_s + (B_t - B_s)$,

$B_s B_t = B_s^2 + Bs(Bt - Bs)$

$cov(B_s, B_t) = E[B_s B_t]$ (as $E(B_i) = 0$)

so $cov(B_s, B_t) = E[B_s^2] + E[B_s(B_t - B_s)]$,

as all increments of Brown motion are independent the second term in the RHS

$= E[B_s]E[B_t - B_s] = 0*0 = 0$. Now

$cov(B_s, B_t) = E(B_s^2) = Var(B_s) = s$.

Similarly if $t \le s$ we get $= t$.

My question is, could I not have done this argument the same way without assuming that $s \le t$ in the first place? I mean, if $s \le t$ why cant I say

$B_s = B_t + (B_s - B_t)$ and continue like this to get the answer with a max instead of a min?

Is it because $B_t - B_s$ is only an increment if $t \ge s$? I mean otherwise we don't have independency or something?

Answer

To answer your question. If you had for $t \le s$ we would get t instead of s in our previous $cov(B_s, B_t) = s$ Combining this, we would still end up with $cov(B_s, B_t) = \min(s,t)$

Another way to do this is:

Suppose $B_c(t) := B(c^2 t)/c$,

Then:

$$cov(B_c(s), B_c(t)) = E[(B(c^2 s)/c)(B(c^2 t)/c)] =$$
$$c^{-2} \min(c^2 s, c^2 t) = \min(s,t) = cov(B - S, B_t)$$

Question

Local Maximum of Brownian motion.

Given two positive rational number a,b How to show that almost surely Brownian motion

attains a local maximum at some time in (a,b) ?

Answer

Without loss of generality, we can assume $a=0$ (since the process $W_t := B_{t+a} - B_a$ is again a Brownian motion). Denote by $\Omega_{\max}$ the set of $w \in \Omega$ such that the path $(0,b) \quad t \mapsto B_t(w)$ does **not** attain a local maximum in $(0,b)$.

Let $w \in \Omega_{\max}$. Since the Brownian motion has continuous paths there are two possibilities:

- $(0,b) \quad t \mapsto B_t(w)$ does not attain a local minimum. This implies the monotonicity of the mapping.
- - $(0,b) \quad t \mapsto B_t(w)$ does attain a local minimum. In this case, we can find a partition $0 = s_0 < \ldots < s_n = b$ such that $B_{\cdot}(w)|_{[s_{j-1},s_j]}$ is monotone for $j = 1,\ldots,n$.

Thus, we conclude that the path is of bounded variation, i.e.

$\mathrm{VAR}(B_{\cdot}(w),b)$:

$$\sup_{\Pi} \sum_{t_j - t_{j-1} \in \Pi} |B_{t_j}(w) - B_{t_{j-1}}(w)| \le \sum_{j=1}^{m} |B_{s_j}(w) - B_{s_{j-1}}(w)| < \infty$$

where the supremum is taken over partitions
$\Pi = \{0 = t_0 < \ldots < t_m = b\}$.

On the other hand, it is known that the total variation $\mathrm{VAR}(B_\cdot, b)$ is equal to ∞ a.s., so we conclude $(\Omega_{\max}) = 0$.

Question

Harmonic functions and Brownian motion

How can I prove that harmonic functions have the mean-value property using Brownian motion B_t ?

I know that I need to use the fact that $B_{t \wedge \tau}$ is a martingale where τ is a stopping time which denotes the first time the Brownian motion leaves a ball with radius r around B_0

Answer

Fix a ball $B(x, r)$ and consider $\tau = \inf t > 0, | B_t - x | = r$.

Now compute by Itô's formula

$$f(B_\tau) = f(B_0) + \int_0^\tau f'(B_s) dB_s + \frac{1}{2} \int_0^\tau \Delta f(B_s) ds$$

Take expectations:

$$[f(B_\tau)] = f(B_0) + \left[\int_0^\tau f'(B_s) dB_s\right] + \frac{1}{2} \int_0^\tau \Delta f(B_s) ds$$

Since $\left[\int_0^\tau f'(B_s)dB_s\right]=0$ [8] and $\Delta f \equiv 0$ we obtain that

$$[f(B_\tau)]=f(B_0)$$

Now it suffices to take a Brownian motion starting at x and to note by rotation invariance that

$$[f(B_\tau)]=\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} f(x)dS$$

and therefore you obtain the mean property

$$f(x)= \quad [f(B_\tau)]=\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} f(x)dS$$

remark: It is important to note that τ is a finite almost surely stopping time and that $|B_\tau - x|=r$

---

[8] $\left[\int_\circ^\tau f'(B_s)dB_s\right]=\circ$ this follows from Ito calculus: imagine $f(t,\omega)=f(B(t))$ is constant, you get from such integral $B_\tau - B_\circ$ .It f is constant by parts, you get $\sum_{j=1}^{n} f(t_j)B(t_{j+1})-B(t_j)$ wich has zero expectation. Now you just need to approach f with such elementary functions the zero expectation is preserved.

Question

Brownian motion process and brownian bridge process

Let U be a brownian bridge process , and let

$B(t)=(1+t)U\left(\frac{t}{1+t}\right)$, $t\geq 0,$ show that B is a brownian motion on $t\geq 0$?

Answer

Suppose $s<t$ and note that $\frac{s}{1+s}<\frac{t}{1+t}$. Then:

$$Cov(B_s,B_t)=$$

$$cov\left((1+s)X(\frac{s}{1+s}),(1+t)X(\frac{t}{1+t})\right)=$$

$$(1+s)(1+t)Cov(X(\frac{s}{1+s}),X(\frac{t}{1+t}))$$

The brownian bridge has covariance $s-st$ for $s<t$.

So $Cov(B_s,B_t)=(1+s)(1+t)(\frac{s}{1+s}-\frac{s}{1+s}\frac{t}{1+t})=s$

Question

Brownian motion - characteristic function

Let me remind first the construction of Brownian motion.

Fix a vector $x \in \mathbb{R}^n$ and define

$$p(t,x,y) := (2\pi t)^{-\frac{n}{2}} \cdot \exp\left(-\frac{|x-y|^2}{2t}\right), \text{ for } y \in \mathbb{R}^n,\ t > 0.$$

Then for $0 \le t_1 \le t_2 \le \ldots \le t_k$ we define a measure $\nu_{t_1,\ldots,t_k}$ on $\mathbb{R}^{nk}$ by

$$\nu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = \int_{F_1 \times \ldots \times F_k} p(t_1,x,x_1) \prod_{j=1}^{k-1} p(t_{j+1}-t_j, x_j, x_{j+1}) dx_1$$

where

the following conventions are used $dy = dy_1 \ldots dy_k$ for Lebesgue measure[9] and $p(0,x,y)dy = \delta_x(y)$ (the unit point mass at x).

By Kolmogorov's extension theorem applied to the probability measures $\nu_{t_1,\ldots,t_k}$ (which easily satisfies all the assumptions of that theorem) there exists a probability space $(\Omega, \quad , P^x)$ and a

---

[9] Given a subset $E \subseteq \mathbb{R}$ with the length of an (open, closed, semi-open) interval $I = [a,b]$ given by $l(I) = b - a$ the Lebesgue outer measure $\lambda^*(E)$ is defined by $\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty} l(I_k) : (I_k)_{k\in\mathbb{N}}\right.$ is a sequence of open intervals with $\left.\right\}$

stochastic process $\{B_t\}_{t\geq 0}$ on Ω such that the finite distributions of B_t are given by

$$(\ )\quad P^x(B_{t_1}\in F_1,\ldots,B_{t_k}\in F_k)=$$

$$\int_{F_1\times\ldots\times F_k} p(t_1,x,x_1)\prod_{j=1}^{k-1} p(t_{j+1}-t_j,x_j,x_{j+1})dx_1\ldots dx_k.$$

Problem

I want to show that for the random variable $Z=(B_{t_1},\ldots,B_{t_k})\in\mathbb{R}^{nk}$ there exist a vector $M\in\mathbb{R}^{nk}$ and a non-negative definite matrix $C\in \mathrm{M}_{nk\times nk}(\mathbb{R})$ such that

$$E^x\left[\exp(i<u,Z>)\right]=\exp(-\frac{1}{2}<Cu,u>+i<u,M>)$$

for all $u=(u_1,\ldots,u_{nk})\in\mathbb{R}^{nk}$ (left hand side stands for the characteristic function of the random variable Z). Moreover, M is the mean value of Z and C is a covariance matrix of Z .

I was trying to calculate it explicitly by writing the left hand side by its definition and applying the ( ) formula, but the integral which I've got is not so nice and I cannot conclude what I want to.

This exercise is nothing else, but showing that B_t is a Gaussian process (so it is not hard to guess what should be M and C in this case).

I hope you know some tricks how to compute such an integral in an easy way. Thanks in advance for any help.

Answer

Denote $t_0 = 0$ and $x_0 = 0$ for notational convenience. Then

$$(\exp(i\langle u, Z\rangle) =$$

$$= \int_{\mathbb{R}^{nk}} \exp(i\langle u, x\rangle) \prod_{j=1}^{k} p(t_j - t_{j-1}, x_j - x_{j-1}) \mathrm{d}x_1 \cdots \mathrm{d}x_k$$

Now change variables $x_k = \sum_{i=1}^{k} s_k$. This has unit Jacobian[10], you get

$$(\exp(i\langle u, Z\rangle) =$$

$$= \int_{\mathbb{R}^{nk}} \prod_{k=1}^{k} \exp(i s_j \sum_{m=j}^{k} u_q)) \prod_{j=1}^{k} p(t_j - t_{j-1}, s_j) \mathrm{d}s_1 \cdots \mathrm{d}s_k$$

Now integration with respect to each s_j can be carried out independently and will produce $\exp(Q_j(u))$, where Q_j is a quadratic multivariate polynomial in u . Hence the characteristic function of k-dimensional time-slice distribution of this Brownian motion process is going to be $\exp(Q(u))$.

[10] In vector calculus, the **Jacobian matrix** is the matrix of all first-order partial derivatives of avector-valued function. When the matrix is a square matrix, both the matrix and its determinant are referred to as the**Jacobian** in literature. $\mathrm{J} = \frac{\mathrm{df}}{\mathrm{dx}} = \begin{bmatrix} \frac{\partial \mathrm{f}}{\partial \mathrm{x}_1} & \cdots & \frac{\partial \mathrm{f}}{\partial \mathrm{x}_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1} & \cdots & \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{f}_m}{\partial \mathrm{x}_1} & \cdots & \frac{\partial \mathrm{f}_m}{\partial \mathrm{x}_n} \end{bmatrix}$

Now $Q(u) = \langle Cu, u \rangle + i \langle u, M \rangle$, where C is covariance matrix and M is mean vector. The free term will be missing, because c.f. is one for zero vector u.

Question

Considering Brownian bridge as conditioned Brownian motion

Let B be a standard Brownian motion. Define a Brownian bridge b by $b_t = B_t - tB_1$. Let $'$ be the law of this process.

According to [Wikipedia][11]

A Brownian bridge is a continuous-time stochastic process B(t) whose probability distribution is the conditional probability distribution > of a Wiener process W(t) (a mathematical model of Brownian motion)

given the condition that B(0) = B(1) = 0.

Surely it makes no sense to condition on a probability 0 event? So I'm trying to show that $'$ is the weak limit as $\to 0$ of Brownian motion conditioned upon the event $\{| B_1 | \leq \}$. How do we prove this?

[11] http://en.wikipedia.org/wiki/Brownian_bridge

Answer

Brownian motion B_t over the interval $[0,1]$ can be decomposed into two independent terms. That is, the process $X_t = B_t - tB_1$ and the random variable $Y = B_1$. As these are joint normal, to prove that they are independent, it is enough to show that the covariance $\mathrm{Cov}(X_t, Y) = \mathrm{Cov}(B_t, B_1) - t\mathrm{Var}(B_1) = t - t$ vanishes.

The distribution of B conditional on $|B_1| <$ is just the same as that of X plus the independent term Y (conditioned on $|Y| <$). As goes to zero, this converges to the distribution of X, which is a Brownian bridge.

Question

Ito's formula and $2-D$ Brownian motion

Let $a \in R$, $B = (B^1, B^2)$ a brownian motion.

$$X_t = e^{B_t^1}\left(\int_0^t e^{-B_s^1} dB_s^2 + a\int_0^t e^{-B_s^1} ds\right)$$

Show there is a brownian motion β such that

$$X_t = \int_0^t \sqrt{1+X_s^2}\, d\beta_s + \int_0^t (a + \frac{1}{2}X_s) ds$$

For the moment:

$$U_t = e^{B_t^1}$$

By the Ito's formula:

$$U_t = 1 + \int_0^t U_s dB_s^1 + (1/2)\int_0^t U_s ds dU_t$$
$$= U_t dB_t^1 + (1/2) U_t dt dX_t$$
$$= dB_t^2 + X_t dB_t^1 + (a + (1/2)X_t)dt$$

But how to show

$$dB_t^2 + X_t dB_t^1 = \sqrt{1 + X_t^2}\, d\beta_t \ ?$$

Answer

Define

$$\beta_t = \int_0^t \frac{dB_t^2 + X_t dB_t^1}{\sqrt{1 + X_t^2}}.$$ Then

$\langle \beta, \beta \rangle_t = t$. By Levy's characterization, $\{\beta_t \mid t \geq 0\}$ is a Brownian motion.

Question

Probability Inequality for Brownian motion

Let B be a standard Brownian motion. How can we show that

$$(B_t < x, B_s < x) \geq \quad (B_t < x) \quad (B_s < x)$$

for $0 < s < t$ and $x > 0$ without actually computing the probabilities (i.e, by using the Markov property or something else)?

Answer

Let $X = B_t - B_s$, $Y = B_s$, and $u : b \mapsto \mathbf{1}_{b<x}$, then X and Y are independent, u is non increasing, and one wants to show that $E[u(X+Y)u(Y)] \quad E[u(X+Y)]E[u(Y)]$.

Let $v_Z : b \mapsto u(b+z)$ for some fixed z, then u and v_Z are both non increasing hence, by the usual coupling inequality, $E[v_Z(Y)u(Y)] \quad E[v_Z(Y)]E[u(Y)]$, that is,

$E[u(z+Y)u(Y)] \quad E[u(z+Y)]E[u(Y)]$. Let μ denote the distribution of X, then, by the independence of X and Y,

$\int E[u(z+Y)u(Y)]\mathrm{d}\mu(z) = E[u(X+Y)u(Y)]$, and

$\int E[u(z+Y)]\mathrm{d}\mu(z) = E[u(X+Y)]$, hence $(*)$ follows.

Edit: Re the coupling inequality, let ξ and ξ' denote i.i.d. random variables and u and v non increasing functions, then, almost surely,

$(u(\xi)-u(\xi'))(v(\xi)-v(\xi'))\quad 0$, since, on $|\xi>\xi']$, both factors are non positive and on $[\xi<\xi']$, both are nonnegative. Integrate and develop the resulting product, then $E[u(\xi)v(\xi)]=E[u(\xi')v(\xi')]$ because ξ and ξ' are i.d. and $E[u(\xi)v(\xi')]=E[u(\xi')v(\xi)]=E[u(\xi)]E[v(\xi)]$ because ξ and ξ' are i.i.d. The coupling inequality $E[u(\xi)v(\xi)]\quad E[u(\xi)]E[v(\xi)]$ follows.

Question

Brownian Motion and Progressive Process

Let B_t be a Brownian motion. Define sign function as follows.

$sign(0)=0$ and $sign(x)=\dfrac{x}{|x|},\forall x\neq 0$. I do not know how to show the following two questions, especially on the progressive measurability one. Could anyone provide some detailed explanation on how to go about with them, please? Thank you!

1. Show that $sign(B_t)$ is a progressive process.

2. Show that $sign(B_t)\cdot B_t$ is a Brownian motion.

Answer

A direct way to show that the process X defined by $X_t = \operatorname{sign}(B_t)$ is progressively measurable is to consider, for every positive integer n,

$$X_t^n = \frac{nB_t}{\sqrt{1+n^2B_t^2}}.$$

Each process X^n is adapted with continuous paths hence progressively measurable. Furthermore, for every t, $X_t^n \to X_t$ almost surely, when $n \to \infty$, hence X is progressively measurable.

(This answers item 1. Item 2. is absurd, for example because $X_t B_t = |B_t|$ and $t \mapsto E(|B_t|)$ is not constant hence |B| is not a Brownian motion (much more is known but this simple argument suffices).)

Question

Brownian motion and stochastic integration

How do I compute the following expectation?

