1. Let $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Explain why the vector $A_\theta \begin{bmatrix} x \\ y \end{bmatrix}$ is a rotation of $\begin{bmatrix} x \\ y \end{bmatrix}$ by θ. Justify this with trigonometry. Hint, one way goes by converting to polar coordinates then converting back. Two useful identities are the ones for $\cos(u+v)$ and $\sin(u+v)$.

   Solution:

   $$A_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x\cos(\theta) - y\sin(\theta) \\ y\cos(\theta) + x\sin(\theta) \end{pmatrix}.$$

   The vector $\begin{bmatrix} x \\ y \end{bmatrix}$ has length $r_0 = \sqrt{x^2+y^2}$. So, we can represent the vector (x, y) in polar coordinates by the points $(r_0, \arctan(y/x)$. To rotate in polar we simply add θ to the angle. So the rotated point is $(r_0, \arctan(y/x)+\theta)$. Now we convert back to standard coordinates (x_θ, y_θ). The new x-coordinate is $x_\theta = r_0\cos(\arctan(y/x)+\theta)$. Using the sum formula: $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$ we can write

   $$\begin{aligned} x_\theta &= r_0\cos(\arctan(y/x)+\theta) \\ &= r_0(\cos(\arctan(y/x))\cos(\theta) - \sin(\arctan(y/x))\sin(\theta). \end{aligned}$$

   This isn't so bad, because $\cos(\arctan(y/x)) = x/r_0$ and $\sin(\arctan(y/x)) = y/r_0$. We can then write the above as

   $$x_\theta = r_0((x/r_0)\cos(\theta) - (y/r_0)\sin(\theta)) = x\cos(\theta) - y\sin(\theta).$$

   This is the answer we get from applying A_θ to $\begin{bmatrix} x \\ y \end{bmatrix}$. The conversion for y_θ is similar.

   (b) For any $\theta, \psi \in \mathbb{R}$, show that $A_\theta A_\psi = A_{\theta+\psi}$ and explain what this means geometrically.

   Solution:

   $$\begin{aligned} A_\theta A_\psi &= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\psi)\cos(\theta) - \sin(\psi)\sin(\theta) & -\cos(\theta)\sin(\psi) - \cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\psi) + \cos(\psi)\sin(\theta) & \cos(\psi)\cos(\theta) - \sin(\psi)\sin(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{pmatrix}. \end{aligned}$$

   The last line follows from using summation identities with sin and cos.

   (c) What is the inverse matrix of A_θ? Explain why.

   Solution: The inverse of A_θ is $A_{-\theta}$ because this will rotate any vector back to its original position. Also, from part (b) we see that $A_\theta A_{-\theta} = A_0 = \mathbf{I}$.

2. Ada is learning about moments of inertia in her physics class and sees an opportunity to put her linear algebra skills to the test. Given a vector $\vec{b} \in \mathbb{R}^3$, she wants to construct the inertia matrix B that when multiplied by any vector $\vec{x}$ yields the cross product $\vec{b} \times \vec{x}$.

(a) If

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

then what is $\vec{b} \times \vec{x}$?

Solution:

$$\vec{b} \times \vec{x} = \begin{bmatrix} b_2x_3 - b_3x_2 \\ -b_1x_3 + b_3x_1 \\ b_1x_2 - b_2x_1 \end{bmatrix}.$$

(b) Use your answer in part (a) to deduce B. The entries should only involve b_i.

Solution:

$$B = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

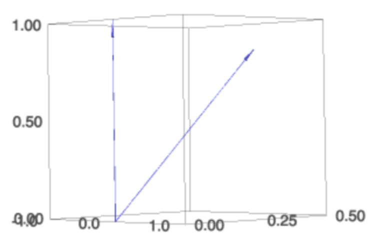
(c) Is the linear transformation defined by multiplication by B a bijection? Make sense of your answer based on what you know about cross products.

Solution: We observe that $B\vec{b} = \vec{0}$, so the transformation is not one-to-one and therefore not a bijection. This makes sense because the cross product will produce a vector in the unique span of the normal vectors to $\vec{b}$ and $\vec{x}$. But when the two vectors are parallel, there are infinitely many directions normal to the pair of vectors, and the cross product will just be $\vec{0}$.

3. Erika, a computer animator, wants to create a scene where a character waves gracefully back and forth in three dimensions. She knows that the rotation matrices about the x,y and z axes are given by:

$$A_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad A_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad A_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) She wants the vector $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to rotate with respect to the x-axis to make an angle of $\pi/3$ radians with the y-axis. Write down the matrix, A, that causes this transformation. (Be careful!)



Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\pi/6) & -\sin(-\pi/6) \\ 0 & \sin(\pi/6) & \cos(-\pi/6) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}$$

(b) What is $A\vec{e}_3$?

Solution: $A\vec{e}_3 = \begin{bmatrix} 0 \\ -\sin(-\pi/6) \\ \cos(-\pi/6) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{bmatrix}$

(c) Describe in words what A^6 does to a vector.

Solution: This will rotate a vector $(-6\pi/6) = -\pi$ around the x-axis.

(d) She also wants the hand to rotate a little around the y axis. She would like $A\vec{e}_3$ to rotate by $\pi/4$ with respect to the y-axis. Write the matrix, B, that corresponds to this transformation.

Solution:

$$B = \begin{bmatrix} \cos(\pi/4) & 0 & -\sin(\pi/4) \\ 0 & 1 & 0 \\ \sin(\pi/4) & 0 & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$$

(e) What is $B(A\vec{e}_3)$?

Solution:

$$\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\sqrt{3}}{2}\begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/4 \\ 1/2 \\ \sqrt{6}/4 \end{bmatrix}.$$

(f) Explain why $B(A\vec{e}_3) = (BA)\vec{e}_3$. Do this without doing any calculations.

Solution: BA represents the matrix that simultaneously does both B and A. Since these are just rotations, it doesn't change the answer to do both simultaneously.

(g) Now she wants to animate this smoothly. She calculates that

$$BA = \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{2}\sqrt{3} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{3}\sqrt{2} & -\frac{1}{2} & \frac{1}{4}\sqrt{3}\sqrt{2} \end{pmatrix}.$$

Suppose she would like the wave to take 1 second and appear smooth. Erika knows that the function $f(t) = (1-t)a + bt$ is such that $f(0) = a$ and $f(1) = b$ and it smoothly interpolates between a and b as t ranges from 0 to 1.

Using this, write a collection of matrices $C(t)$ such that $C(0) = \mathbf{I}$ and $C(1) = BA$.

Solution:

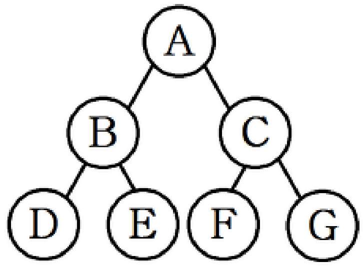
$$C(t) = \begin{pmatrix} (1-t)1 + \frac{1}{2}\sqrt{2}t & 0 & -\frac{1}{2}\sqrt{2}t \\ \frac{1}{4}\sqrt{2}t & (1-t)1 + \frac{1}{2}\sqrt{3}t & \frac{1}{4}\sqrt{2}t \\ \frac{1}{4}\sqrt{3}\sqrt{2}t & -\frac{1}{2}t & (1-t)1 + \frac{1}{4}\sqrt{3}\sqrt{2}t \end{pmatrix}.$$

(h) What would the collection $C(t)^{-1}$ animate?

Solution:

It would reverse the wave back to the vector $\vec{e}_3$.

4. Label the binary tree with 7 nodes as follows:



(a) Write the adjacency matrix, Q, for a binary tree with 7 nodes.

Solution:

$$Q = \begin{pmatrix} & A & B & D & E & C & F & G \\ A & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ B & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ D & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ C & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ F & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ G & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(b) Now suppose that a random walker moved randomly from node to node with equal probability. Write the transition probability matrix, P.

Solution:

$$P = \begin{pmatrix} & A & B & D & E & C & F & G \\ A & 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ B & 1/3 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ D & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ C & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\ F & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ G & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(c) What does the entry in Row A and Column F of Q^{10} represent?

Solution: The number of paths of length 10 that start at A and end at F.

(d) Here is P^{10}.

$$P^{10} = \begin{pmatrix} A & B & D & E & C & F & G \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{275}{486} & 0 & 0 & \frac{211}{486} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{97}{486} & \frac{97}{486} & 0 & \frac{65}{486} & \frac{65}{486} \\ \frac{1}{3} & 0 & \frac{97}{486} & \frac{97}{486} & 0 & \frac{65}{486} & \frac{65}{486} \\ 0 & \frac{211}{486} & 0 & 0 & \frac{275}{486} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{65}{486} & \frac{65}{486} & 0 & \frac{97}{486} & \frac{97}{486} \\ \frac{1}{3} & 0 & \frac{65}{486} & \frac{65}{486} & 0 & \frac{97}{486} & \frac{97}{486} \end{pmatrix}.$$

Where is a random walk started at C most likely to be after 10 steps? Explain why.

Solution: It is most likely to be back at C because $275/486 > 211/486$. These two numbers represent the probability C is at C or at B after 10 steps. All other places are impossible to reach in 10 steps.