Student Name : ______________________ Student Number: ____________

MIDTERM TWO

READ THE DIRECTIONS

ONCE YOU START, MAKE SURE YOUR EXAM HAS 5 PAGES (including the coverpage)

Page 1	14	
Page 2	13	
Page 3	10	
Page 4	13	
Total	**50**	

- Show all work for full credit.
- BOX in your answer to each question.
- Unless otherwise indicated always use EXACT numbers. (i.e $\sqrt{\pi}$ instead of 1.77).
- You may use a scientific calculator during this examination; graphing calculators and other electronic devices are not allowed and should be turned off for the duration of the exam.
- If you use trial-and-error, a guess-and-check method, or numerical approximation when an exact method is available, you will not receive full credit.
- You may use one double-sided, hand-written, 8.5 by 11 inch page of notes.
- You have 60 minutes to complete the exam - so, spend on average $\leq$ 15 minutes per page.

1. Take the derivative of each function.

   (a) (3 points) $f(x) = \sin(\cos(\tan(e^x)))$

   $$f'(x) = \cos(\cos(\tan(e^x)))(-\sin(\tan(e^x)))(\sec^2(e^x))(e^x)$$

   (b) (4 points) $g(x) = \ln(x)^{\ln(x)}$

   $$g'(x) = \ln(x)^{\ln(x)}\left(\frac{1}{x}\ln(\ln x) + \ln(x)\frac{1}{x\ln x}\right)$$

   (c) (3 points) $h(x) = \arctan(1/x) + \arctan(x)$

   $$\frac{1}{1+(1/x)^2}(-\frac{1}{x^2}) + \frac{1}{1+x^2}$$

   (d) (4 points) $F(x) = \dfrac{(x+1)(x+2)^2(x+3)^3(x+4)^4}{\pi^x}$. *If you prefer, you can write $F(x)$ in your solution.* Take ln of both sides, simplify the right hand side using properties of ln then take derivative and multiple $F(x)$ to other side.

   $$F'(x) = F(x)\left(\frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3} + \frac{4}{x+4} + \right)$$

2. A sharknado whirls through the city with parametric equations for $0 \le t \le 2\pi$.

$$x(t) = t + \sin t$$
$$y(t) = t - \sin t$$

(a) (3 points) Find all times when the tangent line is horizontal.
Solve $y'(t) = 1 - \cos t = 0$.

$$t = 0, 2\pi$$

(b) (5 points) Find the three times $0 \le t \le 2\pi$ when the speed of the sharknado is equal to 2.
Speed has formula

$$\begin{aligned} S(t) &= \sqrt{x'(t)^2 + y'(t)^2} \\ &= \sqrt{(1+\cos t)^2 + (1-\cos t)^2} \\ &= \sqrt{1 + 2\cos t + \cos^2 t + 1 - 2\cos t + \cos^2 t} \\ &= \sqrt{2 + 2\cos^2 t}. \end{aligned}$$

Solve $2 = \sqrt{2 + 2\cos^2 t}$ by squaring both sides and simplifying to get

$$1 = \cos^2 t.$$

This has solutions

$$t = 0, \pi, 2\pi$$

(c) (5 points) A news van drives in the city with the sharknado and has coordinates $(x(t), y(t)) = (1 + t, 1 + t)$. Find a time $0 \le t \le \pi$ when the sharknado is moving directly towards the van.
The sharknado is moving in the direction of its tangent line. So we need to find a time when the tangent line to the sharknado goes through the point $(1 + t, 1 + t)$. We can write the equation of the sharknado tangent as

$$y - (t - \sin t) = \frac{1 - \cos t}{1 + \cos t}(x - (t + \sin t)).$$

We want to go through $y = 1 + t$ and $x = 1 + t$, so plug in and solve.

$$\begin{aligned} 1 + t - t - \sin t &= \frac{1-\cos t}{1+\cos t}(1 + t - t - \sin t)) \\ 1 - \sin t &= \frac{1-\cos t}{1+\cos t}(1 - \sin t)) \\ (1 - \sin t)(1 + \cos t) &= (1 - \cos t)(1 - \sin t) \\ 1 - \sin t\cos t - \sin t + \cos t &= 1 - \cos t - \sin t + \sin t \cos t \\ 2\cos t &= -2\sin t \\ \cos t &= -\sin t. \end{aligned}$$

So $t = \frac{3\pi}{4}$. *Note that the algebra was only worth 1 point. Setting up the correct equation was worth the other 4 points.*

3. Let C be the curve defined by the relation

$$xy^2 + 2x^2y - 2 = x.$$

(a) (2 points) Find all y such that the point $(1, y)$ is on the curve C.
Plug in $x = 1$ to the curve and solve $y^2 + 2y - 2 = 1$. This has solutions

$$y = 1, y = -3$$

(b) (5 points) Find the equation of the tangent line at each point from part (a).
Differentiate the relation with $\frac{d}{dx}$.

$$\begin{aligned} y^2 + 2yxy' + 4xy + 2x^2y' &= 1 \\ 2yxy' + 2x^2y' &= -y^2 - 4xy \\ y' &= \frac{-y^2 - 4xy}{2xy + 2x^y} \end{aligned}$$

Next, plug in the points $(1, 1)$ and $(1, -3)$ to get two slopes.

$$y'(1, 1) = -1, \text{ and } y'(1, -3) = -1.$$

Then use point-slope form with the correct points.

$$y - 1 = -(x - 1) \qquad \text{and} \qquad y + 3 = -(x - 1)$$

(c) (3 points) Using tangent line approximation, approximate a number b such that the point $(b, \frac{9}{10})$ is on the curve C.
Plug in $9/10$ for y in the tangent line based at $(1, 1)$. So solve

$$\frac{9}{10} - 1 = -(x - 1)$$

for x to get

$$\frac{11}{10} = x$$

4. (8 points) A circular puddle is evaporating so that its circumfrence is changing at B cm/sec with B an unknown constant. Find the rate B so that the area of the puddle is shrinking at 1 cm^2/sec when the circumference is 10π.

There are lots of ways to solve this. Here is the quickest. We know that $A = \pi r^2$ and $C = 2\pi r$. We can solve $r = \frac{C}{2\pi}$ and plug this into the area formula to write

$$A = \pi \left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}.$$

Differentiate with respect to time to get

$$\frac{dA}{dt} = \frac{2C}{4\pi}\frac{dC}{dt}.$$

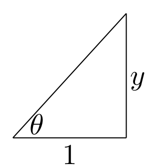
Plug in $C = 10\pi$ and $\frac{dA}{dt} = -1$ to get

$$-1 = \frac{2(10\pi)}{4\pi}\frac{dC}{dt}.$$

This has solution $\frac{dC}{dt} = -1/5$ and so

$$B = -\frac{1}{5}\ cm/s$$

5. (4 points) In a right triangle with fixed base 1. The height of the other leg is given by $\theta = \arctan(y)$. Use tangent line approximation at $y = 1$ to estimate θ when $y = 1.1$.



Differentiate $\theta(y) = \arctan(y)$ with respect to y to get

$$\theta'(y) = \frac{1}{1+y^2}.$$

Since $\theta(1) = \pi/4$ and $\theta'(1) = \frac{1}{2}$, the tangent line at $y = 1$ has equation

$$\theta - \pi/4 = \frac{1}{2}(y-1).$$

Plug in $y = 1.1$ to get

$$\theta = .05 + \pi/4.$$