San Francisco State University
ECON 715

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Fall 2016

Midterm Exam 2

October 27, 2016

Time: 1 hour, 30 minutes

Name: ______________________________

Instructions:
1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (5 points). Prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

This is a limit of the form $\frac{0}{0}$. Using L'Hôpital's rule,

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$$

2. (10 points). Prove that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

Remark: This is an important result in finance. When periodic interest rate is r, and the interest is compounded n times per period, the amount after one period is: $(1+r/n)^n$. For example, r is usually annual percentage rate (APR), and $n = 12$ if the interest compounds monthly, and $n = 360$ if interest compounds daily, etc. If interest compounds every instant, the amount can be approximated by e^r - continuously compounded formula.

We can rewrite the required function in a way that allows the use of substitution rule and L'Hôpital's rule:

$$\left(1+\frac{r}{n}\right)^n = \exp\left(n\ln\left(1+\frac{r}{n}\right)\right) = \exp\left(r\frac{\ln\left(1+\frac{r}{n}\right)}{r/n}\right) = \exp\left(r\frac{\ln(1+x)}{x}\right)$$

where $x = r/n$. Note that

$$\begin{aligned}\lim_{n\to\infty} x &= 0\\ \lim_{x\to 0}\frac{\ln(1+x)}{x} &= 1 \quad \text{[We proved previously, using L'Hôpital's rule]}\end{aligned}$$

Thus, the required limit can be written as

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{x\to 0}\exp\left(r\frac{\ln(1+x)}{x}\right) = \lim_{y\to 1}\exp(ry) = e^r$$

3. (10 points). Suppose the production function of a firm is $F(K,L) = A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}}$, $\theta \in (0,1)$, $\rho \le 1$, where Y is output, A is productivity parameter, K is physical capital, and L is labor. This is known as the Constant Elasticity of Substitution (CES) production function.

(a) Prove that the above production function has Constant Returns to Scale (CRS).

Let $\lambda > 0$.

$$\begin{aligned} F(\lambda K, \lambda L) &= A[\theta(\lambda K)^\rho + (1-\theta)(\lambda L)^\rho]^{\frac{1}{\rho}} \\ &= A[\lambda^\rho \theta K^\rho + \lambda^\rho (1-\theta) L^\rho]^{\frac{1}{\rho}} \\ &= \lambda A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}} = \lambda F(K,L) \end{aligned}$$

(b) Prove that if this firm pays its inputs their marginal product as wages, then the firm's real profit must be zero ($\pi = F(K,L) - F_K(K,L)\cdot K - F_L(K,L)\cdot L = 0$). Precisely state any theorem used in your proof, and explain how it applies to the given question.

Direct proof:

$$\begin{aligned} & F_K(K,L)\cdot K + F_L(K,L)\cdot L \\ = \; & \frac{1}{\rho} A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}-1} \rho\theta K^{\rho-1} K + \frac{1}{\rho} A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}-1} \rho(1-\theta)L^{\rho-1} L \\ = \; & A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}-1} \theta K^\rho + A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}-1} (1-\theta)L^\rho \\ = \; & A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}-1} [\theta K^\rho + (1-\theta)L^\rho] \\ = \; & A[\theta K^\rho + (1-\theta)L^\rho]^{\frac{1}{\rho}} = F(K,L) \end{aligned}$$

Thus,

$$\pi = F(K,L) - F_K(K,L)\cdot K - F_L(K,L)\cdot L = 0$$

Proof based on Euler theorem for homogeneous functions:

Theorem: Let $F(K,L)$ be homogeneous function of degree n. Then $nF(K,L) = F_K(K,L)\cdot K + F_L(K,L)\cdot L$. Since Constant Returns to Scale means homogeneity of degree 1, the required result follows from Euler's theorem for $n = 1$.

4. (10 points). Suppose a firm has a total cost function of $TC(Q)$, where Q is output level. Let the average total cost function be $ATC(Q) = TC(Q)/Q$, and the marginal cost function $MC(Q) = TC'(Q)$. Prove that $ATC(Q)$ is increasing in Q if and only if $MC(Q) > ATC(Q)$.

$$\frac{d}{dQ}ATC(Q) = \frac{d}{dQ}\left(\frac{TC(Q)}{Q}\right) = \frac{TC'(Q)\,Q - TC(Q)}{Q^2} = \frac{TC'(Q) - TC(Q)/Q}{Q}$$

Thus

$$\begin{aligned} \frac{d}{dQ}ATC(Q) &> 0 \\ &\iff \underbrace{TC'(Q)}_{MC(Q)} - \underbrace{TC(Q)/Q}_{ATC(Q)} > 0 \\ &\iff MC(Q) - ATC(Q) > 0 \end{aligned}$$

The last inequality is the required result: $MC(Q) > ATC(Q)$.

