MÄLARDALEN UNIVERSITY
SWEDEN

School of Education, Culture and Communication

Division of Applied Mathematics

BACHELOR THESIS IN MATHEMATICS / APPLIED MATHEMATICS

Lattice approximations for Black-Scholes type models in Option Pricing

by

Anne Karlén and Hossein Nohrouzian

Kandidatarbete i matematik / tillämpad matematik

DIVISION OF APPLIED MATHEMATICS

MÄLARDALEN UNIVERSITY

SE-721 23 VÄSTERÅS, SWEDEN

MÄLARDALEN UNIVERSITY
SWEDEN

School of Education, **Culture and Communication**
Division of Applied Mathematics

Bachelor thesis in mathematics / applied mathematics

Date:

2013-06-05

Project name:

Lattice approximations for Black-Scholes type models in Option Pricing

Authors:

Anne Karlén and Hossein Nohrouzian

Supervisor:

Professor Sergei Silvestrov

Examiner:

Anatoliy Malyarenko

Special thanks to:

Anatoliy Malyarenko, Jan Röman and Karl Lundengård

Comprising:

15 ECTS credits

The first named author of this thesis has written and is responsible for Sections 2.7, 2.8 , 2.9 and the whole Chapter 5. The second named author has written and is responsible for the rest of Chapter 2, the whole Chapters 3, and Chapter 4, and the appendix. All the remaining parts of this thesis were written by the two authors together.

Abstract

This thesis studies binomial and trinomial lattice approximations in Black-Scholes type option pricing models. Also, it covers the basics of these models, derivations of model parameters by several methods under different kinds of distributions. Furthermore, the convergence of binomial model to normal distribution, Geometric Brownian Motion and Black-Scholes model is discussed. Finally, the connections and interrelations between discrete random variables under the Lattice approach and continuous random variables under models which follow Geometric Brownian Motion are discussed, compared and contrasted.

Contents

Chapter 1

Introduction

Investing in the market always contains some risk. People invest some of their capital in the market with the aim of obtaining maximum profit at minimum possible risk. There are different levels of risk aversion. People who want to do some business in the future and are afraid of loosing money due to changed conditions (e.g. increased price of raw material or unfavorable exchange rates), might want to secure themselves through hedging (eliminating the risk). That can be obtained with the purchase of options. An option is a contract which states that the holder of the option is allowed to buy (or sell) a certain asset at a predetermined price at predetermined date(s). Buying options seems to be a secure way of investing in the market. But are they always profitable? No. This would mean arbitrage opportunities (free lunches) and consequently everyone would invest in them. The question is how these options should be priced in order to avoid arbitrage opportunities, i.e., the possibility to earn profit without risk. From the course *Introduction to financial mathematics* we already knew that there exist formulas for fair option pricing, but the parameters were always given to us when we were supposed to use these formulas. The question arose how the parameters actually are determined and when we discovered that there are many different papers concerning option pricing and derivation of parameters, and that different papers present different results, the idea for the topic of our thesis was born. The problem is that there are unpredictable factors, namely the evolution of the price of the underlying asset and the probability of attaining these prices. The price can go up or down, but by how much? And what is the probability to move in either direction? It is not totally reasonable to assume that the probability is 50% each. Additionally there could be no price change at all, so we have a third possibility for the evolution of the price. Intuitively we would expect the probability of no price change to be smaller than the probabilities of up and downward movements. So unpredictable parameters such as random variables or even a stochastic process are involved in option pricing and therefore it is a complex and complicated field to examine.

Option pricing is a part of financial analysis which deals with different areas of mathematics such as probability, stochastic processes, ordinary differential equations and stochastic differential equations. So how do we compute the fair price for an option? Well, first of all there are different kinds of options in the market which are constructed differently. However, there

is a common algorithm for calculating the fair price of any option and this algorithm states that the fair price of an option is its discounted expected payoff. When we talk about expected value in simple cases, we know that it is the average or mean of a sample or population. But finding expected values of some processes is not that easy. Moreover there are some controversial questions. How to calculate this expected value? Is it always possible to calculate the price of an option analytically? How should one translate mathematical formulas to computer language? If it is not possible to calculate the price of an option analytically, which numerical method should be used to estimate the price of the option? What is the error of estimations? What are the definitions for different options and how does one calculate their payoffs? To answer some of these questions, we will try to explain the basic definitions and ideas about options and how to price them. This process is strongly related to our knowledge which we have obtained by studying the *Bachelor's Program in Analytical Finance* at *Mälardalen Högskola*. We will go through the basic ideas which are vital for understanding the algorithm of pricing options specifically in computer language and when we talk about simulating some process for which we can find the result by numerical methods [6].

