Neoclassical Growth Model

Harvard Economics 1011B
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Overview of growth and income differences

- Kaldor facts.
- Solow model.
  - Growth from capital accumulation and exogenous technology.
- **Neoclassical growth model.**
  - Growth from equilibrium capital accumulation and exogenous technology.
  - Efficiency result.
- Confronting neoclassical growth theory with evidence.
- Other and deeper theories of cross-country growth differences.
- Growth over time.
- Cross-country welfare differences beyond GDP.

Outline

1. Overview of neoclassical growth model
2. Setup
3. Equilibrium
4. Analysis

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Overview

- Same setup as Solow, except endogenous consumption, investment.
- For simplicity, abstract from population and technology growth: $n = g = 0$.
- Competitive equilibrium.
- Planner's problem.

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Household

- Solve:

$$
\begin{aligned}
& \max_{c_t, a_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & a_{t+1} = (1+r_t)a_t + w_t + \Pi_t - c_t, \\
\text{initial wealth} \quad & a_0.
\end{aligned}
$$

- Present value Lagrangian:
  $\mathscr{L} = \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + \lambda_t \left((1+r_t)\, a_t + w_t + \Pi_t - (a_{t+1} + c_t)\right)\right).$
- FOC:

$$
\begin{aligned}
c_t: \quad & u'(c_t) = \lambda_t, \\
a_{t+1}: \quad & \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1}(1 + r_{t+1}).
\end{aligned}
$$

- Euler equation:

$$
u'(c_t) = \beta(1 + r_{t+1}) u'(c_{t+1}).
$$

Production firm

- Firm hires labor at wage w_t and rents capital at rate r_t^K:
$$\Pi_t^F = \max_{K_t, L_t} F(K_t, L_t) - w_t L_t - r_t^K K_t.$$
- FOC:
$$\begin{aligned} K_t: &\quad F_{K_t} = r_t^K. \\ L_t: &\quad F_{L_t} = w_t. \end{aligned}$$
- Equilibrium profits:
$$\begin{aligned} \text{CRS:} &\quad F(\lambda K, \lambda L) = \lambda F(K, L), \\ \text{Differentiate w.r.t } \lambda: &\quad F_K(\lambda K, \lambda L) K + F_L(\lambda K, \lambda L) L = F(K, L), \\ \lambda = 1: &\quad F_K K + F_L L = F(K, L), \\ \text{FOC:} &\quad r^K K + wL = F(K, L). \end{aligned}$$
- Intuition: if $\Pi_t^F < 0$, firm would shut down. If $\Pi_t^F > 0$, competitor would enter and compete away profits.

Investment firm

- An investment firm chooses investment I_t each period, taking as given K_0.
- Marginal revenue from last dollar invested: $r^K_{t+1} + 1 - \delta$.
- Alternative return in savings account: $1 + r_{t+1}$.
- Optimum: $r_{t+1} = r^K_{t+1} - \delta$.
- We will derive this rigorously later in the course.

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Definition

A competitive equilibrium consists of an allocation $\{c_t, I_t, L_t, K_{t+1}\}_{t=0}^{\infty}$ and prices $\{w_t, r_t^K, r_t\}_{t=0}^{\infty}$ such that:

1. $\{c_t\}_{t=0}^{\infty}$ solves the households problem, taking prices as given.
2. L_t, K_t solve the firm's problem for every t, taking prices as given.
3. I_t solves the capitalist's problem for every t, taking prices as given.
4. Goods markets clear: $F(K_t, L_t) = c_t + I_t \ \forall t$.
5. Labor market clears: $L_t = 1 \ \forall t$.
6. Capital obeys the law of motion: $K_{t+1} = (1-\delta)K_t + I_t$.

