MORE BIRTHDAY SURPRISES

MORTON ABRAMSON, York University, Toronto, and W. O. J. MOSER, McGill University, Montreal

There does not seem to appear in the literature a solution to the following generalization of the well-known birthday problem. Suppose a "year" has n days. If $k \geq 1$ is given, and p people are chosen at random, what is the probability $B_k(n, p)$ that the birthdays of every pair are at least k days apart? We obtain the answer as a special case of a more general result. For other birthday problems see [1], [2]. Note that the first and last days of the year are considered to be consecutive, so that, for example, with $n = 365$, December 29 and January 3 are 5 days apart. Of course

$$B_1(365, p) = (365)(364) \cdots (365 - p + 1)/(365)^p$$

is the probability that (in an ordinary year) no two of the people have the same birthday.

The birthdays of the p people are described by a p-permutation

$$(1) \qquad b_1, b_2, \cdots, b_p \qquad b_i \in \{1, 2, \cdots, n\}, \quad b_i \neq b_j \text{ if } i \neq j.$$

We seek the number of permutations (1) satisfying

$$(2) \qquad k \leq |b_i - b_j| \leq r, \qquad 1 \leq i \neq j \leq p,$$

and this is $p!$ times the number of p-choices

$$(3) \qquad 1 \leq x_1 < x_2 < \cdots < x_p \leq n$$

satisfying

$$(4) \qquad k \leq x_{i+1} - x_i, \qquad i = 1, 2, \cdots, p - 1, \ x_p - x_1 \leq r.$$

Corresponding to each p-choice (3) satisfying (4) there is a p-choice (obtained by setting $y_i = x_i - (i-1)(k-1)$, $i = 1, 2, \cdots, p$)

$$(5) \qquad 1 \leq y_1 < y_2 < \cdots < y_p \leq n - (p - 1)(k - 1)$$

satisfying

$$(6) \qquad y_p \leq r + y_1 - (p - 1)(k - 1)$$

(obtained from $x_p - x_1 \leq r$ upon replacing x_p by $y_p + (p-1)(k-1)$ and x_1 by y_1). We count these in the two cases: (I) $y_1 \geq n - r$; (II) $y_1 < n - r$. If (I) holds, (6) is redundant and the p-choices are

$$n - r \leq y_1 < y_2 < \cdots < y_p \leq n - (p - 1)(k - 1);$$

there are

$$(7) \qquad \binom{n - (p - 1)(k - 1) - n + r + 1}{p}$$

of these. If (II) holds, the p-choices satisfy

$$1 \leqq y_1 < n - r, \qquad y_1 < y_2 < \cdots < y_p \leqq r + y_1 - (p - 1)(k - 1);$$

there are

$$(8) \qquad (n - r - 1) \binom{r + y_1 - (p - 1)(k - 1) - y_1}{p - 1}$$

of these. Adding (7) and (8) we have the

$$(9) \qquad \frac{n - (p - 1)(k - n + r)}{p} \binom{r - (k - 1)(p - 1)}{p - 1}$$

p-choices (3) satisfying (4). (This proof has been included to make this note self-contained; the result is contained in [3; formula (37)'] and [4; formula (19) with l' large and $l = n - r$].) Hence,

$$(10) \qquad \frac{p!}{n^p} \cdot \frac{n - (p - 1)(k - n + r)}{p} \binom{r - (k - 1)(p - 1)}{p - 1}$$

is the probability that the birthdays (1) satisfy (2). With $r = n - k$ in (10) we have

$$B_k(n, p) = \frac{(p - 1)!}{n^{p-1}} \binom{n - p(k - 1) - 1}{p - 1}.$$

The values of $1 - B_k(365, p)$ are listed for some values of p and k. For example, $1 - B_2(365, p)$ is the probability that (in an ordinary year) some pair have the same or adjacent birthdays. Letting $s(k)$ denote the smallest p for which $1 - B_k(365, p) \geqq \frac{1}{2}$ we see the surprisingly small number of people for which, with probability $\geqq \frac{1}{2}$, some pair have birthdays less than k days apart.

$$1 - B_k (365, p)$$

p \ k	1	2	3	4	k	$s(k)$
5	.02713	.07971	.13013	.17844	1	23
8	.07433	.20873	.32604	.42812	2	14
9	.09462	.26042	.39901	.51433	3	11
10	.11694	.31472	.47209	.59648	4	9
11	.14114	.37056	.54328	.67210	5	8
12	.16702	.42693	.61090	.73952	6	8
13	.19440	.48287	.67363	.79778	7	7
14	.22310	.53749	.73053	.84663	8	7
21	.44368	.83603	.95537	.98890	9	6
22	.47569	.86378	.96774	.99313	10	6
23	.50729	.88791	.97709	.99586		
24	.53834	.90864	.98401	.99757		

References

1. Robert E. Greenwood, and Arthur Richert, Jr., A birthday holiday problem, J. Combinatorial Theory, 5 (1968) 422–424.

2. M. S. Klamkin and D. J. Newman, Extensions of the birthday surprise, J. Combinatorial Theory, 3 (1967) 279–282.

3. R. Lagrange, Sur les combinaisons d'objets numérotés, Bull. Sci. Math., 87 (1963) 29–42.

4. W. O. J. Moser and Morton Abramson, Enumeration of combinations with restricted differences and cospan, J. Combinatorial Theory, 7 (1969) 162–170.

ON A PROBLEM OF J. C. MOORE

ALEX HELLER, City University of New York

J. C. Moore has proposed an interesting problem concerning the "contractibility" of continuous maps.

If X and Y are topological spaces, a map $f:X\to Y$ is *contractible* if X and Y have contracting homotopies $\xi_t:X\to X$ and $\eta_t:Y\to Y$ (i.e., homotopies of the identity with a constant map) such that $f\xi_t=\eta_t f$ for all t. Thus a contractible map has contractible domain and range. Moore asks whether the converse is true.

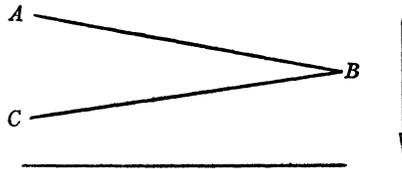The following picture may suggest what is involved:

FIG. 1

The map is the projection indicated by the arrow. It is contractible: AB and CB need only be shrunk simultaneously to B.

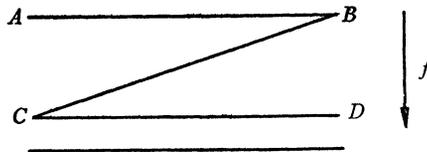Moore further suggests that the map $f:X\to Y$ indicated by the following picture:

FIG. 2

is *not* contractible.

This suggestion seems plausible: it is hard to see how the homotopy ξ_t can deform the image of X away from A and D while B and C are covered. But