5. (15 points). Suppose that the demand function for some good is given by $x = Ap_x^{-1.5}p_y^{0.5}I$, where p_x is the price of x, p_y is the price of some other good y, and I is buyers' income.

(a) Based on the given information, x is a gross complement/substitute/unrelated to good y. Circle the correct answer and provide a brief explanation.

The demand for x is increasing in p_y, that is, when y is more expensive, buyers substitute it with good x.

(b) Calculate the price elasticity of demand for x, the cross price elasticity of demand for x with respect to p_y, and income elasticity of demand for x. Denote these elasticities with η_{x,p_x}, η_{x,p_y} and $\eta_{x,I}$.

Taking logs of the demand function

$$\ln x = \ln A - 1.5 \ln p_x + 0.5 \ln p_y + \ln I$$

The required elasticities are:

$$\begin{aligned} \eta_{x,p_x} &= \frac{\partial \ln x}{\partial \ln p_x} = -1.5 \\ \eta_{x,p_y} &= \frac{\partial \ln x}{\partial \ln p_y} = 0.5 \\ \eta_{x,I} &= \frac{\partial \ln x}{\partial \ln I} = 1 \end{aligned}$$

(c) Calculate the approximate % change in the demand for x as a result of a 1% simultaneous increase in p_x, p_y and I.

$$\begin{aligned} \%\Delta Q &= \eta_{x,p_x} \cdot \%\Delta p_x + \eta_{x,p_y} \cdot \%\Delta p_y + \eta_{x,I} \cdot \%\Delta I \\ &= -1.5 \cdot 1\% + 0.5 \cdot 1\% + 1 \cdot 1\% = 0 \end{aligned}$$

Remark: This has to be the case with any demand function. When prices and income increase by the same proportion, the budget constraint does not change, and the demand does not change as well. We therefore say that demand (for any good) is always homogeneous of degree zero.

6. (10 points). Given the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

(a) Prove that

$$\lim_{\sigma \to 1} u(c) = \ln(c)$$

Since the above limit has $\frac{0}{0}$ form (i.e. the numerator and the denominator have limit of 0 as $\sigma \to 1$), we can apply L'Hôpital's rule:

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1-\sigma} = \lim_{\sigma \to 1} \frac{-\ln(c)\, c^{1-\sigma}}{-1} = \ln(c)$$

Here we used the rule of derivatives $(c^x)' = \ln(c)\, c^x$, together with the chain rule.

(b) Calculate the Arrow-Pratt measure of Relative Risk Aversion (RRA).

$$RRA = -\frac{u''(c)}{u'(c)} c = -\frac{-\sigma c^{-\sigma-1}}{c^{-\sigma}} c = \sigma$$

7. (15 points). Let the utility function be $u(x,y) = [\alpha x^{\sigma} + (1-\alpha) y^{\sigma}]^{\frac{1}{\sigma}}$, $\sigma \leq 1$.

(a) Write the equation of a generic indifference curve with utility level $\bar{u}$, and find its slope dy/dx.

Indifference curve:

$$[\alpha x^{\sigma} + (1-\alpha) y^{\sigma}]^{\frac{1}{\sigma}} = \bar{u}$$

Note, the above is implicit function. Using the implicit function theorem:

$$\begin{aligned}
\frac{dy}{dx} u(x,y) &= -\frac{u_x(x,y)}{u_y(x,y)} = -\frac{\frac{1}{\sigma}[\alpha x^{\sigma} + (1-\alpha) y^{\sigma}]^{\frac{1}{\sigma}-1} \sigma \alpha x^{\sigma-1}}{\frac{1}{\sigma}[\alpha x^{\sigma} + (1-\alpha) y^{\sigma}]^{\frac{1}{\sigma}-1} \sigma (1-\alpha) y^{\sigma-1}} \\
&= -\frac{\alpha x^{\sigma-1}}{(1-\alpha) y^{\sigma-1}} = -\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}
\end{aligned}$$

(b) Find the marginal rate of substitution between goods x and y ($MRS_{x,y}$).