W(T) is a standard brownian motion (i.e.) $W(T) \sim N(0,T)$

$$E\left[W(T)\int_0^T s\,dW(s)\right]$$

I know that Brownian motion of disjoint time intervals are independent.

Does this means that,

$$E\left[\left(W(T)-W(t)\right)\int_0^t s dW(s)\right]$$
$$=E\left[\left(W(T)-W(t)\right)\right]E\left[\int_0^t s dW(s)\right]=0$$

However, when I try to let $W(T)=W(T)-W(t)+W(t)$, the whole expectation becomes more complicated.

Answer

Use the three following facts:

$$(i)\int_0^T s\mathrm{d}W_s = TW_T - \int_0^T W_s \mathrm{d}s$$
$$(ii)\ E[W_T^2]=T$$
$$(iii)\ E[W_T W_s]=\min(s,T)$$

Question

Normal probability and Brownian motion

Let X_t be a Brownian motion with parameter σ. Find the probability in terms of $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{\alpha^2}{2}} d\alpha$

How would I do this for $P(X_2>2)$?

I know that $P(X_2 > 2) = 1 - P(X_2 < 2)$ but where do I go from there? X_2 is a normal random variable with mean 0 and variance 2σ.

Answer

A normal random variable X with mean μ and variance σ^2 is related to the standard normal random variable Z (zero mean and unit variance) with affine transform:

$X \overset{d}{=} \mu + \sigma Z$ where $\sigma > 0$. Hence

$$\Pr(X \quad x) = \Pr\left(\mu + \sigma Z \quad x\right) = \Pr\left(Z \quad \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Question

Absolute value of Brownian motion

I need to show that $R_t = \dfrac{1}{|B_t|}$ is bounded in ^{2} for $(t \geq 1)$, where B_t is a $3-dimensional$ standard Brownian motion.

I am trying to find a bound for $[\int_{t=1}^{\infty} R_t^2]$.

Asymptotically B_t^i is between $\sqrt{t}$ and t. I also know that $|B_t| \to \infty$, but the rate is not clear.

Hints would be helpful.

Answer

Since B_t and $\sqrt{t}B_1$ are identically distributed, $\mathrm{E}(R_t^2) = t^{-1}\mathrm{E}(R_1^2)$, hence $\mathrm{E}(R_t^2) \quad \mathrm{E}(R_1^2)$ for every $t \quad 1$ and it remains to show that $\mathrm{E}(R_1^2)$ is finite. Now, the density of the distribution of B_1 is proportional to $\mathrm{e}^{-\|x\|^2/2}$ and B_1 has dimension 3 hence the density of the distribution of $Y = \| B_1 \|$ is proportional to $\varphi(y) = y^{3-1}\mathrm{e}^{-y^2/2} = y^2\mathrm{e}^{-y^2/2}$ on $y \geq 0$. Since the function $y \mapsto y^{-2}\varphi(y) = \mathrm{e}^{-y^2/2}$ is Lebesgue integrable, the random variable $Y^{-2} = R_1^2$ is integrable.

On the other hand, $\mathrm{E}\left(\int_1^{+\infty} R_t^2 \mathrm{d}t\right)$ is infinite.

The distribution of B_1 yields the distribution of $Y = \| B_1 \|$ by the usual change of variables technique. To see this, note that in dimension n and for every test function u,

$$\mathrm{E}(u(Y)) \propto$$

$$\int_{\mathbb{R}^n} u(\| x \|)\mathrm{e}^{-\|x\|^2/2}\mathrm{d}x \propto \int_0^{+\infty}\int_{S^{n-1}} u(y)\mathrm{e}^{-y^2/2}y^{n-1}\mathrm{d}\sigma_{n-1}(\theta)\mathrm{d}y,$$

where σ_{n-1} denotes the uniform distribution on the unit sphere S^{n-1} and $(y,\theta) \mapsto y^{n-1}$ is proportional to the Jacobian of the transformation $x \mapsto (y,\theta)$ from $\mathbb{R}^n \quad \{0\}$ to $\mathbb{R}_+^* \times S^{n-1}$. Hence,

$$\mathrm{E}(u(Y)) \propto \int_0^{+\infty} u(y) y^{n-1} \mathrm{e}^{-y^2/2} \mathrm{d}y,$$

which proves by identification that the distribution of Y has a density proportional to $y^{n-1}\mathrm{e}^{-y^2/2}$ on $y \geq 0$.

Question

Converge to Brownian Motion problem

Consider the following sequence of SDEs:

$$dX_t^n = \sin(nX_t^n)dt + dW_t, X_0^n = 0$$

Show that the solutions X^n converge in finite dimensional distribution to Brownian Motion.

I have been working on this for a long time and can't get anywhere. Any help would be appreciated. Thank you.

Answer

Surely not an exam proof, what is going to come.

We will use Theorem 5.4.1 from Ethier and Kurtz, Markov Processes, Wiley, 1986 (EK86); the important part is quoted below.

Let us first write down the generator A_n of the process $X^{(n)}$

$$A_n f(x) = \sin(nx) f'(x) + \frac{1}{2} f''(x), x \in \mathbb{R}.$$

The expected limit process, Brownian motion, has generator $Af(x) = \frac{1}{2} f''(x)$, $x \in \mathbb{R}$. Both definitions are understood for functions $f \in C_c^\infty(\mathbb{R})$, the compactly supported, infinitely often differentiable functions on $\mathbb{R}$.

Lemma 1: The sequence $\{X_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ is tight in the Skorohod space $D([0,\infty),\mathbb{R})$.

Lemma 2: For any $f \in C_c^\infty(\mathbb{R})$ we find a function f_n such that

$$\| f_n - f \|_\infty \to 0, \qquad \| A_n f_n - Af \|_\infty \to 0$$

as $n \to \infty$.

Proof of your statement : Let $Y = (Y_t)_{t \geq 0}$ be a limit point of the sequence $\{X_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ which exists by Lemma 1. Then Lemma 2 (combined with Theorem 5.4.1 in EK86) tells us that Y solves the martingale problem related to the generator A. That means that Y is a Brownian motion. Since this holds for any limit point Y, we know that $\{X_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ converges weakly in Skorohod space to Brownian motion. This implies converges in finite dimensional distributions by Theorem 3.7.5 in EK86.

Proof of Lemma 2:

Fix $f \in C_c^2(\mathbb{R})$ and let $a := \inf \operatorname{supp} f$, $A := \sup \operatorname{supp} f$ such that $f(x) = 0$ for all $x \notin [a, A]$.

$$f_n(x) :=$$

$$f(x) - 2\int_a^x \exp(2n^{-1}\cos(ny))\left(-C + \int_a^y \exp(-2n^{-1}\cos(nz))\sin(nz) f'(z)dz\right).$$

Here $C = C(n) = \int_a^A \exp(-2n^{-1}\cos(nz))\sin(nz)f'(z)dz$.

It is easy to see that $A_n f_n(x) - Af(x) = 0$ for any $x \in \mathbb{R}$. So we are left with the verification of $\sup_x |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Note that since $1-(2/n) \le \exp(-n^{-1}\xi) \le 1+(4/n)$ for any $\xi \in [-2,2]$, $n \ge 4$ we have

$$\int a^y \exp(-2n^{-1}\cos(nz))2\sin(nz)f'(z)dz =$$

$$-\int_a^y \exp(-2n^{-1}\cos(nz))f''(z)dz + f'(y)\exp(-2n^{-1}\cos(ny))$$

$$-f'(a)\exp(-2n^{-1})$$

$$= -\int_a^y f''(z)dz + f'(y) - f'(a) + \left(n^{-1}\|f''\|_\infty + n^{-1}\|f'\|_\infty\right) =$$

$$\left(n^{-1}\|f''\|_\infty + n^{-1}\|f'\|_\infty\right)$$

for any $y \in \mathbb{R}$. This allows to say that for $x \in [a, A]$:

$$f(x) - f_n(x)$$
$$= 2\int_a^x \exp(-n^{-1}\cos(ny))dy \cdot \left(n^{-1}\|f''\|_\infty + n^{-1}\|f'\|_\infty\right)$$
$$\le 4|A-a| \cdot \left(n^{-1}\|f''\|_\infty + n^{-1}\|f'\|_\infty\right) \to 0$$

as $n \to \infty$. If we are not in the interval of support, i.e. $x \notin [a, A]$, then we have for $x \ge A$:

$$f(x) - f_n(x) = -f_n(x) =$$

$$2\int_a^x \exp(2n^{-1}\cos(ny))(-C + \int_a^y \exp(-2n^{-1}\cos(nz))\sin(nz)f'(z)dz)$$

$$|A-a|\cdot\left(n^{-1}\|f''\|_\infty+n^{-1}\|f'\|_\infty\right)+$$

$$2\int_A^x \exp(2n^{-1}\cos(ny))\left(-C+\int_a^A\cdots dz+\int_A^y\cdots dz\right).\sqrt{b^2-4ac}$$

The first term vanishes as before uniformly in x, which leaves us with the terms of the last line. By definition of C we are only left with the last term which is zero anyway, since we are not in the support of f anymore. A similar, even easier reasoning applies for $x<a$. The previous lines include saying that f_n is a bounded function.

Proof of Lemma 1:

Use Remark 5.4.2 in EK86 which requires standard conditions on A which are fulfilled here and the compact containment condition for $\{X_t^{(n)})_{t\geq 0}:n\in\mathbb{N}\}$. For the compact containment condition hold define two processes Z^i, $i=-1,1$ solving $dZ_t^i=i\ dt+dB_t$. Then Theorem 5.2.18 in Karatzas and Shreve, Brownian Motion and Stochastic Calculus, 1991 allows to say we can construct probability spaces $(\Omega^n,\ \ ^n,P^n)$ such that $Z_t^{-1}\leq X_t^n\leq Z_t^1$ for all $t\geq 0$ P^n almost surely for all $n\in\mathbb{N}$.

Theorem 5.4.1 in EK86

Let $A\subset C_b(\mathbb{R})\times C_b(\mathbb{R})$ and $A_n\subset\ \ _b(\mathbb{R})\times\ \ _b(\mathbb{R})$, the bounded measurable functions, $n\in\mathbb{N}$. Suppose that for each $(f,g)\in A$ there is an $(f_n,g_n)\in A_n$ such that

$\|f_n-f\|\to 0,\quad \|g_n-g\|\to 0.$ If for each $n\in\mathbb{N}$, $X^{(n)}$ is a solution of the A_n martingale problem with paths in the Skorohod

space and $X_n \Rightarrow X$, then X is a solution to the A -martingale problem.

Note: The authors of EK86 prefer to denote a generator $A : D(A) \to \ _b(E)$ by its graph $\{(f, Af) : f \in D(A)\} \subset (\ _b(E), \ _b(E))$.

Question

Ito's Isometry using Brownian Motion

Let B_t be standard Brownian Motion. Could someone please help me to show that $E[(\int_0^t B_s dB_s)^2] = \int_0^t E[B_s^2]ds$

I am sure that it has something to do with Ito's formula but I am pretty new to this stuff so some help would be greatly appreciated.

Answer

The RHS is direct to evaluate. The variance of B_s is s so:

$$\int_0^t E[B_s^2]ds = \int_0^t s ds = \frac{t^2}{2}$$

You are right about use Ito's Lemma for the LHS.

By Ito:

$$B_t^2 = t + 2\int_0^t B_s dB_s$$

So

$$E[(\int_0^t B_s dB_s)^2] =$$

$$E\left[\left(\frac{B_t^2 - t}{2}\right)^2\right] =$$

$$\frac{1}{4}(E[B_t^4] - 2tE[B_t^2] + t^2) =$$

$$\frac{1}{4}(3t^2 - 2t \cdot t + t^2)$$

Question

local maximum of brownian motion

I have a question:

Given two disjoint intervals [a,b] and [c,d], how to prove almost surely we have

$$\sup_{t \in [a,b]} B_t \neq \sup_{t \in [c,d]} B_s$$

where B is a standard brownian motion. I have no idea about this problem. Does someone have an idea?

Answer

If you write $X_t = B_{a+t} - B_a$ for $0 \le t \le b-a$ and $Y_s = B_{c+s} - B_c$ for $0 \le d-c$, then $(X_t)_{t\in[0,b-a]}$ and $(Y_s)_{s\in[0,d-c]}$ are independent Brownian motions, also independent of B_a and B_c. So their suprema $A = \sup_{t\in[0,b-a]} X_t$ and $C = \sup_{s\in[0,d-c]} Y_s$ are independent random variables, which are also independent of B_a and B_c. Now

$$\left[\sup_{t\in[a,b]} B_t = \sup_{t\in[c,d]} B_t\right] = [B_a + A = B_c + C] = [A - C = B_c - B_a].$$

This is always 0 since $B_c - B_a$ is continuous (actually, $N(0, c-a)$-distributed) and independent of $A - C$ (which is also continuous unless both $a = b$ and $c = d$.)

Question

Conditioned Maximum of Brownian Motion

Let W_t be a Brownian motion and

$$M_t = \max_{o<s<t} W_s$$

Can anyone give me some insights on how to prove:

$$P[M_t > a \,|\, W_t = M_t] = \exp(-a^2/2t) \quad ; \quad a > 0$$

Many thanks in advance.

Answer

The joint distribution of (B_t, M_t) is well-known. The probability density reads:

$$f_{(B_t, M_t)}(x, y) = \sqrt{\frac{2}{\pi}} \frac{2y - x}{t^{3/2}} \exp\left(-\frac{(2y - x)^2}{2t}\right)[y > 0, x \quad y]$$ w

here $[y > 0]$ denotes Iverson bracket.[12]

From here the distribution of $(M_t, M_t - B_t)$ is easy to read off:

$$f_{(M_t, M_t - B_t)}(x, y) = \sqrt{\frac{2}{\pi}} \frac{x + y}{t^{3/2}} \exp\left(-\frac{(x + y)^2}{2t}\right)$$ Now finding

$[y > 0, x > 0]$

the conditional probability density is also straightforward. Assuming $y > 0$:

[12] In mathematics, the **Iverson bracket**, named after Kenneth E. Iverson, is a notation that denotes a number that is 1 if the condition in square brackets is satisfied, and 0 otherwise. More

exactly, $[P] = \begin{cases} 1 & \text{if } P \text{ is true;} \\ 0 & \text{otherwise.} \end{cases}$

where *P* is a statement that can be true or false. This notation was introduced by Kenneth E. Iverson in his programming language APL,[1] while the specific restriction to square brackets was advocated by Donald Knuth to avoid ambiguity in parenthesized logical expressions.
(Kenneth E. Iverson, *A Programming Language*, New York: Wiley, p. 11, 1962.)

$$f_{M_t|M_t-B_t}(x\mid y)=$$

$$\frac{f_{M_t,M_t-B_t}(x,y)}{f_{M_t-B_t}(y)}=$$

$$\frac{x+y}{t}\exp\left(-\frac{x(x+2y)}{t}\right)[x>0]$$

This will now allow you to find the quantity of interest:

$$\left(M_t>a\mid B_t=M_t\right)=$$

$$\left(M_t>a\mid M_t-B_t=0\right)=$$

$$\lim_{y\to 0^+}\int_a^\infty f_{M_t|M_t-B_t}(x,y)\mathrm{d}x=$$

$$\lim_{y\to 0^+}\exp\left(-\frac{a(a+2y)}{2t}\right)=\mathrm{e}^{-\frac{a^2}{2t}}$$

Question

beginner's question about Brownian motion

I have just started learning about stochastic processes and I am confused with the notion of Brownian motion. The text defines (linear) Brownian motion under measure as $B=(B_t;t\geq 0)$ where each B_t is some random variable such that:

- $t\to B_t$ is a continuous function
- B_t is distributed as $N(0,t)$.

- $B_{s+t} - B_s$ is distributed as $N(0,t)$

I have some questions regarding this definition. What is the sample space (space of events) on which each random variable B_t is defined? What does it mean that $t \to B_t$ is continuous? Is it that for every ω in the sample space $t \to B_t(\omega)$ is continuous? What kind of values is B_t taking?

Hopefully I formulated my questions clearly. Any help is appreciated.

Answer

The Brownian motion B is a mapping $I \times \Omega \quad (t,w) \mapsto B(t,w) \in \mathbb{R}$ where $I \subseteq \mathbb{R}$ (usually $I = [0,\infty)$ or $I = [0,1]$) and $(\Omega, \quad , \quad)$ is an arbitrary probability space. For fixed $w \in \Omega$ the mapping

$$I \quad t \mapsto B(t,w) \in \mathbb{R}$$

is called sample path. And you are correct: "$t \mapsto B_t$ continuous" means that the sample paths $t \mapsto B_t(w)$ are continuous for all $w \in \Omega$.

