San Francisco State University
ECON 715

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Final Exam

December 17, 2019

Time: 120 minutes

Name: ______________________________

Instructions:

1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (25 points). Suppose you own a bottle of rare wine, the value of which is estimated t years from now according to the function

$$V(t) = Ke^{f(t)}$$

where $f(0) = 0$, $f'(t) > 0\ \forall t$, and $f''(t) < 0\ \forall t$.

(a) Derive the growth rate of the value of your wine, as a function of t.

The growth rate is:

$$\frac{V'(t)}{V(t)} = \frac{Ke^{f(t)} f'(t)}{Ke^{f(t)}} = f'(t)$$

(b) Suppose that you want to sell the wine when its present value is maximized. Write the optimization problem of maximizing the present value of the wine, and derive the first order necessary condition for optimum.

Optimization problem is:

$$\max_t PV(t) = V(t)\, e^{-rt}$$

First order necessary condition for maximum is:

$$\begin{aligned} \frac{d}{dt} PV(t) &= V'(t)\, e^{-rt} - rV(t)\, e^{-rt} = 0 \\ &\Rightarrow V'(t) - rV(t) = 0 \end{aligned}$$

(c) Provide economic intuition of the first order necessary condition from the previous section.

The first order condition can be written as

$$\frac{V'(t)}{V(t)} = f'(t) = r$$

This means that the owner should keep the asset until the growth rate of its value equalizes to the interest rate.

(d) Assuming that the first order necessary condition characterizes a unique global maximum, prove that higher interest rate r, shortens the optimal holding time.

Write the F.O.N.C. as implicit function:

$$F(t, r) = f'(t) - r = 0$$

Then, by the implicit function theorem:

$$\frac{dt}{dr} = -\frac{F_r}{F_t} = -\frac{-1}{f''(t)} = \frac{1}{f''(t)} < 0$$

Notice that the last inequality holds because it is given that $f''(t) < 0 \ \forall t$.

(e) Provide economic intuition for the above result (higher interest rate implies shorter holding time).

The interest rate is the opportunity cost of holding the asset for additional time. The owner can sell the asset, and invest the money at interest rate r. Thus, higher interest rate, means that the opportunity cost of holding the asset is higher, and the seller would like to sell it earlier.

2. (25 points). Consider a monopoly that sells a single product to n segmented markets. The revenue function in market i, as a function of quantity of product sold in that market, is $R_i(Q_i)$, with $R_i'(Q_i) > 0$ and $R_i''(Q_i) < 0$. Assume that the total cost function, $C(Q) = C\left(\sum_{i=1}^{n} Q_i\right)$, is increasing and strictly convex.

   (a) Write the optimization problem of the monopoly, and derive the first order necessary condition.

   Profit maximization problem:

$$\max_{Q_1,...,Q_n} \pi(Q_1, ..., Q_n) = \sum_{i=1}^{n} R_i(Q_i) - C(Q)$$

   First order necessary condition:

$$\frac{\partial}{\partial Q_i}\pi(Q_1, ..., Q_n) = R_i'(Q_i) - C'(Q) = 0 \quad \forall i = 1, ..., n$$

   (b) Provide economic interpretation of the first order necessary condition from the previous section.

   The above condition states that the Marginal Revenue (MR) in all markets must be equal to the common marginal cost.

(c) Show that the monopoly's selling price in market i is equal to the markup rate μ_i above the marginal cost, i.e.

$$p_i = [1 + \mu_i]\, MC$$

where the markup rate is $\mu_i = 1/\left(|\eta_i| - 1\right)$, and η_i is the price elasticity of demand in market i.

