Math 308, Midterm 2 Name: ______________________

May 9, 2016

Instructions.

- Vectors are indicated with arrows, as in

  Let $\vec{v}$ be a vector in $\mathbb{R}^n$.

- You are allowed a two-sided sheet of notes in your own handwriting.
- No calculators.
- There are 5 problems on 6 pages. Make sure your exam is complete.
- **Any cheating observed during, or noticed afterwards when comparing exams, will result in a 0 on this exam. Moreover, such an exam cannot be dropped. This means you lose 28% percent of your total raw grade. It will be difficult to pass the class if this occurs.**

Question	Points	Score
1	17	
2	14	
3	6	
4	4	
5	9	
Total:	50	

[3 points] 1. (a) Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Circle which expressions make sense to compute. You do not need to actually compute anything.

(i) $C^T C$ **(ii)** $B^T A$ (iii) $(CA)(AB)$ (iv) A^{-1} (v) $(CA)^{-1}$

[2 points] (b) Suppose that $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = C\vec{x}$. Fill in the blanks:

$$T\colon \mathbb{R}^{--} \to \mathbb{R}^{--}.$$
$$S\colon \mathbb{R}^{--} \to \mathbb{R}^{--}.$$

Solution:

$$T\colon \mathbb{R}^3 \to \mathbb{R}^2.$$
$$S\colon \mathbb{R}^2 \to \mathbb{R}^2.$$

[4 points] (c) Give an example of a linear transformation, T, that is *neither* one-to-one nor onto. Provide two different vectors, $\vec{x}$ and $\vec{y}$ such that $T(\vec{x}) = T(\vec{y}) = \vec{0}$, and a vector not in Range(T).

Solution: The map $T(x) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x}$ has range $\begin{bmatrix} t \\ 0 \end{bmatrix}$ with $t \in \mathbb{R}$. It sends both $\vec{0}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ to $\vec{0}$. The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in the range.

[8 points] (d) Suppose T_1, T_2 and T_3 are linear transformations with matrices A_1, A_2 and A_3. The row reduced forms of these matrices after being augmented by an arbitrary vector in codomain are given below. Circle whether the maps T are one-to-one or onto. Also, describe explicitly the vectors in the range.

(i) $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & x+y \\ 0 & 1 & -1 & -1 & y \end{array}\right)$ **onto** one-to-one

RANGE:

Solution: $\mathbb{R}^2$.

(ii) $\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & x_1 \\ 0 & 1 & -\frac{2}{3} & 0 & -x_1 + \frac{1}{3}x_3 \\ 0 & 0 & 1 & 0 & 2x_1 - x_2 \\ 0 & 0 & 0 & 0 & -\frac{1}{3}x_1 - \frac{4}{3}x_2 + \frac{2}{3}x_3 + x_4 \end{array}\right)$ onto one-to-one

RANGE:

Solution: All $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $0 = \frac{1}{3}x_1 - \frac{4}{3}x_2 + \frac{2}{3}x_3 + x_4$

(iii) $\left(\begin{array}{cc|c} 1 & 8 & x_1 \\ 0 & 1 & \frac{7}{48}x_1 - \frac{1}{48}x_2 \\ 0 & 0 & \frac{7}{48}x_1 - \frac{49}{48}x_2 + x_3 \\ 0 & 0 & \frac{235}{16}x_1 - \frac{237}{16}x_2 + x_4 \end{array}\right)$ onto **one-to-one**

RANGE:

Solution: All $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $0 = \frac{7}{48}x_1 - \frac{49}{48}x_2 + x_3$ and $0 = \frac{235}{16}x_1 - \frac{237}{16}x_2 + x_4$.

2. Let P be the plane $x + y + z = 0$ and Q be the plane $x - y + 2z = 0$. An animator is attempting to find a linear transformation of $\mathbb{R}^3$ that maps P to Q.

[2 points] (a) Is it possible to have a linear transformation $\mathbb{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ that is *not* onto but maps P to Q? Explain with a sentence.

Solution: Yes. There is a linear transformation that maps all of $\mathbb{R}^3$ to the plane Q. For instance, projecting to the $x - y$-plane then rotating to make Q.