Question

Prove increment of Brownian motion is Brownian motion

I am trying to solve the following exercise in Oksendal's book:

Let B_t be Brownian motion and fix $t_0 \geq 0$. Prove that $\overline{B_t} := B_{t_0+t} - B_{t_0}; \quad t \geq 0$ is a Brownian motion.

I try to prove it by definition. In the book, Brownian motion is defined as

$$P^x(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k) = \int_{F_1 \times \cdots \times F_k} p(t_1, x, x_1) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 \ldots dx_k,$$

where

$$p(t, x, y) = (2\pi t)^{-n/2} \cdot \exp(-\frac{|x-y|^2}{2t})$$

Hence for simplicity, we only need to show

$$P^0(\overline{B}_{t_1} \in F_1, \overline{B}_{t_2} \in F_2) = \int_{F_1 \times F_2} p(t_1, 0, x_2) p(t_2 - t_1, x_3, x_2) dx_2 dx_3$$

Suppose B_t starts at x_0. By definition of $\overline{B_t}$

$$P^0(\overline{B}_{t_1} \in F_1, \overline{B}_{t_2} \in F_2) = P^0(B_{t_0+t_1} - B_{t_0} \in F_1, B_{t_0+t_2} - B_{t_0} \in F_2) \quad (1)$$

Then I don't know how to handle it. I know I should use B_t 's finite dimensional distribution. So if I can write (1) as

$$P^{x_0}[\bigcup_{x_1\in\mathbb{R}^n}\left(B_{t_0}=x_1, B_{t_0+t_1}\in F_1+x_1, B_{t_0+t_2}\in F_2+x_1\right)] \quad (2)=$$

$$\int_{\mathbb{R}^n\times(F_1+x_1)\times(F_2+x_1)} p(t_0,x_0,x_1)p(t_1,x_1,x_2)p(t_2-t_1,x_2,x_3)dx_1dx_2dx_3$$

Then apply change of variable formula and Fubini's theorem[13], I can get the result.

But I don't know how to justify (2) and the last equality. I think it should be true because it's the conditional probability. But the book didn't introduce this concept and since $\mathbb{R}^n$ is uncountable, I don't know how this is valid in terms of measure theory.

---

[13] In mathematical analysis **Fubini's theorem**, introduced by Guido Fubini (1907), is a result that gives conditions under which it is possible to compute a double integral using iterated integrals. One may switch the order of integration if the double integral yields a finite answer when the integrand is replaced by its absolute value.

$$\int_X\left(\int_Y f(x,y)\mathrm{d}y\right)\mathrm{d}x=\int_Y\left(\int_X f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

$$=\int_{X\times Y} f(x,y)\mathrm{d}(x,y)$$

As a consequence it allows the order of integration to be changed in iterated integrals. Fubini's theorem implies that the two repeated integrals of a function of two variables are equal if the function is integrable. **Tonelli's theorem** introduced by Leonida Tonelli (1909) is similar but applies to functions that are non-negative rather than integrable.

- *Fubini, G. (1907), "Sugli integrali multipli", Rom. Acc. L. Rend. (5)* ***16*** *(1): 608–614, Zbl 38.0343.02* Reprinted in *Fubini, G. (1958), Opere scelte* ***2****, Cremonese, pp. 243–249*

Answer

Why aren't you applying the characterization: "Any continuous real-valued process (X_t) that is zero-mean Gaussian process with covariance $\operatorname{cov}(X_t, X_s) = t \wedge s$ is a Brownian motion" ? (Rogers and Williams, page 4).

I write B'_t for your new process. It is a Gaussian process.

Then B'_t has continuous sample paths (since B_t does). Also

$$E(B'_t) = E(B_{t_o+t}) - E(B_{t_0}) = 0.$$

Finally for $s < t$

$$\operatorname{cov}(B'_t, B'_s) = E((B_{t_o+t} - B_{t_o})(B_{t_o+s} - B_{t_o})) =$$
$$\min(t_0 + t, t_o + s) + \min(t_0, t_0) - \min(t_0 + s, t_0) + \min(t_0 + t, t_0) =$$
$$(t_0 + s) + t_0 - t_0 - t_0 = s = \min(s, t).$$

In the second line I have expanded the brackets, used linearity of expectation, and the characterization applied to (B_t).

Question

An Integral Involving Brownian Motion

Let B_t $(t \geq 0)$ be a Brownian motion on $\mathbb{R}^3$. That is, $B_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$, where each $B_t^{(i)}$ is a Brownian motion on $\mathbb{R}$. Let Y be a Borel subset of $\mathbb{R}^3$.

I am being asked to show that

$$\left(\int_0^\infty I(\{t : B_t \in Y\})(t)dt\right) = c\int_Y \frac{dy}{|B_0 - y|}.$$ for some constant c .

Here $I(A)$ denotes the indicator function of a set A .

Using Fubini's Theorem on the left-hand side, I reduced the equation to

$$\int_0^\infty (B_t \in Y)dt = c\int_Y \frac{dy}{|B_0 - y|}.$$

Unfortunately, I'm not sure what to do now. I would appreciate any help.

Thanks.

EDIT: As some people have pointed out, the expectation and probability on the left-sides of both equation should probably be conditioned on B_0 . The professor has been a bit sloppy about this with Brownian motion.

Answer

I suspect that it is either the expectation of the right hand-side or the conditional expectation with respect to B_0 on the left side. But the equation as it is cannot be since

the right side is stochastic and left side deterministic. Assuming that you were asked

$$\left(\int_0^\infty I(\{t : B_t \in Y\})(t)dt \mid B_0\right) = c\int_Y \frac{dy}{|B_0 - y|}.$$

You can still use Fubini, so the question reduces to calculate:

$$P(B_t \in Y \mid B_0) = P(B_t - B_0 \in Y - B_0 \mid B_0)$$

(recall that $P(A|Y)=E(1_A|Y)$). Since the increment $B_t - B_0$ is independent of B_0 and follows a multivariate Normal $(0, diag(t,t,t))$ the conditional probability is equal to

$$\int_{Y-B_0} \frac{1}{(2\pi t)^{3/2}} e^{-\frac{s_1^2+s_2^2+s_3^2}{2t}} ds_1 ds_2 ds_3$$

Then you do the change $u = s - B_0$, Fubini again, integrating with respect to t and you should get the result.

Question

Computing expectation of brownian motion

I need to compute the following:

$$E\left[B_t \int_0^t B_s^2 ds\right] \text{ for } t\geq 0$$

Where B_t is a standard Brownian motion.

I'm thinking this is really obvious, But I cannot get my head round it. I thought about using Fubinis theorem to move the expectation inside of the integral to get an s, but I don't think that works seeing as there's a product.

Answer

Since B_t is symmetrical, B_t and $-B_t$ have the same distribution, $B_t \int_0^t B_s^2$ and $-B_t \int_0^t B_s^2$ also have the same distribution. Therefore

$$E\left[B_t \int_0^t B_s^2 ds\right] = -E\left[B_t \int_0^t B_s^2 ds\right]$$

It is 0.

Question

Joint moments of Brownian motion

My approach to [this SE question][1] uses the following joint moments of

Brownian motion. For $n = 1, 2$ they are obvious and well-known, the others

are not terribly hard to work out. Is there a reference where these

formulas are given, or/and is there a pattern to the coefficients?

Fix $t_1 \le t_2 \le t_3 \le \cdots \le t_n$.

For odd values of n we have $[W(t_1)W(t_2)\cdots W(t_n)] = 0$

while for even values of n we get

$$[W(t_1)W(t_2)] = t_1$$

$$[W(t_1)W(t_2)W(t_3)W(t_4)] = 2t_1t_2 + t_1t_3$$

$$[W(t_1)W(t_2)W(t_3)W(t_4)W(t_5)W(t_6)] =$$
$$2t_1t_2t_5 + t_1t_3t_5 + 4t_1t_2t_4 + 2t_1t_3t_4 + 6t_1t_2t_3$$

I suppose everything about Brownian motion

has been worked out, but I can't find this in any of my books.

It's not very important, but I'm just curious![14]

Answer

Another formula for this joint moment can be found as Lemma 4.5 in a paper by J. Rosen and M.B. Marcus [Annals of Probability, vol. 20, no. 4 (1992) pp. 1603-1684]; the authors refer to it as "well-known". The formula is this (for even n): The joint moment $E[W(t_1)W(t_2)\cdots W(t_n)]$ is the sum over all *pairings* $\{\{a(1),b(1)\},\{a(2),b(2)\},\ldots,\{a(n/2),b(n/2)\}\}$ of $\{1,2,\ldots,n\}$

$$\prod_{i=1}^{n/2} E[W(t_{a(i)},t_{b(i)})].$$

A pairing is simply a partition of $\{1,2,\ldots,n\}$ into $n/2$ doubletons.

Conjecture:([15])

---

[14] http://math.stackexchange.com/questions/27841/how-to-calculate-e-int-0tw-sdsn-n-geq-2

[15] By considering $t_1 = t_2 = \ldots = t_{2n}$, the sum of the coefficients is the 2n th moment of $W(1)$, $(2n-1)!!$, which is the number of paths of length 2n in Young's lattice from the empty partition to itself.

I haven't yet proven the following, but I think induction should work.

Question

Asymptotic behaviour of brownian motion

Let $(B_t)_{t\geq 0}$ be a brownian motion, i want to show that

$$\frac{B_t}{t^p} \xrightarrow[t\to\infty]{a.s.} 0,$$

for all $p > \frac{1}{2}$.

I was told to use that

Conjecture:

The indices of the terms with positive coefficients correspond to Dyck words, so the number of terms in $E[W(t_۱)W(t_۲)...W(t_{۲n})]$ is the n th Catalan number. $t_۱t_۲t_۴$ corresponds to the Dyck word $++-+--$ with pluses in positions $(۱,۲,۴)$.

The coefficient of the term corresponding to a Dyck word is the number of paths of length 2n in Young's lattice from the empty partition to itself such that the partition at each step has size equal to the height of the Dyck path.

The coefficient of $t_{i_۱}t_{i_۲}...t_{i_n}$ is $\prod_{k=۱}^{n}(۲k-i_k)$. For example, the coefficient of $t_۱t_۲...t_n$ equals $\prod_{k=۱}^{n}(۲k-k)=n!$ and paths in Young's lattice which ascend n times and then descend n times correspond to pairs of standard tableaux of the same shape, which are in bijection with permutations of $\{۱,۲,...,n\}$.

(A Dyck path is a staircase walk from $(0,0)$ to (n,n) that lies strictly below (but may touch) the diagonal $y=x$. The number of Dyck paths of order n is given by the Catalan number)

$$X_t = \frac{B_t^2 - t}{(t+1)^{2p}} \xrightarrow[t\to\infty]{a.s.} 0,$$

which i have proven already, but i can't see how $X_t \to 0 a.s.$ implies the first claim. Can someone give me a hint on how those two are related?

Answer

If $p > 1/2$ then $\frac{t}{(t+1)^{2p}} \to 0$. Combined with $X_t \to 0$, this implies $X_t + \frac{t}{(t+1)^{2p}} \to 0$.

But $X_t + \frac{t}{(t+1)^{2p}} = \frac{B_t^2}{(t+1)^{2p}} = \left(\frac{B_t}{(t+1)^p}\right)^2 \to 0.$

It follows that

$\frac{B_t}{(t+1)^p} \to 0$ and thus

$$\frac{B_t}{t^p} = \frac{B_t}{(t+1)^p}\frac{(t+1)^p}{t^p} \to 0 \cdot 1 = 0$$

Question

Brownian motion transition density question

Let $Y_t = M_t - W_t$ where M_t is the running maximum of brownian motion and W_t is brownian motion. I want to show that

$P^0[Y_{t+s} \in dy \mid Y_t = x] = p(s, x, y) + p(s, x, -y)$ where p is the transition density of brownian motion.

This problem is in Karatzas and Shreve (page 124) and actually the solution is there as well. However, the solution is not easy to understand. One part of the solution reads as follows:

For $b > m \geq w$, $b \geq a$, $m \geq 0$

$$P^0[W_{t+s} \in da, M_{t+s} \in db \mid W_t = w, M_t = m]$$

$$= P^0[W_{t+s} \in da, max_{0 \leq u \leq s} W_{t+u} \in db \mid W_t = w, M_t = m]$$

$$= P^W[W_s \in da, M_s \in db]$$

How is this step justified? Don't we need (W, M) to be a Markov family for this? How can we show that this is a Markov family?

Thank you very much.

Answer

A simple way to prove that (W, M) is a Markov process is to note that, for every $t \quad 0$ and $s \quad 0$, $W_{t+s} = W_t + \bar{W}_s$, where $\bar{W} = (\bar{W}_s)_{s \quad 0}$ is a Brownian motion starting from 0 and

independent of the sigma-algebra $_t = \sigma(W_u; 0 \quad u \quad t)$.

Furthermore, considering $\bar{M}$ the maximum process of $\bar{W}$, one has $M_{t+s} = \max\{M_t, W_t + \bar{M}_s\}$, for every $s \quad 0$, hence the distribution of $(W_{t+s}, M_{t+s})_{s \quad 0}$ conditionally on $_t$ depends on (W_t, M_t) only, as desired.

Question

Conditional distribution in Brownian motion

I need to prove the following:

Let X be a Brownian motion with drift μ and volatility σ. Pick three time points $s < u < t$. Then, the conditional distribution of B_t given $X_s = x$ and $X_t = y$ is normal; in fact

B_T

I know the conditional distributions of Gaussian are also Gaussian. I also have formulae to find $(X \mid Y = y)$ and $\mathrm{Var}(X \mid Y = y)$ when X and Y are jointly Gaussian. However I cannot find a way to apply those results here. Could someone help me please?

Answer

Consider the theorem [16]on conditional expectation of multivariate Gaussian distribution. Let $x = X_u$ and t. Let $X_0 = 0$, so that

$\mu_X = 0$ and $\mu_Y = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

$$\Sigma_{XX} = Cov(X_u, X_u) = \sigma^2 u \Sigma_{XY} = E(XY) =$$

$$[E(X_u X_s) \quad E(X_u X_t)] =$$

$$\sigma^2 [s \quad u] \Sigma_{YY} = E(YY)$$

$$= \begin{bmatrix} E(X_s^2) & E(X_s X_t) \\ E(X_t X_s) & E(X_t^2) \end{bmatrix} =$$

$$\sigma^2 \begin{bmatrix} s & s \\ s & t \end{bmatrix}$$

Plugging the formula, the random variable $X_u \mid (X_s = x, X_t = y)$ is also Gaussian with

- **mean**: $\Sigma_{XY} \Sigma_{YY}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{t-u}{t-s} x + \frac{u-s}{t-s} y$

- **variance**:

$$\Sigma_{XX} + \Sigma_{XY} \Sigma_{YY}^{-1} \Sigma_{YX} =$$

$$\sigma^2 u + \sigma^2 [s \quad u] \begin{bmatrix} s & s \\ s & t \end{bmatrix}^{-1} \begin{bmatrix} s \\ u \end{bmatrix} = \sigma^2 \frac{(t-u)(u-s)}{t-s}$$

[16] *Eaton, Morris L. (1983). Multivariate Statistics: a Vector Space Approach. John Wiley and Sons. pp. 116–117. ISBN 0-471-02776-6*

Question

Scaling property for Brownian motion

Define Brownian motion as a continuous process (B_t) with independent increments, such that $B_{s+t} - B_s$ has normal distribution with mean 0 and variance t. How do you use the independence of increments to prove that for $t > 0$ and $s_1 < \cdots < s_n$

$$(B_{s_1 t}, \ldots, B_{s_n t}) =_d (t^{1/2} B_{s_1}, \ldots, t^{1/2} B_{s_n})$$

where $=_d$ means that both vectors have the same distribution.

Only using the independence of increments, not that the process is Gaussian.

Answer

(I'll only write this down for $n = 2$.) Let f be a bounded continuous function of two real variables; let Z_1 and Z_2 be independent standard normal random variables. Then

$$E[f(B_{s_1 t}, B_{s_2 t})] =$$
$$E[f(B_{s_1 t}, B_{s_1 t} + (B_{s_2 t} - B_{s_1 t}))] =$$
$$E[f(\sqrt{s_1 t} Z_1, \sqrt{s_1 t} Z_1 + (\sqrt{s_2 t - s_1 t} Z_2))] =$$
$$E[f(\sqrt{t}\sqrt{s_1} Z_1, \sqrt{t}\sqrt{s_1} Z_1 + \sqrt{t}(\sqrt{s_2 - s_1} Z_2))] =$$
$$E[f(\sqrt{t} B_{s_1}, \sqrt{t} B_{s_1} + \sqrt{t}(B_{s_2} - B_{s_1}))] =$$
$$E[f(\sqrt{t} B_{s_1}, \sqrt{t} B_{s_2})]$$

Of course it is crucial here that (because B_s is normal with mean 0 and variance s) B_s has the same distribution as $\sqrt{s}Z$, where Z is standard normal. Because f was arbitrary, this shows that $(B_{s_1 t}, B_{s_2 t})$ has the same distribution as $(\sqrt{t}B_{s_1}, \sqrt{t}B_{s_2})$.

