Solow Model

Harvard Economics 1011B
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Overview of growth and income differences

- **Kaldor facts.**
- **Solow model.**
  - Growth from capital accumulation and exogenous technology.
- Neoclassical growth model.
  - Growth from equilibrium capital accumulation and exogenous technology.
  - Efficiency result.
- Confronting neoclassical growth theory with evidence.
- Other and deeper theories of cross-country growth differences.
- Growth over time.
- Cross-country welfare differences beyond GDP.

Outline

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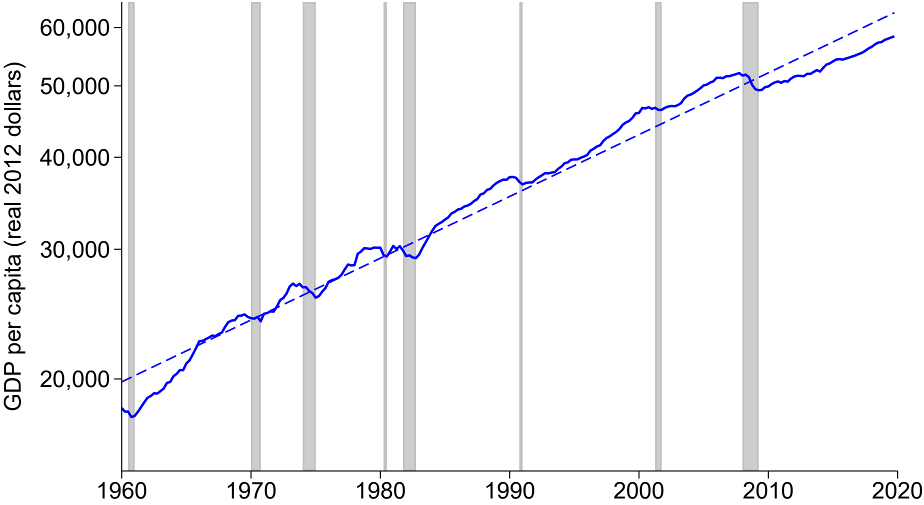
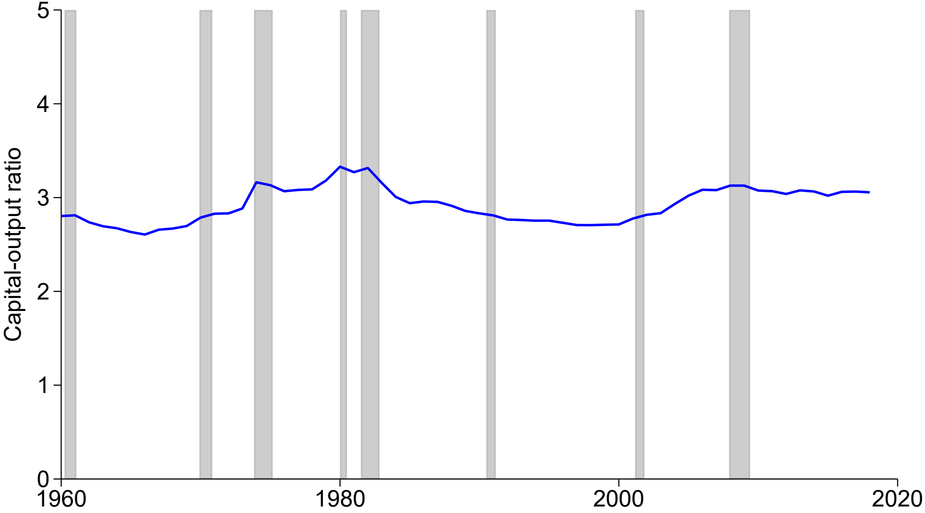
Stylize facts

1. Rate of growth of GDP per capita is stable over the long-run.
2. Capital to output ratio is stable over the long-run.
3. Labor share of income and capital share of income are constant.