Outline

Difference equations

1. Consumption growth, $u(c) = c^{1-\frac{1}{\sigma}}/(1-\frac{1}{\sigma})$:

$$\begin{aligned} \text{Euler:} \qquad & c_t^{-\frac{1}{\sigma}} = \beta(1+r_{t+1})c_{t+1}^{-\frac{1}{\sigma}}, \\ r_t = r_t^K - \delta = F_{K,t} - \delta: \qquad & c_t^{-\frac{1}{\sigma}} = \beta(1+F_{K,t+1}-\delta)c_{t+1}^{-\frac{1}{\sigma}}, \\ \text{Rearrange:} \qquad & c_{t+1} = [\beta\,(1+F_{K,t+1}-\delta)]^{\sigma}\,c_t. \end{aligned}$$

2. Goods market clearing and capital law of motion:

$$K_{t+1} = (1-\delta)K_t + F(K_t,1) - c_t.$$

Two *difference equations* in K and c. Given c_t, K_t, they characterize c_{t+1}, K_{t+1}:

$$\begin{aligned} c_{t+1} &= [\beta\,(1+F_{K,t+1}-\delta)]^{\sigma}\,c_t, \\ K_{t+1} &= (1-\delta)K_t + F(K_t,1) - c_t. \end{aligned}$$

Balanced growth path

$$
\begin{aligned}
c_{t+1} &= [\beta(1 + F_{K,t+1} - \delta)]^{\sigma} c_t, \\
K_{t+1} &= (1-\delta)K_t + F(K_t, 1) - c_t.
\end{aligned}
$$

- BGP requires constant c and K (why?):

$$
\begin{aligned}
c_{t+1} = c_t \Rightarrow \qquad & \beta^{-1} = 1 + F_{K,bgp} - \delta, \\
F_{K,bgp} = \alpha K^{\alpha-1}: \qquad & K_{bgp} = \left(\frac{\alpha}{\beta^{-1} - 1 + \delta}\right)^{\frac{1}{1-\alpha}}, \\
& c_{bgp} = K_{bgp}^{\alpha} - \delta K_{bgp}.
\end{aligned}
$$

- The saving rate along the BGP is:

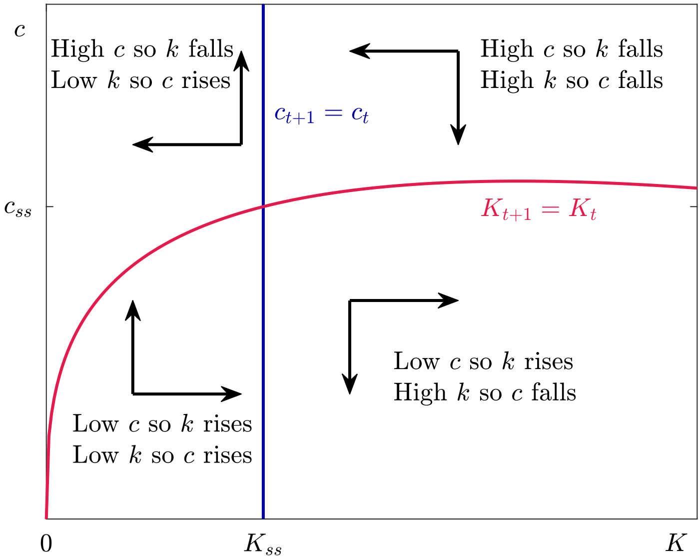
$$
c_{bgp} = (1 - s_{bgp})K_{bgp}^{\alpha} \Rightarrow s_{bgp} = \delta K_{bgp}^{1-\alpha} = \frac{\delta\alpha}{\beta^{-1} - 1 + \delta}.
$$

