Math 340, Midterm 2 Name: ________________________

April 10, 2018

Instructions.

- You are allowed one side of handwritten notes.
- No calculators.
- The last page is a table for Φ.
- There are 5 problems on 6 pages. Make sure your exam is complete.

Page	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Useful facts

- $e \approx 2.718281828$
- Law of total probability: $P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$ with $B_1, \dots, B_n$ a partition.
- $E[\text{Geo(p)}] = 1/p$.

[6 points] 1. (a) Mattual, a mutual fund, claims they can with probability more than 1/2 beat the average return of the stock market each month. You decide to start monitoring their gains for the next 25 months. How many of these months would Mattual have to outperform the market for you to be 95% certain of their claim?

It is sufficient to call this number x and write an algebraic equation you would solve.

Solution: Let $X = \sum_1^n 25X_i$ be the number of months they outperform the market. We need x so that $P(X \geq x) = .05$. Rescaling this is the same as

$$P(Y \geq \frac{x - .5 - 25/2}{5/2})$$

with $Y = N(0,1)$. We know that $P(Y \geq 1.65) = .05$. Thus, we would solve

$$\frac{x - .5 - 25/2}{5/2} = 1.65.$$

So $x = 18$.

[4 points] (b) Suppose Mattual beats the market 20 of those 25 months. Give a 95% percent confidence interval for their probability p of outperforming the market.

Solution: $\mathbf{E}\hat{p} = p$ and $\operatorname{var}\hat{p} = p(1-p)/n$. We have observed that $\hat{p} = 4/5$, and want to be inside 2 standard deviations of this. Notice that $\sqrt{(4/5)(1/5)/25} = 2/25$

$$[\frac{4}{5} - 2(2/25), \frac{4}{5} + 2(2/25)] = [.64, .96].$$

2. Let $U_1, U_2, \ldots$ be independent uniform$(0,1)$'s. Recall $EU_1 = 1/2$ and $\mathrm{var}(U_1) = 1/12$.

[3 points] (a) Let $S_n = \sum_{i=1}^n U_i = U_1 + \cdots + U_n$ be the sum of the first n. Use normal approximation to estimate $P(S_{100} > 55)$. You may use the approximation $\sqrt{12} \approx 3.5$.

> **Solution:**
> $P(S_{100} > 55) \approx P(N > (55-50)/(10/\sqrt{12}) \approx P(N > 3.5/2) \approx 1 - .9599 = .0401.$

[1 point] (b) Let $R_n = \prod_{i=1}^n U_i = U_1 U_2 \cdots U_n$ be the product of the first n. What is ER_n?

> **Solution:** $ER_n = (1/2)^n = 2^{-n}$.

[4 points] (c) Apply the law of large numbers to $\log R_n$ to find a number $0 < r < \infty$ such that $R_n \sim r^{-n}$. You may use the fact that $\int_0^1 \log x \, dx = -1$.

> **Solution:**
> $$\log R_n = \sum_1^n \log U_i \approx nE \log U_i = -n$$
> So $R_n \approx e^n = e^{-n}$.

[2 points] (d) What is $\lim_{n\to\infty} R_n / ER_n$?

> **Solution:** $R_n/ER_n \approx e^{-n}/2^{-n} \to 0$ since $e > 2$.

3. Three people are taking turns flipping a coin that lands heads with probability q. This first to flip heads wins. Let p be the probability Player 1 wins.

[2 points] (a) Use the law of total probability to write a recursive relationship for p.

Solution: $p = q + (1-q)^3 p$

[2 points] (b) Solve for p to obtain a formula for p in terms of q. Please simplify using the fact that $(1-q)^3 = -q^3 + 3q^2 - 3q + 1$.

Solution:

$$p = \frac{q}{1-(1-q)^3} = \frac{1}{q^2 - 3q + 3}.$$

[2 points] (c) What is p when $q = 1/2$? Explain why this is larger than $1/2$.

Solution: 4/7. It is larger because P1 wins on the first flip with probability 1/2 but could also win if everyone misses.

[4 points] (d) What are $\lim_{q\to 1} p$ and $\lim_{q\to 0} p$? Explain why the numbers you find makes sense.

Solution: 1 and 1/3. Since all players flip the same coin, the first player must always have the largest probability.

4. 10 ducks are flying past 5 photojournalists. Each takes a photo of a uniformly selected duck. The photo is non-blurry with probability 1/3. Let D be the number of distinct ducks with a non-blurry photograph.

[3 points] (a) Find ED.

Solution: The probability duck i is clearly photographed by photographer j is $(1/3)/10 = 1/30$. So the probability duck i has a clear photo taken is $q = 1 - (29/30)^5$. Since $D = \sum_1^{10} D_i$ we have $ED = 10q \approx 1.56$.

[3 points] (b) Suppose that the 10 ducks are comprised of two representatives from 5 different species. (So there are 5 different pairs of ducks). Say the photojournalists successfully photograph 7 **different** ducks selected uniformly at random. Let X be the number of complete pairs that they photograph. What is EX? (It is okay to use p to represent any complicated probabilities.)

Solution: Let X_i be 1 if both ducks from pair i are photographed and 0 otherwise. We have $P(X_1 = 1) = C_{2,2}C_{8,5}/C_{10,7} = 7/15 = p$. It follows that $EX = 5p = 7/3$.

[4 points] (c) Find var(X). Feel free to not simplify and to use variables to represent any complicated expressions you obtain.

Solution: Notice that

$$\text{cov}(X_1 X_2) = C_{4,4}C_{6,2}/C_{10,8} - p^2$$

$\text{var}(X) = 10p(1-p) + 2C_{5,2}\text{cov}(X_1, X_2)$.

5. Al has a coin and Betty has a coin. One lands heads with probability 1/3, the other is fair. Al and Betty are equally likely to have the biased coin. Let A be the event that Al has the biased coin.

[4 points] (a) After they each flip their coins once Al flips tails and Betty flips heads. Call this event B. What is $P(A \mid B)$?

Solution:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} = \frac{(2/3)(1/2)(1/2)}{(1/3)(1/2)(1/2) + (1/2)(2/3)(1/2)}$$

This is equal to 2/3.

[4 points] (b) Al now flips his coin an additional 18 times. Let X be the number of heads he flips. What is $E(X \mid B)$?

Solution: $E(X \mid B) = 18(1/3)P(A \mid B) + 18(1/2)P(A^c \mid B) = 6(2/3) + 9(1/3) = 7.$

[2 points] (c) Suppose C is the event that both Al and Betty initially flip heads. Without doing any calculations find $P(A \mid C)$.

Solution: 1/2. We don't get any information about who has the biased coin from the event C.

Normal Table

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

To illustrate the use of the table: $\Phi(0.36) = 0.6406$, $\Phi(1.34) = 0.9099$

	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9986	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990