The revenue in any market is $R(Q) = P(Q)\,Q$ (here we omit the market subscript i for ease of notation). Thus, the Marginal Revenue can be expressed as a function of price elasticity of demand:

$$MR = \frac{d}{dQ}R(Q) = \frac{dP}{dQ} \cdot Q + P = P\left[1 + \frac{dP}{dQ}\frac{Q}{P}\right] = P\left[1 + \frac{1}{\eta}\right] = P\left[1 - \frac{1}{|\eta|}\right]$$

Using the $MR = MC$ profit maximization condition:

$$\begin{aligned} P\left[1 - \frac{1}{|\eta|}\right] &= MC \\ P &= \left[\frac{|\eta|}{|\eta| - 1}\right] MC = \left[1 + \frac{|\eta|}{|\eta| - 1} - 1\right] MC \end{aligned}$$

Thus, the selling price in market i is

$$P_i = \left[1 + \frac{1}{|\eta_i| - 1}\right] MC = [1 + \mu_i]\, MC$$

(d) Suppose that Genentech (a pharmaceutical company) sells its medicine to India and U.S., and price elasticities of demand in the two countries are $\eta_{India} = -11$, $\eta_{US} = -1.1$. Find the markup rate (in % above the marginal cost) that Genentech will charge in the two countries.

$$\begin{aligned}
\mu_{India} &= \frac{1}{|\eta_{India}| - 1} = \frac{1}{11 - 1} = 0.1 = 10\% \\
\mu_{US} &= \frac{1}{|\eta_{US}| - 1} = \frac{1}{1.1 - 1} = 10 = 1000\%
\end{aligned}$$

Thus, the company will charge 1000% markup in the U.S. and 10% markup in India.

(e) Prove that the critical value of the profit function is the unique global maximum. Clearly state the theorems used in your proof.

Proof. It is given that $R_i(Q_i)$ is strictly concave $\forall i$, and therefore $\sum_{i=1}^{n} R_i(Q_i)$ is strictly concave (sum of strictly concave functions is str. concave). The cost function is given to be strictly convex, so $-C(Q)$ is strictly concave (f is concave if and only is $-f$ is convex, strict or not). Thus, the profit function is a sum of strictly concave functions and therefore strictly concave. Consequently, the critical point of a strictly concave function is automatically a unique global maximum.

3. (10 points). Prove that any Cobb-Douglas function $u(x_1, ..., x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$, with $\alpha_i > 0 \; \forall i = 1, ..., n$, is **strictly quasiconcave**. Clearly state the theorems used in your proof.

The given function can be written as

$$u(x_1, ...x_n) = \exp\left(\sum_{i=1}^{n} \alpha_i \ln x_i\right)$$

The $\ln(\cdot)$ function is strictly concave ($\frac{d}{dx}\ln(x) = x^{-1}$, $\frac{d^2}{dx^2}\ln(x) = -x^{-2} < 0$), the weighted sum of strictly concave functions $\sum_{i=1}^{n} \alpha_i \ln x_i$ is strictly concave. The exponential function $\exp(\cdot)$ is monotone increasing, and since monotone increasing transformation of strictly concave function is strictly quasiconcave, $u(x_1, ...x_n)$ is strictly quasiconcave.

4. (10 points). Consider the consumer's problem:

$$\max_{x,y} u(x,y) \quad s.t. \quad p_x x + p_y y = I$$

where x and y are quantities consumed of two goods, p_x and p_y are prices, and I is income. Let $V(p_x, p_y, I)$ be the maximum value function associated with this problem (also known as indirect utility function). That is, letting the solution to the problem be $x(p_x, p_y, I)$ and $y(p_x, p_y, I)$, then we define: $V(p_x, p_y, I) \equiv u(x(p_x, p_y, I), y(p_x, p_y, I))$. Using the envelope theorem (**do NOT prove it here**), prove that

$$x = -\frac{\left(\frac{\partial V}{\partial p_x}\right)}{\left(\frac{\partial V}{\partial I}\right)}$$

Clearly state the theorems used in your proof.

The Lagrange function associated with the optimization problem is:

$$\mathcal{L} = u(x,y) - \lambda\left[p_x x + p_y y - I\right]$$

According to the envelope theorem,

$$\frac{\partial V}{\partial p_x} = \frac{\partial \mathcal{L}}{\partial p_x} \quad \text{and} \quad \frac{\partial V}{\partial I} = \frac{\partial \mathcal{L}}{\partial I}$$

where the partial derivatives are taken, ignoring the impact of parameters on the choice variables x and y. That is, x and y can be treated as constant. Thus,

$$\begin{aligned} \frac{\partial V}{\partial p_x} &= \frac{\partial \mathcal{L}}{\partial p_x} = -\lambda x \\ \frac{\partial V}{\partial I} &= \frac{\partial \mathcal{L}}{\partial I} = \lambda \end{aligned}$$