Dynamics

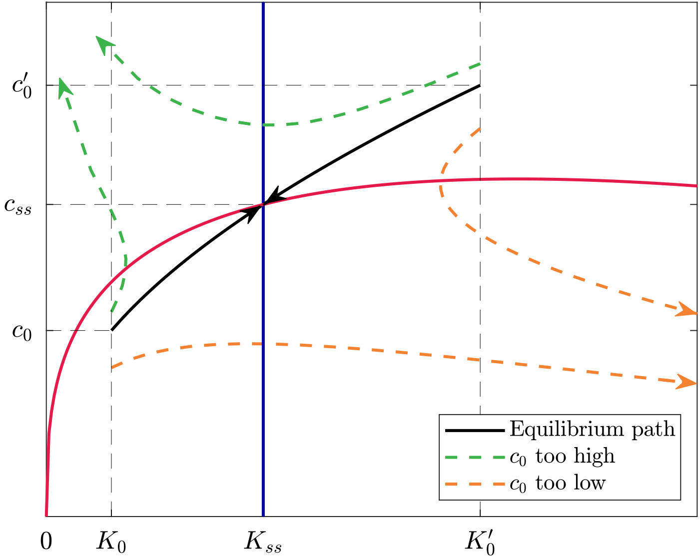
$$c_{t+1} = [\beta(1 + F_{K,t+1} - \delta)]^{\sigma} c_t,$$
$$K_{t+1} = (1-\delta)K_t + F(K_t, 1) - c_t.$$

- $K_{t+1} < K_{bgp} \Rightarrow F_{K,t+1} > F_{K,bgp} \Rightarrow c_{t+1} > c_t$.
  - In (c, K) space, vertical line at K_{bgp} determines dynamics of c.
- $c_t < F(K_t, 1) - \delta K_t \Rightarrow K_{t+1} > K_t$.
  - In (c, K) space, curve $F(K) - \delta K$ determines dynamics of K.
  - Let $g(K) = F(K) - \delta K$, with $g'(K) = F'(K) - \delta,\ g''(K) = F''(K) < 0$.
  - $g(K)$ is inverse U, maximized at $\alpha K^{\alpha-1} = F'(K) = \delta$.

Graphical representation of dynamics



Initial conditions and saddle path



Golden rule revisited

- Recall K_{gr} maximizes consumption along BGP.
- With $n = g = 0$:

$$K_{gr} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

- Therefore, $K_{bgp} \neq K_{gr}$ unless $\beta = 1$.
- For $\beta < 1, K_{bgp} < K_{gr}$.

Is the competitive equilibrium optimal?

Planner's problem

- Suppose instead all agents delegate decisions to a benevolent "social planner" who solves:

$$
\begin{aligned}
& \max_{\{c_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & K_{t+1} = (1-\delta)K_t + F(K_t, 1) - c_t \ \forall t, \\
\text{initial capital} \quad & K_0.
\end{aligned}
$$

- Same objective function as household and same physical constraint.
- Present value Lagrangian:
$\mathscr{L} = \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + \lambda_t \left(K_{t+1} - ((1-\delta)K_t + F(K_t, 1) - c_t)\right)\right)$.
- FOC:

$$
\begin{aligned}
c_t : \quad & u'(c_t) = \lambda_t, \\
K_{t+1} : \quad & \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} (1 + F_{K_{t+1}} - \delta).
\end{aligned}
$$

- Combine and $u(c) = c^{1-\frac{1}{\sigma}} / (1 - \frac{1}{\sigma})$:

$$
c_t^{-\frac{1}{\sigma}} = \beta(1 + F_{K,t+1} - \delta) c_{t+1}^{-\frac{1}{\sigma}}.
$$

Comparison

- In both competitive equilibrium and planner's problem, we obtain:
$$c_t^{-\frac{1}{\sigma}} = \beta(1 + F_{K,t+1} - \delta)c_{t+1}^{-\frac{1}{\sigma}}.$$

- The constraint on the planner's problem coincides with the second dynamic equation of the competitive equilibrium.

- We have proved the competitive equilibrium implements the social optimum.

Intuition

- Economy faces no distortions (e.g. monopoly power), no externalities, and no inequality.
- Therefore, the prices faced by the household and firm perfectly mirror the technology constraints faced by the planner.
- Competitive equilibrium does not maximize BGP consumption, unless household is perfectly patient. This should make sense!
- We will spend a lot of time on departures from this result. But important to keep benchmark in mind.