Question

(Multidimensional) Standard Brownian Motion: Convergence

I know that $\left(X_j - X_{j-1}\right)_{j=1}^{t} \xrightarrow{d} \left(Y_j\right)_{j=1}^{t}$.

Further, we know that $\left(Y_j\right)_{j=1}^{t} =^{d} \left(W(y_j) - W(y_{j-1})\right)_{j=1}^{t}$,

where W is a Standard brownian Motion.

Now, how do we get that

$$\left(X_j\right)_{j=1}^{t} \xrightarrow{d} \left(W(y_j)\right)_{j=1}^{t}.$$

I think, it should be rather obvious, but I do not see it. [17]

[17] http://math.stackexchange.com/questions/897245/convergence-to-standard-brownian-motion/897252?noredirect=1#comment1851909_897252

Answer

It follows straight from the definition of convergence in distribution that

$$(X_j^n - X_{j-1}^n)_{j=1}^t \xrightarrow{d} (W(y_j) - W(y_{j-1}))_{j=1}^t.$$

Define a mapping $g : \mathbb{R}^t \to \mathbb{R}^t$ by

$$g(x) := (x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_t).$$

where $x := (x_1, \ldots, x_t)$. Then g is continuous and

$$g((y_j - y_{j-1})_{j=1}^t) = (y_1, \ldots, y_t) \qquad (1)$$

for any $y = (y_1, \ldots, y_t) \in \mathbb{R}^t$. Applying the continuous mapping theorem yields

$$g((X_j^n - X_{j-1}^n)_{j=1}^t) \xrightarrow{d} g((W(y_j) - W(y_{j-1}))_{j=1}^t).$$

Hence, by (1), this is equivalent to

$$(X_j^n)_{j=1}^t \xrightarrow{d} (W(y_j))_{j=1}^t \qquad \text{as } n \to \infty.$$

Question

Geometric Brownian motion - Volatility Interpretation

A Geometric Brownian motion satisfying the SDE $dS_t = rS_t dt + \sigma S_t dW_t$ has the analytic solution

$$_t = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t\right\}\exp\{\sigma W_t\}$$

Recently in an interview I was asked the following (I am paraphrasing):

> The magnitude of uncertainty of the movement of S_t is represented by σ and is clearly captured in the term $\exp\{\sigma W_t\}$.

But intuitively, why does σ appear again in the term $r - \frac{\sigma^2}{2}$? That is, why are we deducting $\frac{\sigma^2}{2}$ from our drift r? What is the interpretation?

Does anybody know how to interpret it?

Answer

The reason is that the compounded return of an investment has a "drag" when the intermediate returns are more volatile. Check for instance the compounded return of two return series with the same arithmetic mean, but different variance. You will see that the series with a lower variance has a higher compounded return (and

geometric mean). The reason is that if an investment looses 50% of its value, it has to make a return of +100% (very different from +50%) to come back to the initial value.

The value of a bank account with no risk paying a continuously compounded interest rate r is $S_0 e^{rt}$. However, the GBM $S(t)$ is a model for the value of a risky/stochastic investment.

As the exact solution for $S(t)$ shows, the correct interpretation of the value of $S(t)$ as paying a continuously compounded rate of return is with $\gamma = r - \frac{1}{2}\sigma^2$ (also called growth rate) and not r, because $\lim_{t\to\infty} S(t) = S_0 e^{\gamma t}$ Although the mean of $S(t)$ is $E[S(t)] = S_0 e^{rt}$, the median of its distribution is $S_0 e^{\gamma t}$.

While the intuition of an expected value as a proxy for the long-term value is appropriate for random variables with broadly symmetric distributions, such as returns, $dS(t)/S(t)$ (and other stationary series), in the case of the value of $S(t)$ this intuition is flawed.

To see this, notice that if $\gamma > 0$ then $\lim_{t\to\infty} S(t) = \infty$, and if $\gamma < 0$ then $\lim_{t\to\infty} S(t) = 0$. On the other hand, $r > 0$ does not guarantee an increasing value of S in the long run, because if σ is large enough, γ could be negative even if r is positive. Hence, the growth rate γ is a better indicator of the long-term value of an investment than r.

Another way to see this is to notice that if $\gamma > 0$, as t increases the distribution of $S(t)$ becomes more skewed to the right with an ever greater right tail. As a consequence, as time passes the mean of the distribution moves deeper into the right tail, exponentially diverging

from the center of the distribution, i.e. the median. To see this, notice that the ratio of the mean to the median is equal to $e^{\frac{1}{2}\sigma t}$.

Notice that the *compounded* rate of return γ and the *instantaneous* rate of return r $\frac{dS(t)}{S(t)} = rt + \sigma dW_t$ coincide for the value of a bank account with no risk paying interest rate r, i.e. $r = \gamma$ because in such case there is no randomness (aka risk) and hence $\sigma = 0$. But for a risky investment, the compounded rate of return $\gamma < r$!

Question

Convergence to Brownian motion integral

Let X_i be i.i.d with $(X_i) = 0$ and $Var(X_i) = 1, S_n = \sum_{i=1}^{n} X_i$. I would like to show that $\sum_{i=1}^{n} \frac{f(S_i / \sqrt{n})}{n}$ converges to $\int_0^1 f(B_t)dt$ in distribution for every continuous and bounded function f , where B_t is standard Brownian motion.

I would start with the following application of central limit theorem: $S_t^n \to B_t$ in distribution, where $S_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i$. Not sure how to proceed further.

I would be grateful for any suggestion or ideas.

Thanks.

Answer

Recall

>**Theorem (Donsker, 1952)**[18] Let $\{X_i\}$ be a sequence of i.i.d. zero mean square integrable random variable, with variance 1. Let S_n^* the random piecewise linear function defined on [0,1] such that $S_n^*(kn^{-1}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} X_i$. Then $\{S_n^*\}$ converges in law in C[0,1] to $(W_t)_{0 \quad t \quad 1}$.

Then use a Riemann sum approximation.

Question

Hermite Polynomials and Brownian motion

I am asked to prove the following :

Let B_t be a standard brownian motion.

[18] In probability theory, **Donsker's theorem** (also known as **Donsker's invariance principle**, or the **functional central limit theorem**), named after Monroe D. Donsker, is a functional extension of the central limit theorem.
Donsker, M.D. (1951). "An invariance principle for certain probability limit theorems". *Memoirs of the American Mathematical Society, 1951, no. 6.*

The n th Hermite polynomial is

$$H_n(t,x) = \frac{(-t)^n}{n!} e^{x^2/(2t)} \frac{d^n}{dx^n} e^{-x^2/(2t)}.$$

Show that the H_n play the role that the monomials $\frac{x_n}{n!}$ play in ordinary calculus.

In other words show : $dH_{n+1}(t,B_t) = H_n(t,B_t)$

Now my guess is that we can take the integral of both sides and use ito integration ? But I am not sure how to proceed from there.

Answer

First of all, the statement

$$dH_{n+1}(t,B_t) = H_n(t,B_t)$$

doesn't make sense. Or can you explain what you mean by it? I guess, it should read

$$dH_{n+1}(t,B_t) = H_n(t,B_t)dB_t;$$

at least that's what I prove in this answer.

>**Lemma 1**

$$H_{n+1}(t,x) = \frac{x}{n+1} H_n(t,x) - \frac{t}{n+1} H_{n-1}(t,x) \qquad (0)$$

Proof: Denote by $G_n(y) := (-1)^n \exp\left(\frac{y^2}{2}\right) \frac{d^n}{dy^n} \exp\left(-\frac{y^2}{2}\right)$ the "standard" Hermite polynomial. Then it follows from the chain rule

that $H_n(t,x) = t^{n/2}\frac{1}{n!}G_n\left(\frac{x}{\sqrt{t}}\right)$. (1) Using the recursion relation $G_{n+1}(y) = yG_n(y) - nG_{n-1}(y)$ (2) we find

$$H_{n+1}(t,x) \overset{(1)}{=} t^{(n+1)/2}\frac{1}{(n+1)!}G_{n+1}\left(\frac{x}{\sqrt{t}}\right)$$

$$\overset{(2)}{=} t^{(n+1)/2}\frac{1}{(n+1)!}\left[\frac{x}{\sqrt{t}}G_n\left(\frac{x}{\sqrt{t}}\right) - nG_{n-1}\left(\frac{x}{\sqrt{t}}\right)\right]$$

$$\overset{(1)}{=} \frac{x}{n+1}H_n(t,x) - \frac{t}{n+1}H_{n-1}(t,x)$$

>**Lemma 2**

$$\frac{\partial}{\partial t}H_n(t,x) = -\frac{1}{2}H_{n-2}(t,x).$$

Proof: It follows straight from the product rule that

$$\frac{\partial}{\partial t}H_n(t,x) =$$

$$-\frac{(-t)^{n-1}}{(n-1)!}\exp\left(\frac{x^2}{2t}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2t}\right)$$

$$-\frac{(-t)^n}{n!}\frac{x^2}{2t^2}\exp\left(\frac{x^2}{2t}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2t}\right)$$

$$+\frac{(-t)^n}{n!}\frac{1}{2t^2}\exp\left(\frac{x^2}{2t}\right)\frac{d^n}{dx^n}\left(x^2\exp\left(-\frac{x^2}{2t}\right)\right). \quad (3)$$

Applying Leibniz' product rule

$$\frac{d^n}{dx^n}(f\cdot g)=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x)$$ we get

$$\frac{d^n}{dx^n}\left(x^2\exp\left(-\frac{x^2}{2t}\right)\right)=$$

$$x^2\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2t}\right)+2nx\frac{d^{n-1}}{d^{n-1}x}\exp\left(-\frac{x^2}{2t}\right)$$

$$+(n-1)n\frac{d^{n-2}}{d^{n-2}x}\exp\left(-\frac{x^2}{2t}\right)\ /\ (4)$$

Plugging (4) into (3) yields

$$\frac{\partial}{\partial t}H_n(t,x)=$$

$$-\frac{(-t)^{n-1}}{(n-1)!}\exp\left(\frac{x^2}{2t}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2t}\right)$$

$$+\frac{1}{2}\frac{(-t)^{n-2}}{(n-2)!}\exp\left(\frac{x^2}{2t}\right)\frac{d^{n-2}}{d^{n-2}x}\exp\left(-\frac{x^2}{2t}\right)=$$

$$\frac{n}{t}H_n(t,x)+\frac{1}{2}H_{n-2}(t,x)\overset{(0)}{=}$$

$$\frac{n}{t}\left(\frac{x}{n}H_{n-1}(t,x)-\frac{t}{n}H_{n-2}(t,x)\right)+\frac{1}{2}H_{n-2}(t,x)=$$

$$-\frac{1}{2}H_{n-2}(t,x).$$

$**Lemma3**$

$$\frac{\partial}{\partial x}H_n(t,x)=H_{n-1}(t,x).$$

Proof:

$$\frac{\partial}{\partial x} H_n(t,x) =$$

$$\frac{(-t)^n}{n!} \exp\left(\frac{x^2}{2t}\right)\left[\frac{x}{t}\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2t}\right)+\frac{d^{n+1}}{d^{n+1}x}\exp\left(-\frac{x^2}{2t}\right)\right]=$$

$$\frac{x}{t}H_n(t,x)+\frac{n+1}{-t}H_{n+1}(t,x)\overset{(0)}{=}H_{n-1}(t,x).$$

>**Theorem** $dH_{n+1}(t,B_t) = H_n(t,B_t)dB_t$

Proof: The claim follows directly from Itô's formula and Lemma 2+3. Note that Lemma 3 implies

$$\frac{\partial^2}{\partial^2 x} H_{n+1}(t,x) = \frac{\partial}{\partial x} H_n(t,x) = H_{n-1}(t,x).$$

Remark Note that this shows in particular that $(H_n(t,B_t))_{t\geq 0}$ is a martingale (see also this [question]).[19]

Question

Brownian motion and posterior distribution

I am a bit stuck on this question:

[19] http://math.stackexchange.com/q/1214054/

Suppose that $X_t = W_t + \alpha t$, where W is a standard Brownian motion, and let $\quad_t = \sigma(X_u : 0 \le u \le t)$. The drift is constant in time but unknown: it takes one of the values a or $-a$, where $a > 0$. Each of the two possible values has equal (prior) probability.

Want to show that the dynamics of X evolves as follows:

$$dX_t = d\hat{W}_t + a \tanh aX_t \, dt,$$

where $\hat{W}$ is a $\quad_t$-Brownian motion.

>Progress in my solution so far:

>By simple calculations, it can be shown that the posterior for α is of the form $\quad[\alpha = a \mid \quad_t] = \dfrac{\exp\{aX_t\}}{\exp\{aX_t\} + \exp\{-aX_t\}}$.

This shows that $\quad(\alpha \mid \quad_t) = a \tanh aX_t$. I am stuck from here. I have tried applying Girsanov theorem to $\quad(\alpha \mid \quad_t) - \alpha$, but it doesn't really work. Are there any tricks you can think of?

Answer

First of all, note that $\sigma(X_t) = \sigma(W_t)$ since Borel sigma-algebras are translation invariant, so that $\quad_t = \quad_t^W$ for each $t \quad 0$. Now, for the filtered space $\left(\Omega, \quad, (\quad_t)_t \quad 0, \quad\right)$ you have shown

$$\left[\alpha \mid \quad_t\right] = a \tanh aX_t.$$

Consequently, consider a re-write of $\mathrm{d}X_t$ as

$$\mathrm{d}X_t = \left(\mathrm{d}W_t + \left(\alpha - \quad[\alpha \mid \quad_t]\right)\mathrm{d}t\right) + \quad[\alpha \mid \quad_t]\mathrm{d}t,$$

which suggests we define

$$\mathrm{d}\hat{W}_t := \mathrm{d}W_t + \left(\alpha - \quad [\alpha \mid \quad {}_t]\right)\mathrm{d}t. \ / \ (1)$$

Note that, for the continuous process " $-\int_0^t \left(\alpha - \quad [\alpha \mid \quad {}_s]\right)\mathrm{d}s$ ", we have,

$$\int_0^t \left(\alpha - \quad [\alpha \mid \quad {}_s]\right)^2 \mathrm{d}s$$

$$< \int_0^t \left(\left|\alpha\right| + \left| \quad [\alpha \mid \quad {}_s]\right|\right)^2 \mathrm{d}s$$

$$< \int_0^t (a+a)^2 \mathrm{d}s = 4a^2 t < \infty.$$

This means Novikov's[20] Condition is satisfied:

$$\left[e^{\frac{1}{2}\int_0^t \left(\alpha - \quad [\alpha \mid \quad {}_s]\right)^2 \mathrm{d}s} \right] < e^{2a^2 t} < \infty.$$

Therefore, by a corollary of [Girsanov's [21] theorem using the preferred measure suggested by the theorem, $\hat{W}_t$ defined by (1) is a ${}_t^W$ -Brownian motion and, therefore, it is a ${}_t$ -Brownian motion.

---

[۲۰] http://en.wikipedia.org/wiki/Novikov%27s_condition

[21] http://en.wikipedia.org/wiki/Novikov%27s_condition

Question

Brownian motion at exponential time

I want to find the law of B_T, where B_t is a brownian motion, T is exp-distributed with parameter 1, with B_t and T being independent.

My idea is to say that the density of B_T is given by

$$f_{B_T}(x) = \int_0^\infty f_{(B_T,T)}(x,y)dy = \int_0^\infty f_{B_T|T}(x \mid y) * f_T(y)dy$$

where $f_{B_T|T}(x \mid y) = \frac{1}{\sqrt{2\pi y}} * e^{-\frac{x^2}{2y}}$ and $f_T(y) = e^{-y}$, since B_T given $T = y$ is $(0, y)$-distributed.

Is this a proper argument?