After our introduction, we will in Chapter 2 go through the concept of binomial models. We will study how it is possible to price an option using a binomial tree. In the binomial model Chapter, we will start with the definitions of payoff for European and American options [7], which we have studied in the course *"Introduction to Financial Mathematics"*. After that, we will follow our process by studying some probability theorems and definitions [21] which are essential for getting a good understanding for pricing options via the binomial model approach. We obtained this knowledge in our *"Probability"* course. Then, using binomial approach, we will try to explain how the price of American and European options can be calculated. At this step, we will be able to analyze a binomial tree and we will have a system of equations with some unknowns. We will continue our process by calculating them. Firstly, we will try to find the value for risk-neutral probability [4], [7] by constructing a replicating portfolio. Here we will get help from different literature like our knowledge from the courses *"Stochastic Processes"* [12] and its lecture notes [14]. Secondly, we will derive the other unknowns in our system of equations, namely up and down factors in full details and we will study the CRR model (Cox, Ross and Rubinstein model) and its results [4]. After that, we will talk about random walks and transition probabilities which will help us to derive the backward and forward equations for pricing options [12],[14]. Consequently, we will discuss the formula for pricing the option [7]. Finally, we will end Chapter 2, with some example which will show how our process can be helpful to price options, especially when we deal with an American Put option, in which early exercise on a predetermined date is possible.

In Chapter 3, we start to compare and contrast the behavior of a random variable, namely stock price, in discrete and continuous time. Additionally, we will consider the result of Black and Scholes [1] and Merton [15]. We know they assumed that the dynamic of risky security prices follows a Geometric Brownian Motion. We will also follow Cox, Ross and Rubinstein [4] approach to see how as well the sequence of the binomial model converges to Geometric

Brownian Motion. To do so, we will start studying the sequence of the binomial model and its convergence to normal distribution [12],[14],[21]. Then we will show how the sequence of the binomial model converges to the Black-Scholes model under risk neutral probability [12],[14]. After that, we will make a distinction between normal and log-normal random variables. Moreover, we will discuss how stock prices can be treated as log-normal random variables with normal-distribution and how it can be treated as a normal random variable with log-normal distribution. In this part we do some interesting derivations using our knowledge from calculus and probability [21], which are really useful for approximation of some variants of binomial models to the Black-Scholes pricing formula.

In Chapter 4, we will study some different variants of binomial models. We have seen the result and formula for some of these variants in our course *"Analtical Finance I"* [17], but we will try to apply our knowledge to derive the final formulas in details. We will see that for the binomial approach, we will always have two equations for expected value and variance which, depending on our choice of normality or log-normality of our random variable (namely stock price), can be approximated differently with different means and variances. Moreover, we will see that we have a system of two equations and three unknowns. We will see that for example Cox, Ross and Rubinstein [4] chose their third equation like $ud = 1$. In simple cases, we introduce our third equation by $ud = 1$ or $p = 1/2$ and we will approximate the binomial model in a way that the mean and the variance of our models converge to the Black-Scholes formula. Then we will study some other models like Jarrow-Rudd model [10],[9], Tian model [19], Trigeorgis model [20] and Leisen-Reimer model [13]. We will see how the approximation of these different models works and what advantage and disadvantage each model has. There are lots of other models which can be considered, but we will finish this chapter by just considering the models that we have mentioned.

In Chapter 5, we will study the trinomial model. We start off with the basic principles of the trinomial distribution. It is similar to the binomial distribution, but not as widely used. Consequently there was less literature available that covered the subject but due to the similarity with the binomial model, previous knowledge of probability theory and Wackerly [21], the concept and properties of trinomial distribution is derived and explained. It is important so that further parts can be understood.

Directly after the basics of trinomial distribution we study the paper of Boyle from 1988 [2] because it extends Cox, Ross and Rubinstein approach of risk neutral valuation with jumps in two directions (binomial model) into a model with jumps in three directions (trinomial model) with the condition of risk neutral return which in the short term is the risk free interest rate. The probabilities under this process are derived connecting mean and variance of discrete and continuous distributions where the discrete functions are approximations of the lognormal distribution of the underlying asset which is governed by a Geometric Brownian Motion. This model will show that we have two constraints and five unknown parameters, giving no unique parameters for the probabilities.

