Math 308: Midterm 2 Review

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1 Matrices

1.1 Linear Transformations

1. Let A be an $n \times m$ matrix and T the linear transformation given by $T(\vec{x}) = A\vec{x}$.
   (a) T is a linear transformation between which two spaces?

   Solution: From $\mathbb{R}^m$ to $\mathbb{R}^n$.

   (b) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}$, then is T one-to-one?

   Solution: The two column vectors are not scalars of each other, and so are linearly independent. It follows that T is one-to-one.

   (c) Using the same A as above, is T onto? What is the range of T?

   Solution: We set up and reduce the system corresponding to the span of our two column vectors.

$$\begin{bmatrix} 1 & 2 & a \\ 3 & 0 & b \\ 1 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & a \\ 0 & -6 & b-3a \\ 0 & -1 & c-a \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & a \\ 0 & -6 & b-3a \\ 0 & 0 & 3a+b-6c \end{bmatrix}$$

   So our range is all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $0 = 3a + b - 6c$.

2. Using two linear transformations, rotate and scale the vector $\vec{e}_1 \in \mathbb{R}^2$ so that it is mapped to the vector $\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$. Write a collection of matrices $C(t)$ that would animate this smoothly.

   Solution: We need to rotate by $\pi/4$. This is done by the matrix

$$A = \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}.$$

   This gives the vector with size 1 and angle $\pi/4$. That is $\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$. We need to multiple this by 2. So, we use the linear transformation

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The product AB is the matrix

$$AB = \begin{bmatrix} 2\cos\pi/4 & -2\sin\pi/4 \\ 2\sin\pi/4 & 2\cos\pi/4 \end{bmatrix}.$$

We can smoothly animate this from the identity matrix with the collection of matrices:

$$C(t) = \begin{bmatrix} (1-t)1 + 2t\cos\pi/4 & -2t\sin\pi/4 \\ (1-t) + 2t\sin\pi/4 & 2t\cos\pi/4 \end{bmatrix}.$$

3. Give an example of a linear transformation from $\mathbb{R}^4 \to \mathbb{R}^3$ that is onto but not one to one.

Solution: The map $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 4 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vec{x}$. Has two linearly dependent vectors, so must send infinitely many vectors to $\vec{0}$.

4. Find a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Decide if T is one to one or onto.

Solution: Well, $T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We need

$$\begin{bmatrix} a \\ c \end{bmatrix} + 2\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$2\begin{bmatrix} a \\ c \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

This reduces to the system

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \end{array}\right)$$

Which has solution:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right).$$

So, our matrix is:

$$A = \frac{1}{3}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

5. Explain why translation is not a linear transformation, and why by going up a dimension it can be.

Solution: A translation does not map 0 to 0. In one higher dimension (say $\mathbb{R}^{n+1}$) it is possible to embed $\mathbb{R}^n$ so it isn't touching 0. We can then use a linear transformation that shifts this space by the desired amount, thus shifting our vectors in $\mathbb{R}^n$.

6. The map $T(\vec{x}) = \vec{b} \times \vec{x}$ has matrix $B = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$. When augmented with $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and row reduced by a compute we obtain:

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{b_1}{b_3} & \frac{y}{b_3} \\ 0 & 1 & -\frac{b_2}{b_3} & -\frac{x}{b_3} \\ 0 & 0 & 0 & \frac{b_1 x}{b_3} + \frac{b_2 y}{b_3} + z \end{array}\right)$$

What is the span of this linear transformation? Describe why this makes sense geometrically. (*Hint: the computer didn't do anything wrong, but row reduced to something correct but a little misleading. Can you correct it?*

Solution: The span is the plane $0 = \frac{b_1 x}{b_3} + \frac{b_2 y}{b_3} + z$. It is the plane with normal vector $\vec{b}$ that goes through $\vec{0}$. The computer should have multiplied everything by b_3 so that we have the plane $\vec{b} \cdot \vec{x} = 0$.

1.2 Matrix Algebra

7. Give an example of two 2×2 matrices A and B such that $AB \neq BA$.

Solution: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ don't. The top left entry is a 3 for one direction and a 5 for the other.

8. Explain why the following is true, or why it is false:

Most linear transformations $S, T \colon \mathbb{R}^2 \to \mathbb{R}^2$ have the property that $T \circ S = S \circ T$.

Solution: It is false. Linear transformations having this property is equivalent to two matrices commuting with multiplication. This is a fairly stringent condition, for which most matrices will fail. Of course, one that there are infinitely many linear transformations with this property and so most do. However, if one randomly picks two maps, it is rather unlikely.

9. Explain why the following is true, or why it is false:

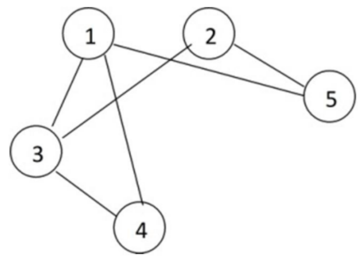
Most linear transformations $S, T \colon \mathbb{R}^2 \to \mathbb{R}^2$ are invertible.

Solution: It is true. This is the same as saying most matrices are invertible. Think about the 2×2 case. It is very unlikely you will choose two vectors on the same line, or in the $\mathbb{R}^3$ case it is very unlikely to choose 3 vectors on the same plane.

10. Give an example of two 3×3 matrices that aren't I and do commute.

Solution: Any two rotation or dilation matrices will work.

11. Write the adjacency matrix for this graph, and without doing any arithmetic write out the adjacency matrix squared.



Solution:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 3 & 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 2 \end{pmatrix}$$

The square is done by counting paths.

1.3 Inverses

12. Explain why a non-square matrix cannot have an inverse.

Solution: An inverse has the property that $AA^{-1} = A^{-1}A = I$, but the two sided multiplication only makes sense for square matrices.

13. What is the inverse of $\begin{bmatrix} 3\cos\theta & -3\sin\theta \\ 3\sin\theta & 3\cos\theta \end{bmatrix}$.

Solution: Take $\frac{1}{3}A_{-\theta}$ with A_θ the rotation matrix.

14. Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I$ but $BA \neq I$.

Solution: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$

15. Suppose that A is an $n \times n$ matrix and that $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$. Does this mean A is invertible?

Solution: Yes. The columns are linearly independent. Big Theorem.