Answer

Maybe more simply, and *assuming that T is independent of (B_t)*, one has, for every bounded measurable function u and every t,

$$E[u(B_T) \mid T = t] = E[u(B_t) \mid T = t] = E[u(B_t)] = \int_{\mathbb{R}} u(x) p_t(x) \mathrm{d}x,$$

where (p_t) is the Brownian transition kernel. Integrating this with respect to the distribution of T yields

$E[u(B_T)] = E[E[u(B_T)\,|\,T]] = \int_0^\infty \left(\int_{\mathbb{R}} u(x)\,p_t(x)\,\mathrm{d}x\right) f_T(t)\,\mathrm{d}t$, where f_T is the density of T. By Fubini, B_T has density g defined by

$$g(x) = \int_0^\infty p_t(x) f_T(t)\,\mathrm{d}t.$$

Question

Definition of the Brownian motion

The way I understood the definition of a Brownian motion B_t in $\mathbb{R}$ is that it consists of two parts:

1. We first define the finite-dimensional distributions

$\nu_{t_1,\ldots,t_n}(A_1,\ldots,A_n)$ Since they satisfy t X wo properties of [Kolmogorov extension theorem](http://en.wikipedia.org/wiki/Kolmogorov_extension_theorem) there is a probability space $(\Omega, \quad ,\mathsf{P})$ and a stochastic process X such that

$\nu_{t_1,\ldots,t_n}(A_1,\ldots,A_n) = \mathsf{P}\{X_{t_1} \in A_1,\ldots,X_{t_n} \in A_n\}$. 2. Second, we claim that trajectories of this process have to be continuous with probability 1. Such claim can be satisfied since by Kolmogorov continuity theorem there is a continuous version Y of a process with such finite-dimensional distributions ν.

What is not clear is the following: in the first step we have already defined the stochastic process X. What do we do in the second

step? We still stay in the same probability space (since we have to define what does the version mean). So does it mean that finite-dimensional distributions of Y are different from that of X ? If they are not different, does it mean that $X = Y$ (considering them as measurable function from Ω to $\mathbb{R}$) or it means that Kolmogorov extension theorem does not provide the uniqueness?

Answer

Keep in mind that here, "stochastic process" means a set of random variables, indexed by the nonnegative reals: $X = \{X_t : t \in [0,\infty)\}$. Kolmogorov's extension theorem guarantees the existence of such a set with the desired finite-dimensional distributions. In particular, the process X has independent normal increments.

For continuity, we wish to produce a measurable set $E \subset \Omega$ with $P(E) = 1$ such that for every $\omega \in E$, the function $[0,\infty) \quad t \mapsto X_t(\omega) \in \mathbb{R}$ is continuous. Unfortunately, it is possible that for the X supplied by Kolmogorov's extension theorem, no such E exists. (The MO post linked by Henry's comment gives an example of a process with the same finite-dimensional distributions as Brownian motion which lacks continuity.)

So we pass to a version Y of X. This means that $Y = \{Y_t : t \in [0,\infty)\}$ is a different set of random variables with the property that for each t, $X_t = Y_t$ almost surely. To be explicit: for every t, there exists a measurable set $E_t \subset \Omega$ with $P(E_t) = 1$ such that for every $\omega \in E_t$, $X_t(\omega) = Y_t(\omega)$. There need *not* be: a single set $E \subset \Omega$ with $P(E) = 1$ such that for all $\omega \in E$ and all t, $X_t(\omega) = Y_t(\omega)$. (One might try to put $E = \bigcap_{t \in [0,\infty)} E_t$, but

since this is an uncountable intersection, we cannot be sure that E has measure 1, or indeed that E is measurable at all.) However, it is easy to check that the finite-dimensional distributions of any such Y are the same as those of X, and in particular Y also has independent normal increments.

Kolmogorov's continuity theorem promises that we can choose a Y which is continuous in the sense of my second paragraph.

You are right that there is no uniqueness in the first step: finite-dimensional distributions do not uniquely determine a process.

Question

Brownian bridge expression for a Brownian motion

Let B_t be a standard Brownian motion in $\mathbb{R}$, then the Brownian bridge on [0,1] is defined as

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{\mathrm{d}B_s}{1-s}$$

for $0 \le t < 1$. Here $Y_0 = a$ and $\lim_{t\to 1} Y_t = b$ a.s. The latter implies

$\lim_{t\to 1}(1-t)\int_0^t \frac{\mathrm{d}B_s}{1-s} = 0$ a.s. and using integration by parts:

$$\lim_{t\to 1}(1-t)\int_0^t \frac{B_s}{(1-s)^2}\mathrm{d}s = B_1 \text{ a.s.}$$

I wonder if the latter formula has been shown to have a particular interesting meaning. Maybe there a known relationship with a Cauchy's integral formula.[22]

Answer

If f is any continuous function, then L'Hopital's rule gives

$$\lim_{t\to 1}(1-t)\int_0^t \frac{f(s)}{(1-s)^2}ds = f(1).$$

In terms of generalized functions, this says that if

$$\mu_t(s) = \frac{1-t}{(1-s)^2}1_{[0,t]}(s),$$

then $\mu_t \to \delta_1$. The Brownian bridge can also be written in terms of generalized functions:

$$Y_t = a(1-t) + bt + \langle \partial B, v_t \rangle,$$

where

[22] In mathematics, **Cauchy's integral formula**, named after Augustin-Louis Cauchy, is a central statement in complex analysis. It expresses the fact that a holomorphic function defined on a disk is completely determined by its values on the boundary of the disk, and it provides integral formulas for all derivatives of a holomorphic function. Cauchy's formula shows that, in complex analysis, "differentiation is equivalent to integration": complex differentiation, like integration, behaves well under uniform limits – a result denied in real analysis.

$$\mathrm{f(a)} = \frac{۱}{۲\pi \mathrm{i}}\oint_\gamma \frac{\mathrm{f(z)}}{\mathrm{z-a}}\mathrm{dz}$$

$$v_t(s) = \frac{1-t}{1-s} 1_{[0,t]}(s),$$

and ∂B is the (random) distributional derivative of the Brownian sample path. Note that $\langle \partial B, v_t \rangle = -\langle \partial v_t, B \rangle$, and

$\partial v_t = (1-t)\delta_0 + \mu_t - \delta_t$. Hence,

$$\lim_{t\to 1}; \langle \partial B, v_t \rangle =$$
$$\lim_{t\to 1}; -\langle (1-t)\delta_0 + \mu_t - \delta_t, B \rangle =$$
$$\lim_{t\to 1}; \langle \delta_t - \mu_t, B \rangle =$$
$$\langle \delta_1 - \delta_1, B \rangle = 0.$$

Question

n times integrated Brownian motion

I have an identity that expresses the n times integrated Brownian motion and I would like to prove that. First, I define what I mean by n times integrated Brownian motion.

$$V_1(t) = \int_0^t B_s \, ds \qquad V_2(t) = \int_0^t V_1(s) ds$$

In general,

$$V_n(t) = \int_0^t V_{n-1}(s) ds$$

The identity that needs proof is as follows.

$$V_n(t) = \frac{1}{n!}\int_0^t (t-s)^n \, dB_s$$

Proof by induction seems appropriate here.

By using the following I show that the identity holds when $n = 1$.

$$tB_t = \int_0^t B_s \, ds + \int_0^t s \, dB_s$$

Then I assume that it holds for some arbitrary $n \geq 1$ and I try to show that the identity holding for $n+1$ is implied.

$$V_{n+1}(t) = \int_0^t V_n(s) ds = \frac{1}{n!}\int_0^t \int_0^s (s-r)^n \, dB_r \, ds$$

Next I apply Ito's lemma to the function $f(t, B_t) = (s-t)^n B_t$ for some $0 < s < \infty$. This gives

$$(s-t)^n B_t = -n\int_0^t (s-r)^{n-1} B_r \, dr + \int_0^t (s-r)^n \, dB_r$$

I set $t = s$ and I substitute the resulting expression for $\int_0^s (s-r)^n \, dB_r$ to the expression for $V_{n+1}(t)$.

$$V_{n+1}(t) = \frac{1}{(n-1)!}\int_0^t \int_0^s (s-r)^{n-1} B_r \, dr \, ds$$

I need to show that

$$\frac{1}{(n-1)!}\int_0^t \int_0^s (s-r)^{n-1} B_r \, dr \, ds = \frac{1}{(n+1)!}\int_0^t (t-s)^{n+1} dB_s$$

Manipulating the RHS using a similar method as above I get

$$n\int_0^t \int_0^s (s-r)^{n-1} B_r \, dr \, ds = \int_0^t (t-s)^n B_s \, ds$$

I cannot show why this is true. I am also not sure if my approach is the right one here. Could someone help please?

Answer

By Fubini's theorem,

$$\int_0^t \int_0^s (s-r)^{n-1} B_r \, dr \, ds =$$

$$\int_0^t \left(\int_r^t (s-r)^{n-1} ds \right) B_r \, dr =$$

$$\frac{1}{n} \int_0^t (t-r)^n B_r \, dr.$$

Combining this with the last equation in your question, finishes the proof.

Remark: Actually, we only have to do the "manipulation" using Itô's formula in order to switch both integrals. Statements as the stochastic Fubini show that this is also possible for stochastic integrals, i.e. it follows directly (e.g. from the stochastic Fubini) that

$$\int_0^t \int_0^s (s-r)^n \, dB_r \, ds = \int_0^t \int_r^t (s-r)^n \, ds \, dB_r.$$

This simplifies calculations a lot.

Question

Brownian motion and hitting frequency

Suppose we have a Brownian motion B_t with $B_0 = 0$ and $B_t - B_s \sim N(0, t-s)$. Every time B_t hits $\pm h$, where h is some "barrier" > 0, I pay someone £1 and the brownian motion resets back to 0, and we start again. Is there a good estimate for how much I would expect to pay over time? (A finite time that is, clearly over arbitrarily large spans of time you'd expect to hit one of $\pm h$ infinitely often.)

I guess equivalently you could say suppose $B_t \sim N(0, \sigma^2 t)$ with B_t smooth in t: roughly how frequently would you expect to hit ± 1 given a reset to 0 each time you hit the barrier? I don't need a fully rigorous answer or even fully accurate, just an estimate is good.

My first thought was that after time t, you "expect" to be roughly $\sigma\sqrt{t}$ away from the origin, so after time $1/\sigma^2$ you expect to be roughly 1 away from the origin: so maybe you expect to hit the barrier approximately every $1/\sigma^2$ units of time. However, I'm very wary of my use of the word "expect" here - the expectation we want for time per unit movement, and obviously expectation doesn't always commute with taking inverses to go from the "movement per unit time" we start with to "time per unit movement". Maybe this inversion is within reason of the correct calculation or maybe it's totally inaccurate. (I'm also cautious about the possibility I may have missed a factor of 2 in there or something like that...)

I'm afraid it's a while since I've done this sort of maths so any help would be greatly appreciated!

Answer

Paying events occur after i.i.d. times distributed as the first hitting time T of $\pm h$ starting from 0. One knows that $(T) = h^2$ (the easiest approach to this result being possibly that $(B_t^2 - t)_{t \quad 0}$ is a martingale). Hence, by the standard law of large numbers, the number of events occuring during a time interval of length t is $(t / h^2) + t\varepsilon_t$, where $\varepsilon_t \to 0$ when t is large.

Question

Mean of exponential Brownian motion

I am new to stochastics and I am trying to compute the expectation of $S_t = e^{\sigma W_t}$, where W_t is a standard Brownian motion and $\sigma > 0$. My attempt (using the log-normal PDF [here](https://en.wikipedia.org/wiki/Log-normal_distribution) and scaling the variance of W_t) is as follows:

$$[S_t] = \int_{\mathbb{R}} e^{\sigma x} \frac{1}{\sqrt{2\pi t \sigma^2} e^{\sigma x}} e^{-\frac{(\sigma x)^2}{2t\sigma^2}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t \sigma}} e^{-\frac{x^2}{2t}} dx ,$$

but I have been told the correct answer is

$$[S_t] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{\sigma x} e^{-\frac{x^2}{2t}} dx .$$

Can someone please point out why my attempt is incorrect?

Answer

It is not clear how you are using or you would use the fact that S_t is lognormal in the integral. What you have is that $S_t = e^{\sigma\sqrt{t}Z}$, where Z is an standard normal, therefore [its mean](https://en.wikipedia.org/wiki/Log-normal_distribution#Arithmetic_moments) is $e^{\sigma^2 t/2}$, which is the same result you get with the second integral.

To clarify, you have to possible paths for computing $E[S_t]$ by using the integral that involves the PDF. We make $X = W_t$.

1. A direct application of the definition of expected value, using the PDF of S_t:

$$E[S_t] = \int_{\mathbb{R}} s_t f_{s_t}(s_t) ds_t \quad / \quad (1)$$

2. Using the PDF of X, noting that $S_t = g(X) = e^{\sigma X}$:

$$E[S_t] = E[g(X)] = \int_{\mathbb{R}} g(x) f_X(x) dx = \int_{\mathbb{R}} e^{\sigma x} f_X(x) dx \quad / \quad (2)$$

The second approach is the one used in your second integral. As I

said, it is not clear how you came up with the integrand in the first integral. A direct use of the fact that S_t is lognormal would be that if $S_t = e^{\mu+\sigma Z}$, where Z is standard normal, then its mean is $e^{\mu+\sigma^2/2}$, which is a result that can be derived using (2).

EDIT: If $Y = e^{\mu+\sigma Z}$, with Z the standard normal, then Y is a lognormal random variable, and [its PDF is given [23]

$$f_Y(y) = \frac{1}{y\sqrt{2\pi}\sigma} e^{-(\ln y-\mu)^2/(2\sigma^2)}, \qquad y > 0$$

In your case $S_t = e^{\sigma X} = e^{\sigma\sqrt{t}Z}$, therefore

$$f_{S_t}(s_t) = \frac{1}{s_t\sqrt{2\pi}\sigma\sqrt{t}} e^{-(\ln s_t)^2/(2\sigma^2 t)}, \qquad s_t > 0$$

and (1) becomes

[23] A random positive variable X is log-normally distributed if the logarithm of X is normally distributed, $(\ln x;\mu,\sigma) = \frac{١}{\sigma\sqrt{٢\pi}}\exp\left[-\frac{(\ln x-\mu)^٢}{٢\sigma^٢}\right]$, $x > ٠$. A change of variables must conserve differential probability. In particular, $(\ln x)d\ln x = (\ln x)\frac{d\ln x}{dx}dx = (\ln x)\frac{dx}{x} = \ln(x)dx$, where $\ln(x;\mu,\sigma) = \frac{١}{x\sigma\sqrt{٢\pi}}\exp\left[-\frac{(\ln x-\mu)^٢}{٢\sigma^٢}\right]$, $x > ٠$ is the log-normal probability density function.*Johnson, Norman L.; Kotz, Samuel; Balakrishnan, N. (1994), "14: Lognormal Distributions", Continuous univariate distributions. Vol. 1, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics (2nd ed.), New York: John Wiley & Sons, ISBN 978-0-471-58495-7, MR 1299979*

$$E[S_t] = \int_0^\infty s_t \frac{1}{s_t\sqrt{2\pi}\sigma\sqrt{t}} e^{-(\ln s_t)^2/(2\sigma^2 t)} ds_t =$$

$u = \ln s_t = \sigma x$

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma\sqrt{t}} e^{-(\ln s_t)^2/(2\sigma^2 t)} ds_t$$

If we let $u = \ln s_t$,

$$E[S_t] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma\sqrt{t}} e^u e^{-u^2/(2\sigma^2 t)} du \Big/ (3)$$

But since $S_t = e^{\sigma X}$, , and replacing this in (3), we get (2).

Question

Expectation of Brownian motion Integral

I want to calculate $\left[\left(\int_0^t B_s \mathrm{d}B_s\right)^3\right]$ where B_t is a standard Brownian motion. Using Ito's formula for $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$

we can find that $\int_0^t B_s \mathrm{d}B_s = \frac{B_t^2}{2} - \frac{t}{2}$.

Then:

$$\left[\left(\int_0^t B_s \mathrm{d}B_s\right)^3\right] = \frac{1}{8} \left[\left(B_t^2 - t\right)^3\right]$$

and

$$[(B_t^2 - t)^3] =$$

$$[B_t^6 - 3tB_t^4 + 3t^2 B_t^2 - t^3] =$$

$$15t^3 - 9t^3 + 3t^3 - t^3 = 8t^3$$

However, I have seen in another question that

$\left[\left(\int_0^t B_s \mathrm{d}B_s\right)^3\right] = 0$. Am I doing something wrong?