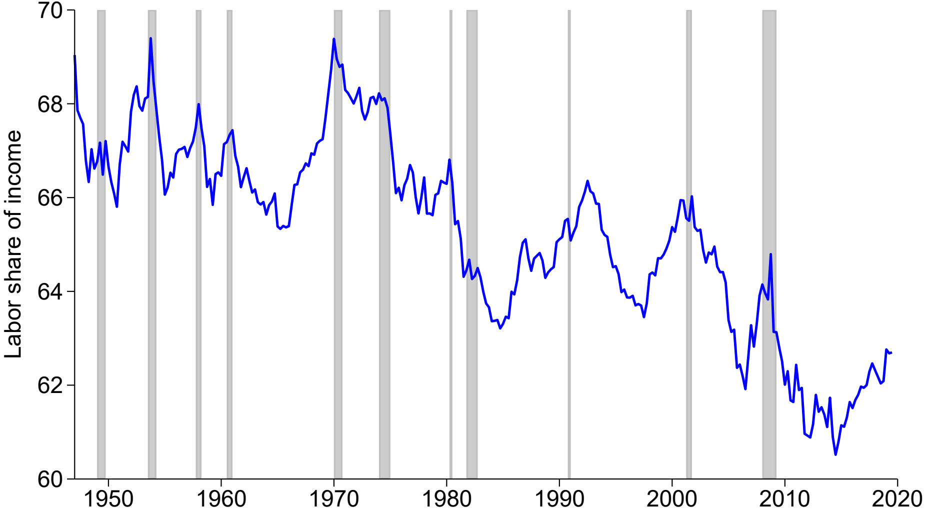
Stable rate of growth



Stable capital-output ratio



STABLE LABOR SHARE



Outline

Production function

- *Aggregate production function*: $Y = F(K, AL)$.
  - Y: output.
  - K: capital used in production.
  - L: labor used in production.
  - A: Technology.

- Constant returns to scale: $F(\lambda K, \lambda AL) = \lambda F(K, AL)$.
- Positive marginal products: $\frac{\partial F(K,AL)}{\partial K} \equiv F_K > 0, \frac{\partial F(K,AL)}{\partial L} \equiv F_L > 0$.
- Diminishing marginal products: $\frac{\partial^2 F(K,AL)}{\partial K^2} \equiv F_{KK} < 0, F_{LL} < 0$.
- Inada conditions: $\lim_{K\to 0} F_K = \infty, \lim_{K\to\infty} F_K = 0$.
- Leading example (Cobb-Douglas):
  $F(K, AL) = K^{\alpha}(AL)^{1-\alpha},\ 0 \leq \alpha \leq 1$ (verify).
- Soon we will introduce t subscripts to index time.
- *Labor-augmenting* technology.

INPUTS

$$\begin{aligned}
\text{Exogenous labor:} \quad & L_{t+1} = (1+n)L_t, \\
\text{Exogenous technology:} \quad & A_{t+1} = (1+g)A_t, \\
\text{Capital accumulation:} \quad & K_{t+1} = (1-\delta)K_t + I_t, \\
\text{Market clearing:} \quad & I_t = sY_t \qquad\qquad \text{why?}
\end{aligned}$$

- Gross savings $sY(t)$= Gross investment $I(t)$. Net investment $K_{t+1} - K_t$ subtracts depreciation δK_t.

- Constant gross savings rate s is key assumption. In neoclassical model, s determined by intertemporal consumption choice.

Intensive form of production function

- Define $y = Y/(AL), k = K/(AL), f(k) = F(k,1)$.
- Then: $y = Y/(AL) = F(K,AL)/(AL) = (AL)F(k,1)/(AL) = f(k)$.
- Partial derivatives:

$$f'(k) = \frac{df(k)}{dk} = \frac{\partial F(k,1)}{\partial k} = \frac{\partial F(K,AL)}{\partial K} > 0,$$
$$f''(k) < 0.$$

- Cobb-Douglas:

$$f(k) = F(k,1) = k^{\alpha}(1)^{1-\alpha} = k^{\alpha},$$
$$f'(k) = \alpha k^{\alpha-1} > 0,$$
$$f''(k) = -(1-\alpha)\alpha k^{\alpha-2} < 0,$$
$$\lim_{k\to 0} f'(k) = \infty,$$
$$\lim_{k\to \infty} f'(k) = 0.$$

- "Detrending trick" to make model easier to analyze.
- Caution: Kurlat uses lower case to denote per capita and $\tilde{x}$ to denote per efficiency unit.