The next section deals with risk neutral probability as well. We will examine if the replicating portfolio can generate the contingent claim if three different movements on the underlying asset are assumed (in contrast to two, which we examine in the binomial lattice). Since we used Kijima [12] and the lecture notes from Stochastic Processes [14] for the replicating portfolio in the binomial model, we find it appropriate to use it for the trinomial variant as well.

Another method for finding parameters suitable to generate contingent claims was established by Kamrad and Ritchken in 1991 [11]. Their idea was to approximate the logarithm of the random variable that describes the return of the underlying asset in one time step. Even here the random variable is discretized in the trinomial lattice and the first two moments of the continuous distribution is matched with those of the discrete distribution.

As we have seen we show several methods where the approximation of the continuous random variable is done by a lattice approach. The next part however uses another method, namely the explicit finite difference approach. We show how the partial derivatives in the Black-Scholes formula can be discretized and further processed in order to find probability densities that satisfy the partial differential equation. Originally Brennan and Schwartz [3] developed this technique, but we studied the notes from Analytical Finance [17] and Hull [7] as well, because the paper itself is structured in a complicated way. For easier understanding we even included graphics in this part.

In the last part we try to find a connection between binomial and trinomial trees based on an observation of one of the authors of this thesis that binomial lattices and trinomial lattices overlap. As we investigated this connection further we found a paper by Derman et al from 1996 [5] which was really helpful and led to yet another discovery, namely that there are tree models with constant volatility, called standard trees, and trees that are constructed in order to match the volatility smile with varying volatility, called implied trees. We give a brief explanation of this detection.

We found it reasonable as well to show the Black-Scholes formula and how it is derived. The appendix covers this subject.

All sources that we have used are either published papers or text books, or lecture notes from professors at Mälardalens Högskola. We had the great opportunity to download the papers from Jstor through our university accounts. We find all references very reliable since they are provided through academic sources and have educated many students and people working in finance and economics, before us.

Chapter 2

Binomial Model

In market the price of stocks move randomly. The price of stocks can go up, down or remain constant between two time intervals. So, the movements of stock prices are stochastic processes. In simple case we can consider a random walk with predefined length of movements. We assume the price of stock can go up or down for a certain amount in each time interval with the probability of p and $1 - p$ respectively. This simple model is called Binomial model. The price of stock at time zero is denoted by S_0 and it is usual to denote the amount of increasing by uS_0 and the amount of decreasing by dS_0. One can start from today, i.e., node one at the time zero and build a binomial tree for a finite time interval. Figure 2.1 illustrates three steps binomial tree graphically. It is obvious that the possible stock prices can be calculated easily at any node. But, how to calculate the fair price of an option? As mentioned before the discounted expected payoff must be considered. In binomial tree at any node we can count all the possible paths to reach that specific node. And then we can formulate our expected payoff. Instead of counting all possible way to reach a specific node we need to explain the formulation of binomial expansion and binomial coefficient which can represent the probability of reaching at any specific node at any step. But, first let start by the most important definitions for option pricing and then we will continue by some probability theorems and definitions as well as binomial expansion.

2.1   Payoff to European and American Options

Let us start with the definition of American and European options[1] [7].

Definition 2.1.1. *European Options are options which give the holder of the options the right, but not the obligation, to exercise them at maturity.*

---

[1]In the market there exist several different kinds of options, like Bermudan Options, which are a part of nonstandard American options, Asian options, Currency Options, Swap Options, Barrier Options and ... [7]
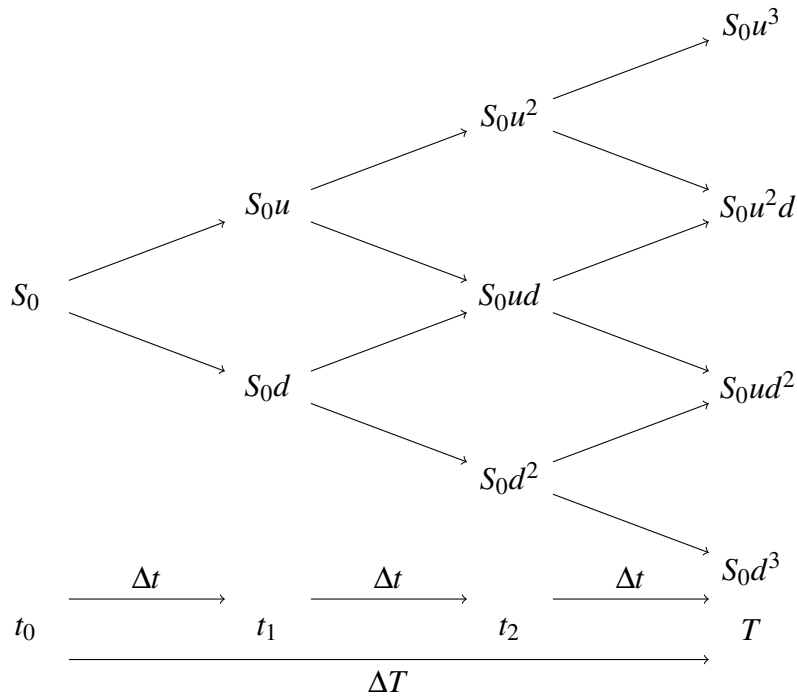
Figure 2.1: **Three Steps Binomial Tree**