Answer

No, I think you are right. The formula

$$\int_0^t B_s \, dB_s = \frac{1}{2}(B_t^2 - t)$$

can be found in almost all books about SDE's. And you could also calculate the given moments using that

$$B_t^2 - t = t \cdot \left(X_t^2 - 1\right)$$

where $X_t := \frac{B_t}{\sqrt{t}}$. is χ_1^2-distributed and the third centered moment of a χ_1^2-random variable is equal to 8 (see [here](http://mathworld.wolfram.com/Chi-SquaredDistribution.html).) Hence

$$X_t^2 \quad ((B_t^2 - t)^3) = t^3 \cdot \quad ((X_t^2 - 1)^3) = 8t^3$$

Remark: If you consider a process $X_t := \int_0^t \sigma(s)dB_s$ where σ is a deterministic(!) coefficient, then X_t is a centered normal random variable and in this case $(X_t^3) = 0$ holds.[24]

Question

Brownian Motion independent increment computation

One can rather easily show that

$$E\left[\sum_{i=0}^{i=n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i})\right] = -T + W_T^2 .$$

What I'm confused about is why we can't simply say that for each i, W_{t_i} is independent of $(W_{t_{i+1}} - W_{t_i})$, so that upon interchanging sums and expectations, and using independence we have

$$E\left[\sum_{i=0}^{i=n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i})\right] = \sum_{i=0}^{i=n-1} E\left[W_{t_i}\right] E\left[W_{t_{i+1}} - W_{t_i}\right] = 0 \quad ?$$

[24] For example, if $[X] = ۰$, we cannot conclude that $[X^۲] = ۰$, etc. An example is $X = Z \sim (۰,۱)$, where $[X] = ۰$ but $[X^۲] > ۰$ since $X^۲$ is never negative.

In this problem, we have naturally partitioned an interval as $0 = t_0 < t_1 < \ldots < t_n = T$, and W_t is a Brownian motion.

Answer

You have an error: you find that the expectation is $-T + E[W_T^2] = -T + T = 0$. That is, your observation is correct. We usually view this as arising from the fact that in the Ito convention, $\int_0^t W_s dW_s$ is a martingale. (So are the integrals of "most" functions against W_t .)[25]

Question

Arcsine law for Brownian motion

Here is the question:

$(B_t, t \geq 0)$ is a standard brwonian motion, starting at 0 .

> $S_t = \sup_{0 \leq s \leq t} B_s$. $T = \inf\{t \geq 0 : B_t = S_1\}$.

---

[25] $(X_t),(Y_t), t \in [a,b]$ ito integration

$$I := \int_a^b X_{t-} dY_t := \lim_{n\to\infty} \sum_{i=۱}^{n} X_{a+(i-۱)h}(Y_{a+ih} - Y_{a+(i-۱)h}), \quad h = \frac{b-a}{n}.$$ *Stratonowitsch-Integral*

$$S := \int_a^b X_t \circ dY_t := \lim_{n\to\infty} \sum_{i=۱}^{n} \frac{۱}{۲}(X_{a+(i-۱)h} + X_{a+ih})(Y_{a+ih} - Y_{a+(i-۱)h}).$$

> Show that T follows the arcsinus law with density

> $g(t) = \frac{1}{\pi\sqrt{t(1-t)}} 1_{]0,1[}(t)$.

I used Markov property to get the following equality:

$$P(T < t) = P(\sup_{t<s<1} B_s < S_t) =$$
$$E(P(\sup_{0<s<1-t} (B_{t+s} - B_t) < S_t - B_t \mid F_t)) =$$
$$P(\hat{S}_{1-t} < S_t - B_t).$$

where $\hat{S}_{1-t}$ is defined for the brownian motion $\hat{B}_s = B_{t+s} - B_t$, which is independant of F_t.

However the reflexion principle tells us that $S_t - B_t$ has the same law as S_t, so we can also write that

$$P(T < t) = P(\hat{S}_{1-t} < S_t).$$

To this point, we can calculate $P(T < t)$ because we know the joint density of $(\hat{S}_{1-t}, S_t)$, but this calculation leads to a complicated form of integral and I can not get the density g at the end.

Do you know how to get the arcsinus law? Thank you.

Answer

Let us start from the formula $(T<t) = (\hat{S}_{1-t} < S_t)$, where , $0 \quad t \quad 1$ and $\hat{S}_{1-t}$ and S_t are the maxima at times $1-t$ and t of two independent Brownian motions.

Let X and Y denote two i.i.d. standard normal random variables, then $(\hat{S}_{1-t}, S_t)$ coincides in distribution with $(\sqrt{1-t}\,|X|, \sqrt{t}\,|Y|)$ hence

$(T<t) = \quad (\sqrt{1-t}\,|X| < \sqrt{t}\,|Y|) = \quad (|Z| < \sqrt{t})$, where

$Z = X / \sqrt{X^2 + Y^2}$. Now, $Z = \sin\Theta$, where the random variable Θ is the argument of the two-dimensional random vector (X, Y) whose density is $e^{-(x^2+y^2)/2} / (2\pi)$, which is invariant by the rotations of center $(0,0)$. Hence Θ is uniformly distributed on $[-\pi, \pi]$ and

$(T<t) = \quad (|\sin\Theta| < \sqrt{t}) =$

$2 \quad (|\Theta| < \arcsin\sqrt{t}) = \frac{2}{\pi}\arcsin\sqrt{t}$. The density of the distribution of T follows by differentiation.

Question

Distribution of Brownian Motion help

If $X = \dfrac{B_1 - B_3 + B_2}{\sqrt{2}}$

Where B_t is brownian motion at time t.

And I want to find the the distribution of X, how would I do so?

$E[X] = 0$ is fairly straight forward.

For variance, however, I run into difficulty in treating:

$$Var(B_1 - B_3 + B_2) =$$
$$Var((B_1 - B_3) + B_2)) =$$
$$Var(B_1 - B_3) + Var(B_2) + 2Cov(B_1 - B_3, B_2)$$

But $Var(B_1 - B_3)$ isn't defined... is it?

Is there something I am missing, another method to calculate the variance of X ?

Answer

You have that $X = \dfrac{B_1 - B_3 + B_2}{\sqrt{2}}$. Since B is a Brownian motion, you know that $B_3 - B_2$ and B_1 are independent. Thus

$$\mathrm{var}(X) = \frac{1}{2}\mathrm{var}(B_1 - B_3 + B_2) =$$
$$\frac{1}{2}\mathrm{var}(B_1 - (B_3 - B_2)) =$$
$$\frac{1}{2}(\mathrm{var}(B_1) + \mathrm{var}(B_3 - B_2)).$$

By definition, the increments $B_s - B_t$ are normally distributed with distribution $N(0, s-t)$ ($s \geq t$), whence $\mathrm{var}(B_s - B_t) = s - t$. Completing the calculation for $\mathrm{var}(X)$, it follows that

$\mathrm{var}(X) = \frac{1}{2}(\mathrm{var}(B_1) + \mathrm{var}(B_3 - B_2)) = \frac{1}{2}(1+1) = 1$. (*Recallthat* B _1 = B_1-B_0, and thus $\mathrm{var}(B_1) = 1 - 0 = 1$).

Question

Is $W_{2t} - W_t$ a brownian motion?

$(W_t)_{t\geq 0}$ is a brownian motion, I have to show that $X_t := W_{2t} - W_t$ is a brownian motion as well.

$W_{2t} = 1/\sqrt{2}W_t \ (by scaling property) then I can write \ 1/\sqrt{2}W_t - W_t$

and its distribution is Normal, with mean 0 and variance $\frac{3-2\sqrt{2}}{2}$.

I would conclude it is a brownian motion.

Answer

For each fixed t, the random variable $X_t = W_{2t} - W_t$ is centered normal with variance t hence indeed distributed like W_t.

But the *process* (X_t) is not a Brownian motion. To see this, note that $X_2 = W_4 - W_2$ and $X_1 = W_2 - W_1$ are the increments of a Brownian motion on some disjoint time intervals hence they are independent while W_2 and W_1 are not (for example, $E(X_2 X_1) = 0$ and $E(W_2 W_1) = 1$).

Question

quadratic variations of Brownian motion squared

I'm trying to refresh my memories about stochastic processes.

We know that Brownian motion has as quadratic variation equals to t. What is the quadratic variation of the Brownian motion *squared* ? Usually for this I would just use Ito's formula and pick out whatever is in front of the dWt, except in that case it doesn't work.

Is there a straightforward way to compute this ? thanks !

Answer

There are several definitions for the quadratic variation:

1. For a "nice" process $(X_t)_{t\geq 0}$ ("nice" means semimartingale), the quadratic variation is defined by

$[X]_t := X_t^2 - X_0^2 - 2\int_0^t X_{s-}dX_s$. For $X_t := B_t^2$, we know from Itô's formula that $d(B_s^2) = 2B_s\,dB_2 + ds$ and

$B_t^4 = 4\int_0^t B_s^3 dB_s + 6\int_0^t B_s^2 ds$. Combining these two equalities yields $[B^2]_t = B_t^4 - 2\int_0^t B_s^2 d(B_s^2) = 4\int_0^t B_s^2 ds$

2. The quadratic variation of a process $(X_t)_{t\geq 0}$ is the limit (in probability) of the sums $S_\Pi(X) := \sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})^2$ as the

mesh size $|\Pi|$ of the partition $\Pi = \{0 = t_0 < \ldots < t_n \le t\}$ tends to 0. For $X_t := B_t^2$, we have

$$S_\Pi(B^2) =$$

$$\sum_{j=1}^{n} (B_{t_j}^2 - B_{t_{j-1}}^2)^2 =$$

$$\sum_{j=1}^{n} (B_{t_j} + B_{t_{j+1}})^2 \cdot (B_{t_j} - B_{t_{j-1}})^2 =$$

$$\sum_{j=1}^{n} (B_{t_j} + B_{t_{j+1}})^2 \left[(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1}) \right]$$

$$+ \sum_{j=1}^{n} (B_{t_j} + B_{t_{j+1}})^2 (t_j - t_{j-1}) =$$

$I_1 + I_2$ Recall that

$$\sum_{j=1}^{n} g(t_j) \cdot (t_j - t_{j-1}) \to \int_0^t g(s) ds \qquad \text{as} n \to \infty$$

for any continuous function g. Using the continuity of g, it is therefore not difficult to see that

$$\sum_{j=1}^{n} (g(t_j) + g(t_{j+1}))^2 (t_j - t_{j-1}) \to 4 \int_0^t g(s)^2 ds.$$

Applying this to the continuous sample paths of the Brownian motion, we find that

$$I_2 \to 4 \int_0^t B_s^2 ds.$$

It remains to prove that I_1 converges to 0. This follows from some straightforward calculations (Hint: Show that the L^2-norm converges to 0 using that $B_t^2 - t$ is a martingale).

Question

Correlated Brownian motion and Poisson process

Is there an easy way to construct, on the same filtered probability space, a Brownian motion W and a Poisson process N, such that W and N are not independent ?

Answer

In case (W,N) has independent increments, then W and N are also independent, since a Brownian Motion has no common jumps with the Poisson process. This of course doesn't say, there is no probability space, where (W,N) has dependent increments, but it gives you a hint, how it might be constructed.

See e.g. Kallenberg - Foundations of Modern Probability 13.6 for a proof

Question

Discontinuous expression cannot be a Brownian motion?

>Let W_t and $\tilde{W}_t$ be two standard independent Brownian motions and for a constant $-1 \le \rho \le 1$, define $X_t := \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$.

>Is $X := (X_t)_{\{t \ge 0\}}$ a Brownian motion?

I know that to show something is a Brownian motion I must check the various criteria, but skipping to one of the later ones it appears that X is discontinuous, e.g. at $\rho = 2$. Does this mean that X cannot be a Brownian motion? It appears to satisfy the other criteria.

Answer

It is a Brownian Motion of correlation ρ if $|\rho| \le 1$ with W_t and $\sqrt{1-\rho^2}$ with $\bar{W}_t$.

One way to show it is by checking is is a martingale with correlation $\mathrm{Corr}[X_t X_s] = [X_t X_s] = s \wedge t$

Question

Martingale property of negative Brownian motion

Let B_t be Brownian motion, with $B_0 = 0$. Next define $M_t = -B_t$.

Have I understood it correctly that M_t is not a Martingale?

$E[M_t] = 0$

$E[M_{t+1} \mid M_t] = -M_t$ and therefore not a Martingale?

Answer

Multiplying by a constant doesn't change the Martingale property: for $t > s$,

$$E[M_t \mid M_s] = E[-B_t \mid M_s] = E[-B_t \mid B_s]$$

$$= -E[B_t \mid B_s] = -B_s = M_s.$$

Question

Probability of exponential of brownian motion

W_t is a brownian motion, I have this exponential value:

$v(t) = e^{0.00025+0.3W_t}$ what's the probability that $v(1) < 0.5$?

By taking natural log on both size, I got
$0.00025 + 0.3W_1 < -ln(2), W_1 = -2.3$

Then the question becomes $P(W_1 < -2.3) = ?$

Answer

You can check from the definition of a Brownian motion that W_1 has the normal distribution $N(0,1)$. So the problem really amounts to figuring out what

$$P(W_1 < -2.3)$$

is for a standard normal random variable W_1. You can look this up in a table. You have that

$$P(W_1 < -2.3) = \Phi(-2.3),$$

where Φ is the cumulative distribution function of the standard normal distribution.

Question

Covariance of Brownian-motion-like processes

We know that $\mathrm{Cov}(B_s, B_t) = \min(s,t)$ if B_t is Brownian motion.

What is $\mathrm{Cov}(B_{f(s)}, B_{f(t)})$ for some injective f ?

How can I write $B_{f(t)}$ in an Ito process form

$dX = \mu(X,t)dt + \sigma(X,t)dB_t$, or Ito integral?

Answer

Let $f:[0,\infty) \to [0,\infty)$ and $s,t \in [0,\infty)$. Then

$cov(B_{f(s)}, B_{f(t)}) = \min\{f(s), f(t)\}$

since $\mathrm{cov}(B_u, B_v) = \min\{u,v\}$

holds for all $u,v \in [0,\infty)$, so in particular for $u := f(s)$, $v := f(t)$.

Concerning your second question:

> Theorem: Let $(X_t)_{t\geq 0}$ a continuous local martingale, $X_0 = 0$. Then

> there exists a Brownian motion $(W_t)_t$ (with respect to a filtration

> ($_t)_{t\geq 0}$) such that $X_t = W_{[X]_t}$ where [X] denotes the compensator of X .

g such that $f(t) = \int_0^t g(s)^2 ds$, then we can define

$$X_t := \int_0^t g(s)dB_s$$

which gives $[X]_t = \int_0^t g(s)^2 ds = f(t)$. By applying the theorem we obtain that there exists a Brownian motion $(W_t)_{t\geq 0}$ such that

$$X_t = \int_0^t g(s)^2 dB_s = W_{[X]_t} = W_{f(t)}$$

This works in particular if f is differentiable and $f(0)=0$, put $g(s) := \sqrt{f'(s)}$. Of course that's not what you were looking for since you wanted to find a representation as an Itô process for a given Brownian motion.

The problem is the following: You could define (as above)

$$X_t := \int_0^t g(s)dB_s$$

where $g(s) := \sqrt{f'(s)}$, $f(0)=0$. Then X_t is a centered Gaussian random variable with variance $f(t)$ and $X_t - X_s$ has variance $f(t)-f(s)$ which means that X_t has the same distribution as the time-changed Brownian motion $B_{f(t)}$. But they are not necessarily the same processes! If you choose for example $f(t) := 2t$ you obtain

$$X_t = \int_0^t \sqrt{2}dB_s = \sqrt{2}B_t$$

and this is clearly not the same process as $B_{f(t)} = B_{2t}$. On the other hand we know that $W_t := \sqrt{2}\cdot B_{\frac{t}{2}}$ is a Brownian motion and obviously $X_t = \sqrt{2}\cdot B_t = \sqrt{2}\cdot B_{\frac{2t}{2}} = W_{2t} = W_{f(t)}$

Remark:

1. Let $X_t := \int_0^t f_s \, dB_s$ where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\left(\int_0^t f(s)^2 ds \right) < \infty$. Then $(X_t)_t$ is a martingale and the compensator is given by $[X]_t = \int_0^t f(s)^2 ds$ If f is deterministic, i.e. $f : \mathbb{R} \to \mathbb{R}$, we know therefore that (X_t) has a deterministic compensator.

2. Let's consider the process $(t, \omega) \mapsto B(f(t), \omega)$ where $f : [0, \infty) \to [0, \infty)$ is a deterministic injective function such that $f(0) = 0$. In this case f determines how fast you run through the trajectories $(t, \omega) \mapsto B(t, \omega)$. For example $f(t) := 2t$ means that you run through the trajectories twice as fast.