Dynamics of k

$$
\begin{aligned}
\text{Definition:} \qquad \Delta k_{t+1} \equiv k_{t+1} - k_t &= \frac{K_{t+1}}{A_{t+1}L_{t+1}} - k_t \\
\text{Cap. accum.} \qquad &= \frac{(1-\delta)K_t + I_t}{A_{t+1}L_{t+1}} - k_t \\
\text{Algebra \& } (I = sy)\text{:} \qquad &= \frac{(1-\delta)K_t + sY_t}{A_t L_t} \frac{A_t L_t}{A_{t+1}L_{t+1}} - k_t \\
\text{Simplify:} \qquad &= \frac{(1-\delta)k_t + sy_t}{(1+n)(1+g)} - k_t \\
\text{Algebra \& prod fxn:} \qquad &= \frac{sf(k_t) - (n+g+\delta+ng)k_t}{(1+n)(1+g)} \\
\text{Approximate:} \qquad &\approx \frac{sf(k_t) - (n+g+\delta)k_t}{1+n+g}.
\end{aligned}
$$

- Key equation of the model. It describes how capital per effective worker — and hence output per worker — evolve.

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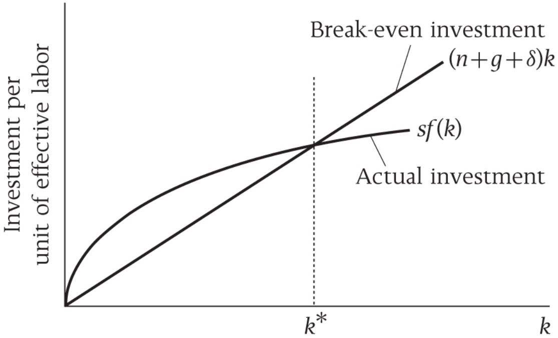
Graphical representation



Balanced growth path: $sf(k^*(t)) = (n+g+\delta)k^*(t)$.

Balanced growth path

- Definition: along a *balanced growth path* (BGP), Y, K, L grow at constant rate.

- Claim: along BGP: y, k are constant (why?).

- Per capita growth rate: $y_t = y = \frac{Y}{AL} \Rightarrow \frac{Y_{t+1}}{L_{t+1}} / \frac{Y_t}{L_t} = A_{t+1}/A_t = 1 + g$.

- Level: $\Delta k_t = 0 \Rightarrow sf(k_{bgp}) = (n + g + \delta)k_{bgp}$.

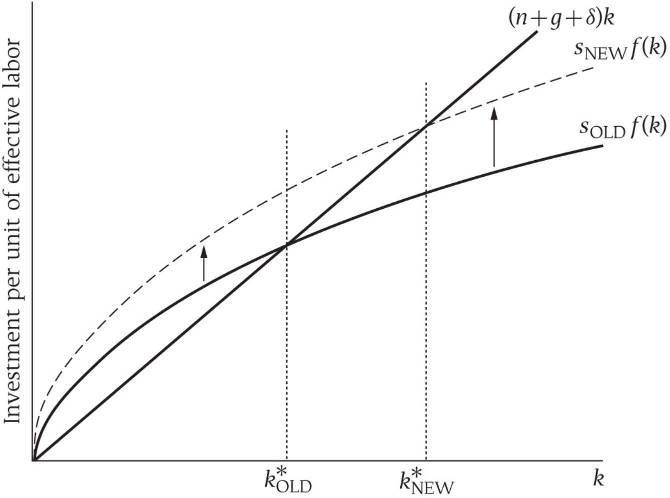
- Cobb-Douglas:

$$
\begin{aligned}
sk_{bgp}^{\alpha} &= (n + g + \delta)k_{bgp} \\
\Rightarrow k_{bgp} &= \left(\frac{s}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}, \\
y_{bgp} &= \left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}}.
\end{aligned}
$$

Sources of growth of output per capita

- Convergence:
  - Suppose economy starts with $k_t < k_{bgp}$.
  - Then $sf(k_t) > (n+g+\delta)k_t$ (why?).
  - Then $k_{t+1} > k_t$.
  - Economy converges to BGP.
- Long-run growth: eventually, growth rate settles down to $1+g$.