[2 points] (b) Let $\vec{u}, \vec{v}$ be two different vectors in P, and $\vec{w}$, $\vec{x}$ be two vectors in Q. If $T(\vec{u}) = \vec{w}$ and $T(\vec{v}) = \vec{x}$, does it always follow that $T(P) = Q$? Explain why or why not.

Solution: Not necessarily. If $\vec{w}$ and $\vec{x}$ are linearly dependent then the image of T won't necessarily be all of Q.

[1 point] (c) It is a fact from Math 126 that $\vec{n} = \langle a, b, c\rangle$ is normal to the plane $ax + by + cz = 0$. Write the normal vectors, $\vec{p}$ and $\vec{q}$ to the planes P and Q, respectively.

Solution: $\vec{p} = \langle 1, 1, 1\rangle$ and $\vec{q} = \langle 1, -1, 2\rangle$

[4 points] (d) Find a one-to-one linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\vec{p}) = \vec{q}$. Explain why it is one-to-one.

Solution: If $T(\vec{x}) = A\vec{x}$ with $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ then we know

$$T(\vec{p}) = A\langle 1, 1, 1\rangle = \vec{a}_1 + \vec{a}_2 + \vec{a}_3 = \langle 1, -1, 2\rangle = \vec{q}.$$

So, the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ will map $\vec{p}$ to $\vec{q}$.

[2 points] (e) Is the linear transformation you wrote in part (d) onto? Explain why or why not.

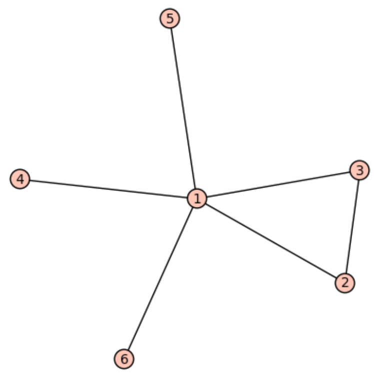
Solution: Yes. Any linear one-to-one linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$ is also onto. This is part of the Big Theorem.

[3 points] (f) Give an example of a linear transformation $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $S(\vec{p}) = \vec{q}$ but S is not one-to-one. Explain why the transformation is not one-to-one.

Solution: Let $S(\vec{x}) = B\vec{x}$ with

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

3. The website www.mathjunge.com has a homepage $\{1\}$ and 5 sub-pages $\{2, 3, 4, 5, 6\}$. They are linked in the following way:



[2 points] (a) Recall that an $n \times n$ adjacency matrix for n nodes has a 1 in the ith row and jth column if nodes i and j are connected. Otherwise it has a 0 in that entry. Write the adjacency matrix, A, for the link structure of these websites.

Solution: $A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

[2 points] (b) Write the entries in the 1st column of A^2.

Solution: $A^2 = \begin{pmatrix} 5 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. It was okay to just give the 1st column.

You could do this either by calculation, or by counting paths of length 2.

[2 points] (c) Below is A^{10}. Explain why the top left entry is the largest.

$$A^{10} = \begin{pmatrix} 5121 & 2573 & 2573 & 1316 & 1316 & 1316 \\ 2573 & 1962 & 1961 & 1257 & 1257 & 1257 \\ 2573 & 1961 & 1962 & 1257 & 1257 & 1257 \\ 1316 & 1257 & 1257 & 869 & 869 & 869 \\ 1316 & 1257 & 1257 & 869 & 869 & 869 \\ 1316 & 1257 & 1257 & 869 & 869 & 869 \end{pmatrix}.$$

Solution: Each entry is the number of paths of length 10 between i and j. Since site 1 is the best connected there are the most paths.

4. An animator wants to smoothly animate a spinning arrow that starts at $\vec{q}$, spins around twice, then ends rotated $\pi/4$ past it's starting point. This should all take 1 second.

[1 point] (a) Rotating twice around, then by $\pi/4$ is equivalent to rotating by what angle? (Hint: what angle corresponds to rotating once around.)

Solution: $4\pi + \pi/4$.

[1 point] (b) A function that smoothly transitions from a to b over one second is $g(t) = bt + (1-t)a$. Write a function that smoothly transitions from 0 to the angle from part (a).

