1 Displacement vs Total vs Net

(i) **Net Displacement:** $\int f(t)dt$. Describes where we end up. Can be negative.

(ii) **Total Displacement:** $\left|\int f(t)dt\right|$. Describes the end change in distance. Always nonnegative.

(iii) **Total Change:** $\int |f(t)|dt$. Describes the total distance traveled. Always nonnegative.

These relate to each other in the following way

$$\int f(t)dt \leq \left|\int f(t)dt\right| \leq \int |f(t)|dt.$$

I definitely explained this incorrectly in class, but this is correct.

1.1 Computing $\int |f(t)|dt$.

Say we want to compute

$$\int_a^b |f(t)|dt.$$

We can't just ignore the absolute value signs. Recall that $|f(t)| = \begin{cases} f(t), & f(t) \geq 0 \\ -f(t), & f(t) < 0 \end{cases}$. We can use this to compute the integral by following these steps.

1. Find where f is zero. Call these points $z_1, \dots, z_n$.
2. Figure out if f is positive or negative on each interval $(a, z_1), (z_1, z_2), \dots, (z_i, z_{i+1}), \dots, (z_n, b)$.
3. Wherever f changes sign split up the integral on that interval. We will end up with something like
$$\int_a^b |f(t)|dt = \int_a^{z_1} |f(t)|dt + \int_{z_1}^{z_2} |f(t)|dt + \cdots + \int_{z_n}^b |f(t)|dt.$$
4. Now use the definition of absolute value to get rid of the absolute value signs. On intervals where f is positive we can ignore them and on intervals where f is negative we multiply by (-1). For example,
$$\int_a^b |f(t)|dt = \int_a^{z_1} f(t)dt + \int_{z_1}^{z_2} -f(t)dt + \cdots + \int_{z_n}^b f(t)dt.$$
5. Since we have eliminated the absolute value signs we can now compute each integral without peril and add up to get our answer.

2 $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt.$

Letting F be the antiderivative of f so that $F'(x) = f(x)$ we have the formula

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x).$$

To derive this formula we use the fundamental theorem of calculus. Define $G(x) = \int_{g(x)}^{h(x)} f(t)dt$. It follows from the FTC that

$$G(x) = F(h(x)) - F(g(x)).$$

Taking the derivative and using the chain rule gives

$$G'(x) = F'(h(x))h'(x) - F'(g(x))g'(x).$$

But $F'(x) = f(x)$. So we know that $F'(h(x)) = f(h(x))$ and $F'(g(x)) = f(g(x))$.