Comparative static: $s \uparrow$



Golden rule capital stock

- Consumption better measure of well-being than output (why?).
- Consumption per effective worker:

$$c_{bgp}(s) = (1-s)y_{bgp}(s) \overbrace{=}^{C-D} (1-s)[s/(n+g+\delta)]^{\frac{\alpha}{1-\alpha}}.$$

- Maximize over s:

$$0 = -\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}\left(\frac{1-s}{n+g+\delta}\right)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}-1}$$

$$\Rightarrow s = \alpha,$$

$$k_{gr} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}.$$

- For $s > \alpha$, "too much" of gross saving offsets depreciation.
- But s isn't a choice in this model.

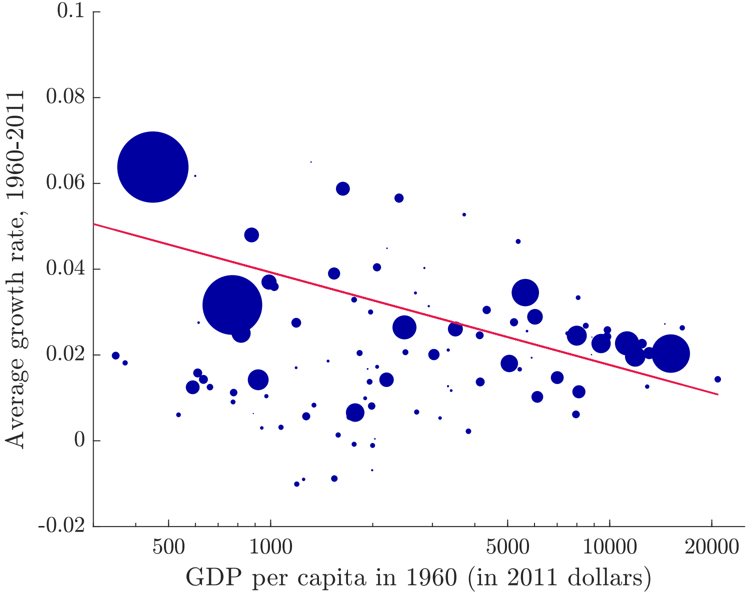
Implications for growth theory

- Income growth and cross-country differences due to exogenous technology and capital accumulation.
- Convergence: countries that start with low levels of capital per worker catch-up.
- Rate of catch-up diminishes as approach frontier.
- At frontier, growth only from exogenous technology.
- Romer 1.6 calculations.

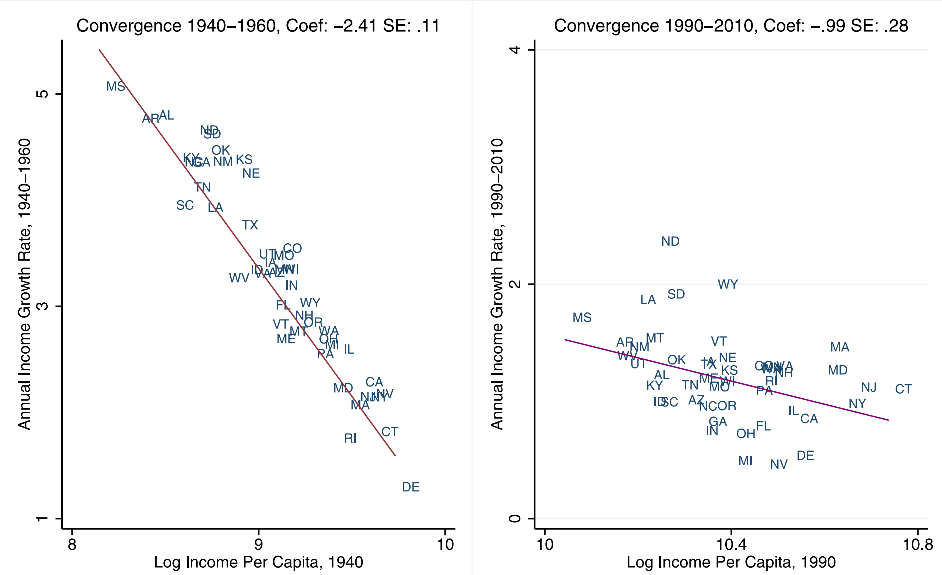
Confronting model with data

1. No "unconditional" convergence.
2. Conditional (or "club") convergence across countries.
3. Convergence across U.S. states (but less than before).

No unconditional convergence



CONDITIONAL CONVERGENCE ACROSS U.S. STATES



Source: Ganong and Shoag (2017). Why Has Regional Income Convergence in U.S. Declined?