Definition 2.1.2. *American Options are options which give the holder of the options the right, but not obligation, to exercise them at any time up to maturity.*

It can be proved that the price of an American put option must be greater or or equal to the price of a European put option [7]. It can also be proved that the price of American call and European call options is the same [7] under the condition that the underlying asset does not pay dividends. As it was explained, the honest price for an option is its discounted expected payoff. For discounting a price it is common to use the risk-free interest rate r with continuous compounding. To explain it mathematically we can write:

$$Price = e^{-r\Delta T}E[payoff] \tag{2.1}$$

To distinguish between the options we can define their payoffs as follow [7]:

Long Call:

$$payoff = \max\{S_T - K, 0\}$$

Short Call:

$$payoff = -\max\{S_T - K, 0\}$$

Long Put:

$$payoff = \max\{K - S_T, 0\}$$

9
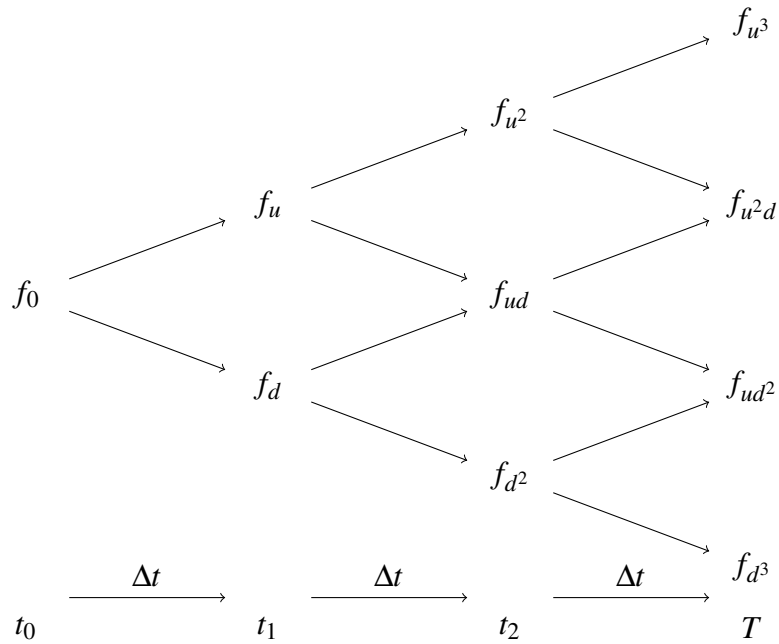
Figure 2.2: **Three Steps Binomial Tree**

Short Put:

$$payoff = -\max\{K - S_T, 0\}$$

where K is the strike price and S_T is the stock price at maturity. As we can see, the payoff equations for both European and American options are the same. Let us denote the payoff at each node by f with its path indexes. The procedure is explained in Figure 2.2. Let us furthermore equip ourselves with some probability theorems and definitions to calculate the price of an option in the binomial model.

2.2 Binomial Expansion

In this part, we will study some related important theorems and definitions [21].

Definition 2.2.1. *Let p and q be any real number, then **Binomial Expansion** of $(p+q)^n$ is:*

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^n q^{n-k} \tag{2.2}$$

*where the **Binomial Coefficients** are:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.3}$$

The binomial coefficients are useful for calculating the probability distribution in a binomial tree. Let us continue with the next theorem [21].

Theorem 2.2.1. *For any discrete probability distribution, the following must be true:*
1. $0 \leq p(y) \leq 1$, for all y
2. $\sum_y p(y) = 1$, where the summation is over all values of y with nonzero probability.