Solution: $f(t) = (4\pi + \pi/4)t$

[2 points] (c) Recall that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates by θ. Write down the rotation matrices $A(t)$ for $0 \le t \le 1$ which accomplish the animator's goal.

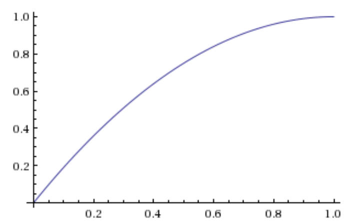
Solution: We want to end at the angle $\phi = 4\pi + \pi/4$. The function $f(t) = \phi t$ has the properties required. So the matrices

$$A(t) = \begin{bmatrix} \cos f(t) & -\sin f(t) \\ \sin f(t) & \cos f(t) \end{bmatrix}$$

animate what we want

(d) For 2 extra credit points, modify your animation so that the vector does the same rotation, but gradually slows down as it spins.

Solution: Instead use the function $g(t) = \phi(-t^2 + 2t)$. This has derivative $-2\phi(t-1)$ which goes to 0 as t goes to 1.

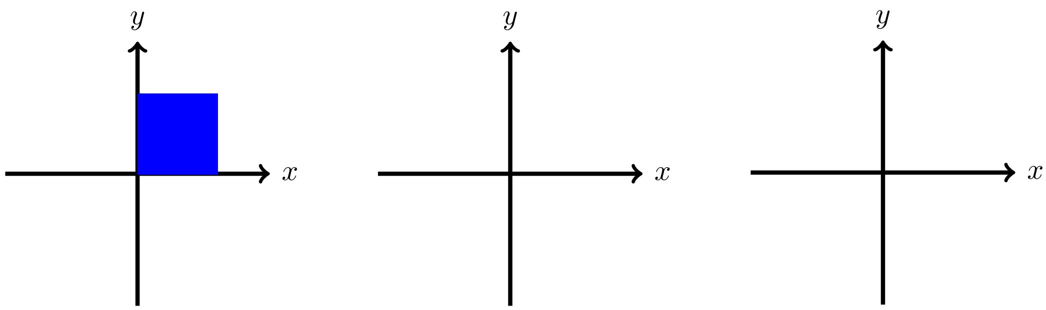


5. Let $f(x) = x + t$ with $x \in \mathbb{R}$ and t a fixed constant.

[2 points] (a) Explain why $f\colon \mathbb{R} \to \mathbb{R}$ is not a linear transformation.

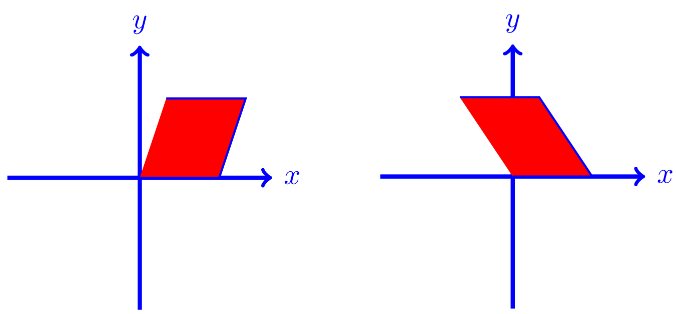
Solution: For fixed $r \in \mathbb{R}$ we have $f(rx) = rx + t \neq rx + tr = rf(x)$. Or you could say that $f(0) \neq 0$ and LT's take 0 to 0.

[3 points] (b) Draw the effect that the linear transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + ty \\ y \end{bmatrix}$ has on the unit square in $\mathbb{R}^2$ when $t = 1$ and when $t = -2$.



$t = 0$ $t = 1$ $t = -2$

Solution:



$t = 1$ $t = -2$

[2 points] (c) Write a formula or a matrix for T^{-1}.

Solution: The matrix for T is $A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. This has inverse:

$$A = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}.$$

[2 points] (d) Write a linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^1$ such that $S\left(\begin{bmatrix} x \\ 1 \end{bmatrix}\right) = x + t$.

Solution: First apply T then apply the map $R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x$. So $R \circ T$ does the trick.