Question

Brownian motion in upper half-plane

I was wondering if there is a way to explicitly calculate the probability that Brownian motion is at two consecutive times s,t in the upper half-plane. So I want to calculate $(B_t > 0, B_s > 0)$ if this is analytically possible.

Even an expression in terms of integrals would be helpful.

Answer

Let $p_t(x, y)$ be the Brownian transition density function $\frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}$, so that its Markov property reads

$$(f(B_{s+h}) \mid B_s) = \int f(y) p_h(B_s, y) dy.$$

We wish to compute $(B_t > 0, B_s > 0)$. In order to use the Markov property in the last paraghraph, rewrite it as $(1_{\{B_s > 0, B_t > 0\}}) = (1_{\{B_s > 0, B_t > 0\}} \mid B_s)$, and notice that you can view $1_{\{B_s > 0, B_t > 0\}} = 1_{[0,\infty)\times[0,\infty)}(B_s, B_t)$ as a function of B_{s+h} with $h = t - s$. Applying the property to the conditional expectation we get

$$(B_t > 0, B_s > 0) = \int 1_{[0,\infty)\times[0,\infty)}(B_s, y) p_{t-s}(B_s, y) dy.$$

The last expectation can be computed directly using the law of B_s, that is, $p_s(0, x)dx$, so finally

$$(B_t > 0, B_s > 0) = \int_0^\infty \int_0^\infty p_{t-s}(x, y) p_s(0, x) dy\, dx.$$

Question

"Continuity" of stochastic integral wrt Brownian motion

I'd like to prove a nice property of a stochastic integral with respect to Brownian motion.

Let $(H_t)_{t\geq 0}$ be a progressive and bounded process that is continuous at 0 and B a standard Brownian motion. Then

$\frac{1}{B_\varepsilon}\int_0^\varepsilon H_s dB_s \to H_0$ as $\varepsilon \to 0$ in probability.

Anyone got some hints ? I'm really puzzled and I don't know where to start.

Answer

Hint Let $\varepsilon > 0$, $\delta > 0$. We have

$$\left(\left|\frac{1}{B_\varepsilon}\cdot\int_0^\varepsilon H_s\, dB_s - H_0\right| > \delta\right) =$$

$$\left(\left|\frac{1}{B_\varepsilon}\cdot\int_0^\varepsilon (H_s - H_0) dB_s\right| > \delta\right) \leq$$

$$\left(\left|\frac{1}{B_\varepsilon}\cdot\int_0^\varepsilon (H_s - H_0) dB_s\right| > \delta, \left|\frac{\sqrt{\varepsilon}}{B_\varepsilon}\right| \leq K\right) + \left(\left|\frac{\sqrt{\varepsilon}}{B_\varepsilon}\right| > K\right) \leq$$

$$\left(\left|\frac{1}{\sqrt{\varepsilon}}\cdot\int_0^\varepsilon (H_s - H_0) dB_s\right| > \frac{\delta}{K}\right) + \left(\frac{|B_\varepsilon|}{\sqrt{\varepsilon}} < \frac{1}{K}\right)$$

$$=: I_1 + I_2$$

for any $K > 0$. Since $\frac{B_\varepsilon}{\sqrt{\varepsilon}} \sim N(0,1)$, we can choose $K > 0$ (independent of ε) such that

$$I_2 \leq \frac{\varepsilon}{4}$$

For the first term I_1 apply Markov's inequality and Itô's isometry to show that it converges to zero as $\varepsilon \to 0$, using the continuity of H at 0.[26]

Question

Law of large numbers for Brownian Motion

Let $\{B_t : 0 \leq t < \infty\}$ be standard Brownian motion and let T_n be an increasing sequence of finite stopping times converging to infinity a.s. Does the following property hold?

$$\lim_{n\to\infty} \frac{B_{T_n}}{T_n} = 0 \text{ a.s.}$$

answer

This is an almost sure property hence the result you are asking to check is equivalent to the following.

[26] **Remark** A detailed proof can be found in Dean Isaacson, Stochastic Integrals and Derivatives (1969).

> Let $f:\mathbb{R}_+ \to \mathbb{R}$ denote a function such that $\lim_{t\to+\infty} f(t)=0$ and (t_n) a sequence of nonnegative real numbers such that $\lim_{n\to\infty} t_n = +\infty$. Then $\lim_{n\to\infty} f(t_n)=0$.

Surely you can prove this. Then fix some ω in Ω, consider the function f defined by $f(t)=B_t(\omega)/t$ and the real numbers $t_n = T_n(\omega)$, and apply the deterministic result.

Question

Integral with respect to brownian motion

Let f be a continuous function on $[0,\infty)$ and B_t be a standard Brownian motion. Define

$$X_t = \int_0^t f(s)dB(s).$$

a) Show that X_t is Gaussian and computer its covariance $C(X_s, X_t)$

b) prove that $e^{(X_t - 1/2C(X_t,X_t))}$ is a continuous time martingale.

I think I know how to show X_t is Gaussian (basically expressing it as a sum of Brownian motion increments each of which is gaussian) is that right?

I don't know how to derive the covariance and show that $e^{(X_t - 1/2C(X_t,X_t))}$ is a martingale, any hint would be appreciated.

Answer

1. Covariance: As @Did pointed out, Itô's formula combined with the polarization identities (i.e. $a \cdot b = \frac{1}{4}((a+b)^2 - (a-b)^2)$) yields

$\left(\int g(r)dB_r \int h(r)dB_r\right) = \left(\int g(r)h(r)dr\right)$ whenenever the expression is well-defined. Setting

$g(r) := 1_{[0,t]}(r)f(r), h(r) := 1_{[0,s]}(r)f(r)$, we find

$(X_t X_s) = \left(\int_0^{s\wedge t} f(r)^2 dr\right) = \int_0^{s\wedge t} f(r)^2 dr$ where the last inequality holds due to the fact that f is deterministic.

Consequently, $C(s,t) = \int_0^{s\wedge t} f(r)^2 dr$.

2. Martingale property: This is a straightforward application of Itô's formula. In fact, we have for $F(t,x) := e^{x-G(t)}$,

$F(t,X_t) - F(0,0) =$

$$\int_0^t F(s,X_s)dX_s - \int_0^t G'(s)F(s,X_s)ds + \frac{1}{2}\int_0^t F(s,X_s)f(s)^2 ds$$

for any differentiable (deterministic) function G. In particular, if we choose $G(t) = \frac{1}{2}C(t,t)$, then this shows

$$F(t,X_t) - 1 = \int_0^t F(s,X_s)dX_s.$$

The right-hand side is a martingale as a stochastic integral with respect to a martingale. This finishes the proof.

Question

Scaled integrated Brownian motion has limit

Let B be a standard Brownian motion and put

$X(t)=\frac{1}{\sqrt{t}}\int_0^t f(B(s))ds$, where $f\in L_1(\mathbb{R}^1)$ and $\int f(x)dx=1$.

Show that $\lim_{t\to\infty} EX(t)=\sqrt{2/\pi}$ and $\lim_{t\to\infty} E[X(t)^2]=1$.

I was thinking to try writing them as Riemann sums, but it seems that it does not work out well. Any hints or help would be appreciated.

Answer

Let

$$X_t:=\frac{1}{\sqrt{t}}\int_0^t f(B_s)ds.$$

Using that $B_s\sim N(0,s)$, $s\geq 0$, and applying Fubini's theorem yields

$$(X_t)=$$

$$\frac{1}{\sqrt{t}}\int_0^t \quad (f(B_s))ds=$$

$$\frac{1}{\sqrt{t}}\int_{\mathbb{R}}\int_0^t f(x)\frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{x^2}{2s}\right)ds\,dx$$

Now we substitute $r:=\frac{\sqrt{s}}{\sqrt{t}}$ and obtain

$$(X_t) = \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\int_0^1 \exp\left(-\frac{x^2}{2r^2 t} \right) dr \right] f(x)dx.$$

From the dominated convergence theorem, it follows that

$$\lim_{t \to \infty} (X_t) = \sqrt{\frac{2}{\pi}} \int f(x)dx = \sqrt{\frac{2}{\pi}}.$$

For the second equation, write

$$\left| \int_0^t f(B_r)dr \right|^2 = \int_0^t \int_0^t f(B_r) f(B_s) dr\, ds,$$

and use that (B_r, B_s) is Gaussian with mean vector $(0,0)$ and covariance matrix

$$\begin{pmatrix} r & r \\ r & s \end{pmatrix}$$

for $r \le s$. By choosing a suitable substitution, we obtain - as in the first part - an integral which can be calculated explicitely.

Question

modulus of continuity of Brownian Motion

Is there any estimate for the following quantity

$E\left(\sup_{\substack{0\le s,t\le 1\\ |t-s|<\delta}} |W_t - W_s|\right)$ for some small $\delta > 0$, where W is a Brownian motion.

I'm hoping for some result of the form

$E(\sup_{\substack{0\le s,t\le 1\\ |t-s|<\delta}} |W_t - W_s|) \le \delta^{\frac{1}{2}}$. Thanks!

Answer

Actually, it turns out that there exists constants c_1 and c_2 such that

$$c_1\left(\delta \log\frac{2}{\delta}\right)^{1/2} \quad \left(\sup_{\substack{0 \quad s,t \quad 1\\ |t-s|<\delta}} |W_t - W_s|\right) \quad c_2\left(\delta \log\frac{2}{\delta}\right)^{1/2}$$

(see Section 4 in [the paper by Fischer and Nappo][27]). Note that they also have good bounds for the p th moment of the modulus of continuity of the Brownian motion.

[27] **Reference:** Fischer, Markus(D-HDBG-A); Nappo, Giovanna(I-ROME) **On the moments of the modulus of continuity of Itô processes.** (English summary) Stoch. Anal. Appl. 28 (2010), no. 1, 103–122.

Question

Expectation of expression involving Brownian motion

How do I compute

$$E\left(tW_t - \int_0^t W_u\,du \,\middle|\, s\right).$$

Given that W_t is standard Brownian motion under the measure P and $\{ \ _t, t \geq 0\}$ denotes its standard filtration.

could someone guide me on the approach to this problem?

Answer

The hard part is finding the conditional expectation of the integral $\int_0^t W_u\,du$. It turns out that there is a "Fubini theorem" for conditional expectation:

> **Proposition**. Suppose X_t is a stochastic process, is a σ -field. Recall by Tonelli's theorem that $E\int_a^b |X_u|du = \int_a^b E\,|\,X_u\,|du$. If this quantity is finite, then

$$E\left[\int_a^b X_u\,du \,\middle|\, \right] = \int_a^b E[X_u \mid \ ]du.$$

I'll let you prove this as an exercise. As a hint, use the definition of conditional expectation to show that the right side is the conditional

expectation of $\int_a^b X_u \, du$; you'll need to use (classical) Fubini's theorem.

(As a side comment, the same proposition holds, with the same proof, if we replace [a,b] by any other σ -finite measure space.)

Given this, the rest of the problem should not be too hard.

Question

change of sign for brownian motion

for a fixed > 0 , I want to show that almost surely (ie with probability 1), a standard brownian motion W_t would change sign over [o,].

I thought about defining a random variable $U_t = sign(W_t)$ which would satisfy :

$(1+U_t)/2$ follows symmetric Bernoulli distribution

But I can't conclude with this.

Am I on a good track?

answer

The reflection principle says that for any > 0 and $x > 0$, we have

$$_0(\sup_{0 \le t \le} W_t \ge x) = 2 \ _0(W \ \ge x) = 2 \ \left(Z \ge x / \sqrt{\ }\right).$$

Letting $x \downarrow 0$ we conclude that

$$_0\left(\sup_{0 \le t \le} W_t > 0 \right) = 2 \quad (Z > 0) = 1.$$

In any time interval, no matter how short, the Brownian motion (W_t) must take a strictly positive value.

Applying this to the Brownian motion $(-W_t)$, we see that (W_t) must also

take a strictly negative value. So the process (W_t) changes sign with probability one.

Question

Laplace Transform of a Brownian motion

If $v(\omega,t):\Omega\times[0,\infty)\to\mathbb{R}$ is a Standard Brownian motion, then for what values of s,ω does the Laplace transform

$I(\omega,s)=\int_0^\infty e^{-st}v(\omega,t)dt$ converge (where s is a complex number).

Answer

The [Law of Iterated Logarithm](http://en.wikipedia.org/wiki/Wiener_process#Quantitative_properties) for the Brownian motion implies that for a.e. ω

$$v(\omega,t) \le C\sqrt{t \log\log(e^e + t)} \qquad \forall t \ge 0$$

Therefore, for a.e. ω the integral $\int_0^\infty e^{-st} v(\omega,t)$ converges for all s with $\operatorname{Re} s > 0$.

On the other hand, the integral diverges almost surely when $\operatorname{Re} s = 0$. Indeed, let $s = iy$. The growth and continuity properties imply that there is an infinite sequence of disjoint time intervals $T_i = [t_i, t_i + 0.1y^{-1}]$ in which $v(\omega,t)$ is contained in some ball $B(a,r)$ with $|a| \ge 2r$. The absolute value of the integral $\int_{T_i} e^{-ity} v(\omega,t)$ is bounded below by a positive constant.

I don't think there can be a useful description of the exceptional values of ω for which the integral fails to converge. The pointwise structure of Ω is usually not important in probability.

Question

Second order Cone and Brownian motion

Denote $D = \{x \in \mathbb{R}^{d+1} : \sqrt{x_1^2 + \ldots + x_d^2} \le x_{d+1}\}$ which is the second order cone (ice-cream cone) in $\mathbb{R}^{d+1}$. My question is that if a standard brownian motion $W_t = (X_t^1, \ldots, X_t^{d+1})$ in $\mathbb{R}^{d+1}$ is started at $x_0 = (1, 0, \ldots, 0)$, will it finally hit D, i.e.,

$$P^{x_0}(\exists t > 0, W_t \in D) = 1.$$

In the case $d=1$, it is true as $\{W_t : t>0\}$ is dense in $\mathbb{R}^2$. For $d>1$, the Brownian motion is transient, i.e., $\|W_t\|_2$ goes to infinity. How can I deal with this situation?

Answer

Let $x \in \mathbb{R}^d$ be any starting point, and define

$S_1 = \{y \in \mathbb{R}^d : \| y \| = 2 \| x \|\}$.

Let τ_1 be the hitting time of S_1 and C

be your cone. The function

$z \mapsto \ _z(B_{\tau_1} \in C)$ is continuous

and strictly positive for z with $\| z \| = \| x \|$ and so, for such z,

$_z(B_{\tau_1} \in C) \geq p > 0.$

Now define $S_n = \{y \in \mathbb{R}^d : \| y \| = 2^n \| x \|\}$

so that, by scaling, $_z(B_{\tau_n} \in C) \geq p > 0$

for z with $\| z \| = 2^{n-1} \| x \|$.

Using the strong Markov property gives

$_x(B_{\tau_n} \notin C \text{ for } n = 1, \ldots, N) \leq (1-p)^N$,

and letting $N \to \infty$ we get $_x(\tau_C < \infty) = 1$.[28]

[28] http://mathoverflow.net/questions/222085/with-probability-1-a-d-dimensional-brownian-motion-will-enter-any-cone-with-p

question

Expected value of average of Brownian motion

For a standard one-dimensional Brownian motion $W(t)$, calculate:

$$E[(\frac{1}{T}\int_0^T W_t\,dt)^2]$$

Note: I am not able to figure out how to approach this problem. All i can think of is that the term $\frac{1}{T}\int_0^T W_t\,dt$ is like 'average'. But not sure how to proceed ahead. I'm relatively new to Brownian motion. I tried searching the forum for some hints..but could not find one. I will really appreciate if you could please guide me in the right direction. Thanks!

Answer

If you recall that $\mathrm{d}(tW_t) = W_t\mathrm{d}t + t\mathrm{d}W_t$ you can write your integral in the other form

$\int_0^T W_t\mathrm{d}t = TW_T - \int_0^T t\mathrm{d}W_t$. If we forget about the factor $\frac{1}{T}$ as it does not affect the derivation much, we obtain

$$\mathsf{E}\left(\int_0^T W_t\mathrm{d}t\right)^2 = \mathsf{E}\left[T^2W_T^2\right] - 2T\mathsf{E}\left[W_T\int_0^T t\mathrm{d}W_t\right] + \mathsf{E}\left(\int_0^T t\mathrm{d}W_t\right)^2$$
$$= T^3 - 2T\int_0^T t\mathrm{d}t + \int_0^T t^2\mathrm{d}t$$

where we applied the Ito isometry a couple of times. Hopefully, the most hard part now is done and you can finish the derivations.[29]

Question

Ito Integral Properties with Brownian Motion

I am working out some of the properties for the Ito integral with Brownian motion and I am trying to use the definition to verify that

$$\int_0^t s\,dB_s = tB_t - \int_0^t B_s\,ds$$

and

$$\int_0^t B_s^2\,dB_s = \frac{1}{3}(B_t^3) - \int_0^t B_s\,ds$$

I am having trouble thinking about how to attack this so any help or suggestions on how to get started would be great!