The definition for binomial distribution is [21]:

Definition 2.2.2. *A random variable K is said to have a **Binomial Distribution** based on n trails with success probability p if and only if*

$$p(k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^n (1-p)^{n-k} \tag{2.4}$$

where

$$k = 0, 1, 2, ..., n \quad and \quad 0 \leq p \leq 1$$

The formula (2.4) defines the probability function for a discrete random variable. Considering Theorem 2.2.1 we say $q = 1 - p$ and we will use it to calculate the price of some options.[2] Moreover, the following definition will help us to calculate the mean or expected value of a discrete random variable [21].

Definition 2.2.3. *Let Y be a discrete random variable with probability function $p(y)$. Then the **Expected Value** of Y, $E(Y)$, is defined to be*

$$E(Y) = \sum_y y p(y) \tag{2.5}$$

if the above series is absolutely convergent.

Finally, we can find the variance of a discrete random variable using the following theorem [21].

Theorem 2.2.2. *Let Y be a discrete random variable with probability function $p(y)$ and mean $E(Y) = \mu$; then*

$$V(Y) = \sigma^2 = E\left[(Y-\mu)^2\right] = E\left(Y^2\right) - \mu^2 \tag{2.6}$$

Remark 2.2.1. *Using Definition 2.2.3 we can see that $E\left(Y^2\right) = \sum_y y^2 p(y)$.*

These theorems and definitions will play a considerable role in lattice approaches.

---

[2]The mean in the discrete binomial distribution is $\mu = np$ and the variance is $\sigma^2 = np(1-p)$ [21].

2.3 Calculating the Price of European Options

As we saw, European options can be exercised only at maturity. Since we know the strike price it is obvious that we can calculate the pay-off when the stock price is known at maturity. Moreover, we can calculate the probability distribution by (2.4) and the expected value by (2.5). Let us consider the Three Step Binomial Tree in Figure 2.1 and Figure 2.2, and see how it works for a European call option.

First, we have a three-step binomial tree. So the binomial expansion is:

$$(p+q)^3 = \sum_{k=0}^{3} \binom{n}{k} p^n q^{n-k} = p^3 + 3p^2 q + 3pq^2 + q^3$$

where $q = 1 - p$. And the price of European call option can be calculated by

$$C_E = e^{-rT} \left[p^3 f_{u^3} + 3p^2 q f_{u^2 d} + 3pq^2 f_{ud^2} + q^3 f_{d^3} \right]$$

Where C is an abbreviation for the price of a European call option and $f_{u^m d^{n-m}}$ represents the payoff at any node. Here, m denotes the number of upward movements and n stands for the number of steps in our binomial tree. For example when we have three steps in the binomial tree, i.e., $n = 3$, we have two upward movements, i.e., $m = 2$, and we can say that the payoff at node $u^2 d$ is:

$$f_{u^2 d} = \max\{S_0 u^2 d - K, 0\}$$

Remark 2.3.1. *Notice that it is possible to do a backward approach to find the price for a European option, but it can be proved that the result will be the same. We will show that in Section 2.9. We will also consider the backward approach for calculating the price of an American put option.*

2.4 Calculating the Price of American Options

As it was mentioned before, with American options the holder of an option has the right, but not the obligation, to exercise the option at any time up to maturity. Considering this fact, we can calculate the fair price of an American option. It is really important to calculate the discounted expected payoff at any possible time for exercising. Let us consider the three-step binomial tree. The holder of an American, say put option, has the right to exercise her option at times t_1, t_2 or T. The question is when the optimal time to exercise that American put option is. Let us do it step by step. At time T we know the payoffs. Using that knowledge, it is possible to go one step back on the nodes, which in our example is time t_2. Now, we have to consider every single node as a one step binomial tree and find the optimum payoff. Considering Figure 2.2 we can formulate the explanation above in mathematical language as follows.

$$optimal\{f_{u^2}\} = \max\left\{f_{u^2}, e^{-r\Delta t}[pf_{u^3} + (1-p)f_{u^2d}]\right\}$$
$$= \max\left\{(K - S_T, 0), e^{-r\Delta t}[pf_{u^3} + (1-p)f_{u^2d}]\right\}$$

$$optimal\{f_{ud}\} = \max\left\{f_{ud}, e^{-r\Delta t}[pf_{u^2d} + (1-p)f_{ud^2}]\right\}$$
$$= \max\left\{(K - S_T, 0), e^{-r\Delta t}[pf_{u^2d} + (1-p)f_{ud^2}]\right\}$$

If we continue doing so, we will be able to calculate the optimal values in all nodes and go back one more time. Then we will use the optimal values of the previous nodes to calculate their discounted expected payoff. We continue doing so until we reach time zero. The discounted expected payoff at time zero will be the fair price of the American option.

$$P_A = e^{-r\Delta t}\left[p \times optimal\{f_u\} + q \times optimal\{f_d\}\right]$$

Now we almost cover every important aspect of the binomial tree. But we still need to know more about the probability p and the u and d factors.