The marginal rate of substitution is the absolute value of the slope of indifference curves:

$$MRS_{x,y} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}$$

(c) Find the elasticity of substitution between goods x and y ($ES_{x,y}$).

$$\begin{aligned}
\frac{1}{ES_{x,y}} &= \frac{\partial MRS_{x,y}}{\partial (y/x)} \cdot \frac{y/x}{MRS_{x,y}} = \left(\frac{\alpha}{1-\alpha}\right)(1-\sigma)\left(\frac{y}{x}\right)^{-\sigma} \cdot \frac{y/x}{\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{y}{x}\right)^{1-\sigma}} = 1-\sigma \\
ES_{x,y} &= \frac{1}{1-\sigma}
\end{aligned}$$

8. (15 points). Consider the following system of nonlinear equations:

$$\begin{aligned} x + 2y + a &= 5 \\ 3x^2 ya &= 12 \end{aligned}$$

The endogenous variables are x and y, while a is an exogenous variables (or parameter).

(a) Suppose that initially, $(x, y, a) = (2, 1, 1)$. Verify that this point is a solution to the given system of equations.

$$\begin{aligned} 2 + 2 \cdot 1 + 1 &= 5 \\ 3 \cdot 2^2 \cdot 1 \cdot 1 &= 12 \end{aligned}$$

(b) Derive the Jacobian matrix, and calculate the Jacobian determinant at the initial point.

The given system can be written as:

$$\begin{aligned} F_1(x, y, a) &= 5 \\ F_2(x, y, a) &= 12 \end{aligned}$$

The Jacobian matrix is (matrix of partial derivatives with respect to the endogenous variables):

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 \cdot 2 \cdot xya & 3x^2 a \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 12 & 12 \end{bmatrix}$$

Jacobian determinant:

$$|J| = 1 \cdot 12 - 2 \cdot 12 = -12 \neq 0$$

(c) Suppose the exogenous parameter changes: $da = 0.03$. Calculate the approximate changes dx, dy, assuming the same initial point $(x, y, a) = (2, 1, 1)$.

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = -J^{-1} \begin{bmatrix} \frac{\partial F_1}{\partial a} \\ \frac{\partial F_2}{\partial a} \end{bmatrix} da$$

The inverse of the Jacobian, evaluated at the given point:

$$J^{-1} = \frac{adj\,(J)}{|J|} = -\frac{1}{12} \begin{bmatrix} 12 & -2 \\ -12 & 1 \end{bmatrix}$$

The partial derivatives of the system with respect to a, evaluated at the given point:

$$\begin{bmatrix} \frac{\partial F_1}{\partial a} \\ \frac{\partial F_2}{\partial a} \end{bmatrix} = \begin{bmatrix} 1 \\ 3x^2 y \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \end{bmatrix}$$

Thus,

$$\begin{aligned} \begin{bmatrix} dx \\ dy \end{bmatrix} &= \frac{1}{12} \begin{bmatrix} 12 & -2 \\ -12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 12 \end{bmatrix} 0.03 = \frac{1}{12} \begin{bmatrix} 12 - 2 \cdot 12 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} 0.03 = \begin{bmatrix} -0.03 \\ 0 \end{bmatrix} \end{aligned}$$

The approximate change in x is $dx = -0.03$, while the approximate change in y is $dy = 0$.

Remark: when solving the system numerically with $a = 1.03$ (using Matlab), the precise changes in endogenous variables are

$$\begin{aligned} \Delta x &= -0.02870131644... \\ \Delta y &= -0.00064934178... \end{aligned}$$

demonstrating that the approximation above was pretty close.

9. (10 points). Briefly explain what the following Matlab commands are doing.

(a) `z = linspace(-10,10,101)`

Creating evenly spaced grid (named `z`) of 101 points between -10 and 10.

(b) `syms x y`

Declaring symbolic variables x and y.

(c) `limit(log(1+x)/x,x,0)`

Finding the limit

$$\lim_{x \to 0} \frac{\ln(1+x)}{x}$$

(d) `diff(x^a*y^b,y)`

Calculating the partial derivative

$$\frac{\partial}{\partial y} x^a y^b$$

(e) `diff(sqrt(x),x,2)`

Calculating the second order derivative

$$\frac{d^2}{dx^2} \sqrt{x}$$