Diving the first by the second, gives the required result (known as Roy's identity):

$$\begin{aligned} \frac{\left(\frac{\partial V}{\partial p_x}\right)}{\left(\frac{\partial V}{\partial I}\right)} &= \frac{-\lambda x}{\lambda} \\ &\Rightarrow \quad x = -\frac{\left(\frac{\partial V}{\partial p_x}\right)}{\left(\frac{\partial V}{\partial I}\right)} \end{aligned}$$

5. (20 points). Suppose Oscar has wealth w and his preferences over risky alternatives are described by Expected Utility Theory, and his utility over certain outcomes is represented by $u(\cdot)$, which is strictly increasing and strictly concave. He can divide his wealth between investment in risky asset, with random return r, and a risk-free asset with guaranteed return r_f. Let the amount invested in risky asset be $x \in [0, w]$.

   (a) Write Oscar's optimal investment problem.

   The amount x is invested in risky asset with return r and the rest $w - x$ is invested in risk-free asset with return r_f. Thus, the future wealth is $x(1+r) + (w-x)(1+r_f) = w(1+r_f) + x(r - r_f)$. The optimal investment problem is therefore:

$$\max_{0 \le x \le w} E\left[u\left(w\left(1+r_f\right) + x\left(r - r_f\right)\right)\right]$$

   (b) Write the first order necessary condition for interior optimum, x^*.

   The first order condition for interior optimum, x^*, is

$$E\left[u'\left(w\left(1+r_f\right) + x^*\left(r - r_f\right)\right)\left(r - r_f\right)\right] = 0$$

(c) Write the second order sufficient condition for global maximum and prove that it is satisfied in this problem.

$$E\left[u''\left(w\left(1+r_f\right)+x\left(r-r_f\right)\right)\left(r-r_f\right)^2\right]<0,\ \forall x$$

The above condition holds for all x since it is given that Oscar is risk-averse ($u(\cdot)$ is strictly concave).

(d) Prove that Oscar will invest positive amount in risky asset ($x^* > 0$) if and only if the expected return on the risky asset is greater than the risk-free return: $E(r) > r_f$.

The condition of positive investment in risky asset, $x^* > 0$, is (slope of utility at $x = 0$ is positive):

$$\begin{aligned} E\left[u'\left(w\left(1+r_f\right)\right)\left(r-r_f\right)\right] &> 0 \\ u'\left(w\left(1+r_f\right)\right)E\left(r-r_f\right) &> 0 \end{aligned}$$

Which holds if and only if

$$E(r) > r_f$$

6. (10 points). Consider the Matlab script below:

```
1
2 -    syms x y p_x p_y I k a
3 -    u(x,y) = a*log(x) + (1-a)*log(y);
4 -    L(x,y,k) = u(x,y) - k*(p_x*x + p_y*y - I);
5 -    [x,y,k] = solve(gradient(L,[x,y,k])==0, [x,y,k]);
6 -    d = subs([x,y],[a,p_x,p_y,I],[0.5,3,0.75,100]);
```

(a) What is the purpose of the entire program?

The program defines a lagrange function corresponding to a consumer utility maximization problem, and solves for the demand.

(b) What is the purpose of the command in line 2?

Declaring symbolic variables.

(c) What is the role of the variable `k` in the above program?

Lagrange multiplier.

(d) What does the function `gradient` in line 5 do?

Differentiates the Lagrange function with respect to `x,y` and `k`, i.e. obtaining the First Order Necessary Conditions for the optimization problem.

(e) What is the purpose of line 6? In particular, what is the purpose of the numerical values in line 6?

The command in line 6 substitutes the numerical values of the parameters into the symbolic solution, to get the numerical demand and lagrange multiplier. Here `a` is the weight on `x` in the utility function, so the utility is $u(x, y) = 0.5 \ln x + 0.5 \ln y$. The values 3,0.75 are the prices p_x, p_y and consumer's income is $I = 100$.