Answer

The first equation is (after the edit) true.

---

[29] Expand the square as

$\left(\int_۰^T W_t\,dt\right)^۲ = \int_۰^T \int_۰^t ۲W_t W_s\,dsdt$, and use Fubini theorem and the identity

s $(W_t W_s) = s$ for every s t to conclude that the expectation you want to compute is

$$\frac{۱}{T^۲}\int_۰^T \int_۰^t ۲s\,dsdt = \frac{T}{۳}.$$

Consider the twodimensional continuous semimartingale (t, B_t), and function $f(x, y) = xy$

we get

$$D_x f(x, y) = y$$
$$D_y f(x, y) = x$$
$$D_1 D_1 f = D_2 D_2 f = 0$$
$$D_1 D_2 f = D_2 D_1 f = 1$$

And therefore ITO's formula gives

$$tB_t = 0 + \int_0^t s \, dB_s + \int_0^t B_s \, ds.$$

The 2nd equation is (after the edit) true as well. You can convince yourself with an argument similar to the one i gave. Apply the 1-dimensional ITO's formula on B_t with $f(x) = x^3$

Question

Bound probability Brownian motion stays in $[-1,1]$

Let $T := \inf\{t > 0 : |B_t| = 1\}$ be a hitting time for standard Brownian motion. I want to show that $\lim_{t \to \infty} e^{\frac{\pi^2}{8} t} \; [T \geq t] = \frac{4}{\pi}$

I had a look at http://math.stackexchange.com/questions/372296/a-bound-for-the-probability-that-a-brownian-motion-stays-in-an-

interval and only managed to get a the bound

$[T \geq t] \leq e^{\frac{-tu^2}{2}} \frac{1}{\cos(u)}$ for all $u \in [0, \frac{\pi}{2})$, but it's not quite enough.

Answer

Observe that $T = \min\{T_1, T_{-1}\}$ where T_1, respectively T_{-1} denote the first hitting time of a Brownian motion B at point 1, respectively -1. Then

$$P(T > t) = P(\min\{T_1, T_{-1}\} > t) = P(T_1 > t, T_{-1} > t)$$

$$= P(T_{-1} > t \mid T_1 > t) P(T_1 > t).$$

Then clearly

$$P(T > t) \leq P(T_1 > t).$$

Now we know that T_1 is Lévy distributed with $\mu = 0$ and $c = 1^2$, see [30]

Hence

$$P(T > t) \leq P(T_1 > t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty s^{-3/2} e^{-\frac{1}{2s}} ds.$$

Now take $e^{\alpha t^\beta}$, $\alpha, \beta \in \mathbb{R}$. Hence

$$e^{\alpha t^\beta} P(T > t) \leq \frac{1}{\sqrt{2\pi}} e^{\alpha t^\beta} \int_t^\infty s^{-3/2} e^{-\frac{1}{2s}} ds.$$

[30] http://en.wikipedia.org/wiki/L%C3%A9vy_distribution.

Then we are interested in finding α, β so that the following limit is a finite constant different from 0. So

$$\lim_{t\to\infty} \frac{\int_t^\infty s^{-3/2} e^{-\frac{1}{2s}} ds}{e^{-\alpha t^\beta}} = "\frac{0}{0}"$$

Then using L'Hôpital's rule and fundamental theorem of calculus we have

$$\lim_{t\to\infty} \frac{\int_t^\infty s^{-3/2} e^{-\frac{1}{2s}} ds}{e^{-\alpha t^\beta}} = \frac{1}{\alpha\beta} \lim_{t\to\infty} \frac{t^{-3/2} e^{-\frac{1}{2t}}}{t^{\beta-1} e^{-\alpha t^\beta}}.$$

Now if $\beta = -1/2$ then

$$\lim_{t\to\infty} \frac{\int_t^\infty s^{-3/2} e^{-\frac{1}{2s}} ds}{e^{-\alpha t^\beta}} = \frac{-2}{\alpha}.$$

In a summary we have

$$\lim_{t\to\infty} e^{\alpha t^{-1/2}} P(T > t) \le \frac{1}{\sqrt{2\pi}} \frac{-2}{\alpha}$$

and you are free to choose the α you like.

I don't know if this estimate is too crude for your purposes. If you want exact limit I guess you have to compute $P(T_1 > t \mid T_{-1} > t)$ explicitly and proceed in the same way.

Question

Conditional Integral of Square of Brownian Motion?

I am struggling to compute the expectation and variance of the following, where $W(s)$ is a standard Brownian motion:

$$X := \int_0^A W(s)^2 ds$$

$$Y := \int_0^A W(s) ds$$

$$E[X \mid Y] = \ ?$$

$$Var[X \mid Y] = ?$$

Answer

Note that, for every s in $(0, A)$, $(W(s), Y)$ is a centered gaussian vector hence there exists some real numbers $(u(s), v(s))$ and some standard normal random variable $Z(s)$ independent of Y such that $W(s) = u(s)Y + v(s)Z(s)$. Thus,

$$X = \int_0^A (u(s)Y + v(s)Z(s))^2 \mathrm{d}s = aY^2 + 2VY + T \quad \text{where}$$

$$a = \int_0^A u(s)^2 \mathrm{d}s$$

$$V = \int_0^A u(s)v(s)Z(s) \mathrm{d}s$$

$$T = \int_0^A v(s)^2 Z(s)^2 \mathrm{d}s.$$

Since V and T are independent of Y, this yields

$$E(X\,|\,Y) = aY^2 + E(T) = Y^2\int_0^A u(s)^2\,\mathrm{d}s + \int_0^A v(s)^2\,\mathrm{d}s.$$

Likewise, $\operatorname{var}(X\,|\,Y) = \operatorname{var}(2VY + T\,|\,Y) = 4E(V^2)Y^2 + \operatorname{var}(T).$

We now indicate how to compute a, $E(T)$, $E(V^2)$ and $E(T^2)$. First, the decomposition of $W(s)$ into a linear combination of Y and $Z(s)$ independent of Y yields

$$E(W(s)Y) = u(s)E(Y^2)$$

$$E(W(s)^2) = u(s)^2E(Y^2) + v(s)^2.$$

Since $E(W(s)Y) = s(A - s/2)$ and $E(Y^2) = A^3/3$, this yields $u(s)$ and $v(s)$, hence the values of a and $E(T) = \int_0^A v(s)^2\,\mathrm{d}s,$

which might be $a = \dfrac{6}{5}\dfrac{1}{A}, \qquad E(T) = \dfrac{1}{10}A^2.$ To compute $E(V^2)$ and $E(T^2)$, it might be better to start from

$$V = \int_0^A u(s)(W(s) - u(s)Y)\,\mathrm{d}s$$

hence

$$T = \int_0^A (W(s) - u(z)Y)^2\,\mathrm{d}s,$$

$$E(V^2) = 2\int_0^A u(s)\int_0^s u(t)E((W(s) - u(s)Y)(W(t) - u(t)Y))\,\mathrm{d}t\,\mathrm{d}s,$$

which yields $E(V^2) = 2\int_0^A u(s)\int_0^s u(t)(t - u(s)u(t)E(Y^2))\,\mathrm{d}t\,\mathrm{d}s,$

from which the value of $E(V^2)$ follows. Likewise for $E(T^2)$, which requires $E(Y^4) = 3E(Y^2)^2$

Question

Is this true about Brownian Motion?

I have the following in my notes and I'm not sure if it's true or not. Any help would be highly appreciated.

If $\{W_t\}_{t\geq 0}$ is a standard Brownian motion stochastic process, $\Delta > 0$ and $n \in \mathbb{N}$ then $W_{n\Delta} - W_{(n-1)\Delta}$ is mean zero and variance Δ normal random variable.

I am using the following definition of Brownian motion, as taken from Harrison's Stochastic flow systems:

A stochastic process $\{W_t\}_{t\geq 0}$ has independent increments if the random variables $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for all $n \geq 1$ and all $0 \leq t_0 \leq \ldots \leq t_n < \infty$. It has stationary independent increments if the distribution of $X_t - X_s$ depends only on $t - s$. A Standard Brownian motion is a stochastic process that has continuous sample paths, stationary independent increments and $X_t \sim N(0,t)$.

That the difference is mean zero, I can see. What I'm not convinced about is that variance will be Δ. Any thoughts?

Answer

$$E[(W_{n\Delta} - W_{(n-1)\Delta})^2] =$$

$$E[W_{n\Delta}^2] - 2E[W_{n\Delta}W_{(n-1)\Delta}] + E[W_{(n-1)\Delta}^2]$$

Use what we know about the second moment of W_t, and rewrite the second term.

$$= n\Delta - 2E\left[\left(W_{(n-1)\Delta} + (W_{n\Delta} - W_{(n-1)\Delta})\right)W_{(n-1)\Delta}\right] + (n-1)\Delta$$

Expand second term and use independence of non overlapping intervals

$$= (2n-1)\Delta - 2E\left[W^2_{(n-1)\Delta}\right] - 2E[(W_{n\Delta} - W_{(n-1)\Delta})]E[W_{(n-1)\Delta}]$$

The third term above has to be 0 by properties we prescribed on the first moment of W_t. Now just do algebra

$$= (2n-1)\Delta - 2(n-1)\Delta = \Delta$$ [31]

Question

Brownian motion recurrence theorems and Hausdorff Dimension

I need help with proving:

1.If $d > 1$ then d-dimensional Brownian motion starting at x has 0 probability to actually hit y. Note that this is different from the

[31] Since X_t is $N(0,t)$ for every t, one knows that $X_0 = 0$. Since the distribution of $X_t - X_s$ depends only on $t-s$, one knows that, for every n, $X_{n\Delta} - X_{(n-1)\Delta}$ is distributed like $X_\Delta - X_0 = X_\Delta$, which is $N(0,\Delta)$. In particular, $X_{n\Delta} - X_{(n-1)\Delta}$ is centered with variance Δ

usual notion of recurrence, where the brownian motion only has to get arbitrarily close.

2.If $d > 2$ then d-dimensional Brownian motion B_t satisfies $|B_t| \to \infty$ as $t \to \infty$ a.s. For this one, I want a specific proof that my professor alluded to but did not give details on. I already know that if you consider a sequence S_n concentric spheres of radius r^n with $r > 1$, then if x is of norm r^N then the probability of a brownian motion starting at x to hit S_{N-1} before S_{N+1} is $1/(r^{(d-2)}+1) < 1/2$. Now, he suggests that this be related to a random walk with drift. I have trouble making this relationship precise. Please assume that the Brownian motion is defined on some probability space. It is okay to assume that it is everywhere continuous, instead of a.s. continuous, though the greater generality the better. I think the strong Markov property has to be used here to elevate this proof idea from the status of being purely motivational. I don't really need other proofs of 2. There's a proof in Durrett[32] that

[32] **Richard Timothy Durrett** is a mathematician known for his research and books on mathematical probability theory, stochastic processes and their application to mathematical ecology and population genetics.

He received his BS and MS at Emory University in 1972 and 1973 and his Ph.D. at Stanford University in 1976 under advisor Donald Iglehart. From 1985 until 2010 was on the faculty at Cornell University, where his students included Claudia Neuhauser. Since 2010, Durrett has been a professor at Duke University.

He was elected to the United States National Academy of Sciences in 2007. In 2012 he became a fellow of the American Mathematical Society.

Durrett is the founder of the Cornell Probability Summer Schools, and he is still its scientific organizer.

is legitimate as far as I can see. I just want to know how this proof works.

3.Also, these questions have motivated me to ask questions about what paths of Brownian motions look like. For some reason, people seem terribly interested in Hausdorff [33]dimension. If anybody can say a few words about why anybody should care about Hausdorff dimension, or how one usually goes about computing the Hausdorf dimension of a Brownian path, please let me know.

Answer

This question is vastly too broad so let me answer part 1. Consider a Brownian motion starting at $x \neq 0$, then one is interested in whether it hits 0 or not. For every positive r, let T_r denote the first hitting time of the sphere centered at 0 with radius r. An idea is to compute $P_x(T_R \quad T_r)$ for every $r < |x| < R$ by solving explicitely a harmonic problem.

Then one can check that $\lim_{r \to 0} P_x(T_R \quad T_r) = 1$, that is, $P_x(T_R < T_0) = 1$, for every $R > |x|$. Now, $T_R \to +\infty$ almost

[33] In topology and related branches of mathematics, a **Hausdorff space**, **separated space** or **T_2 space** is a topological space in which distinct points have disjoint neighbourhoods. Of the many separation axioms that can be imposed on a topological space, the "Hausdorff condition" (T_2) is the most frequently used and discussed. It implies the uniqueness of limits of sequences, nets, and filters.

Hausdorff spaces are named after Felix Hausdorff, one of the founders of topology. Hausdorff's original definition of a topological space (in 1914) included the Hausdorff condition as an axiom.

surely when $R \to +\infty$ hence $P_x(T_0 = +\infty) = 1$. (Note that this is exactly the procedure suggested by your professor to solve part 2.)

Question

A counter example of Brownian Motion

Here is an example in my textbook to illustrate why we need the continuous sample path in the definition of Brownian motion. Let (B_t) be a Brownian motion and U be a uniform random variable on [0,1] . Define $(\tilde{B}_t)$ by

$$\tilde{B}_t = \begin{cases} B_t & \text{if } t \neq U, \\ 0 & \text{if } t = U. \end{cases}$$

Then it is claimed that $\tilde{B}_t$ has the same finite-dimensional distribution as B_t but with discontinuous sample path if $B(U) \neq 0$. I do not understand this example. To be more specific,

1. What does it mean by $t \neq U$ or $t = U$, please? Here, t is a deterministic time and U is a random variable. How can we talk about whether the two are equal or not?

2. Since I do not get the definition of $\tilde{B}_t$, I cannot see why it has the same finite dimensional distribution and discontinuous sample path. Could anyone explain this with more detail, please? Thank you!

Answer

Re 1., the expanded definition (maybe clearer) of the random process $\tilde{B}$ is that, for every ω in Ω, $\tilde{B}_{U(\omega)}(\omega)=0$ and $\tilde{B}_t(\omega)=B_t(\omega)$ if $t \neq U(\omega)$.

In short,

$$\tilde{B}_t = B_t \mathbf{1}_{U \neq t}.$$

Re 2., the almost sure path discontinuity of $\tilde{B}$ should be obvious.

To wit, the event that $\tilde{B}$ is continuous is $[B_U = 0]$, the notation being again a shortcut, this time to the event

$$[\omega \in \Omega \mid B_{U(\omega)}(\omega)=0].$$

Since $P(B_t = 0)=0$ for every t and U is independent of B, one gets indeed that $P(B_U = 0)=0$.

The equality of the finite marginal distributions of B and $\tilde{B}$ follows from the remark that, for every finite subset T of the time interval, the event

$$A=[\omega \in \Omega \mid \forall t \in T, B_t(\omega)=\tilde{B}_t(\omega)],$$

has full probability, since $P(U=t)=0$ for every fixed t and

$$\Omega - A \subseteq [U \in T] = \bigcup_{t \in T} [U=t]$$

Reference

- Brown, Robert (1828). "A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies." Phil. Mag. 4, 161–173. (PDF version of original paper including a subsequent defense by Brown of his original observations, Additional remarks on active molecules.)
- Cohen, Ruben D. (1986). "Self Similarity in Brownian Motion and Other Ergodic Phenomena", Journal of Chemical Education 63, pp. 933–934.
- Svedberg, T. (1907). Studien zur Lehre von den kolloiden Losungen.
-
- Nelson, Edward, Dynamical Theories of Brownian Motion (1967). (PDF version of this out-of-print book, from the author's webpage.)
- Pearson, K. (1905). The Problem of the Random Walk. Nature. 72, 294.
- Risken H., The Fokker–Planck Equation (Springer, Berlin) 1984
- Arratia, Richard Alejandro (1979-01-01). Coalescing Brownian Motions on the Line. University of Wisconsin--Madison.
- A. Fick, Ueber Diffusion, Pogg. Ann. Phys. Chem. 170 (4. Reihe 94), 59-86 (1855)
- http://math.stackexchange.com
- http://mathworld.wolfram.com
- http://www.cafemath.fr/