2.5   Risk-Neutral Probability (The Cox-Ross-Rubinstein Model)

Consider a financial market containing of two different securities, a deterministic bond and a stock which follows a stochastic process.We assume that the market is free of arbitrage. It is possible to prove that the stock price is its discounted expected payoff [7],[17].
In general:
$$S(t) = e^{-rT}E^{p^*}[S(T)]$$

where p^* is called the Risk-Neutral Probability measure or the equivalent martingale measure. Risk-neutral probability measure in the binomial model was originally calculated by Cox, Ross and Rubinstein. They calculated p^* as [4]:

$$p^* = \frac{e^{r\Delta t} - d}{u - d} \tag{2.7}$$

where u and d are up and down factors in the binomial tree, r is the risk-free interest rate and Δt is the time between each two steps in the binomial tree.
Using Definition 2.2.3 and (2.5) the expected stock price in the two step-Binomial tree at time T can be calculated as [7]:

$$E[S(T)] = p^*S_0 u + (1 - p^*)S_0 d \tag{2.8}$$
$$E[S(T)] = p^*S_0(u - d) + S_0 d \tag{2.9}$$

Substituting (2.7) in (2.9) we will get:

$$E[(S(T)] = S_0 e^{rT} \Rightarrow S_0 = e^{-rT}E[S(T)] \tag{2.10}$$

The current result tells us that in a risk neutral world, the expected return on a stock is equal to the risk-free interest rate.

Calculating Risk-Neutral Probability p^*

Consider a market which consists of two types of financial instruments, bonds and stocks. Moreover, there is no possibility of arbitrage. We know that bonds guarantee a certain amount of profit in a specific time period, but the return on the stock is a stochastic process. To begin with, we can write the process of these two securities in mathematical language as follows [12],[14],[17]:

$$B(t) = \begin{cases} e^{r\Delta t} = 1 & \text{,for } t = t_0 = 0 \\ e^{r\Delta t} & \text{,for } t = t_1 = T \end{cases}$$

B represents bonds with deterministic processes and their value at time zero will be one unit of amount of money. At time T it will be their initial value plus the risk-free interest rate continuous compounded and $0 \leq t \leq T$.

As we discussed it previously, stocks follow a stochastic process and this will be as follows:

$$S(t) = \begin{cases} S_0 & \text{,for } t = 0 \\ S(T) & \text{,for } t = T \end{cases}$$

Before going further, it might be crucial to explain our portfolio. Our portfolio is simply our properties. We have invested our money in two categories: stocks and bonds. Let us say that we have decided to invest x percent of our money in bonds and y percent of our money in stocks. x and y can get negative values since its possible to short one of the securities to long the other, but under condition $x + y = 1$. Let us call our portfolio h. So the value of our portfolio at time t is:

$$V(t, h) = xB(t) + yS(t)$$

which represents the value process of portfolio h. We can expand the expected value of the portfolio of bonds and stocks at time t:

$$E[V(t, h)] = \begin{cases} xB(0) + yS(0) & \text{,for } t = 0 \\ xB(T) + yE[S(T)] & \text{,for } t = T \end{cases}$$

which can be simplified as:

$$E[V(t, h)] = \begin{cases} x + yS_0 & \text{,for } t = 0 \\ xe^{r\Delta t} + yE[S(T)] & \text{,for } t = T \end{cases}$$

We have already discussed the possible outcomes of $S(t)$ in the one step binomial model. So the value process at time $t = T$ can be rewritten as:

$$V(t, h) = \begin{cases} xe^{r\Delta t} + yS_0u & \text{,if stock goes up with probability p} \\ xe^{r\Delta t} + yS_0d & \text{,if stock goes down with probability 1-p} \end{cases}$$

Since the proportions of x and y are arbitrary, we can choose them in such a way that the

value of each possible outcome will be equal to value of portfolio at the end of the portfolio [12],[14]. This yields:

$$f_u = xe^{r\Delta t} + yS_0 u \tag{2.11}$$

$$f_d = xe^{r\Delta t} + yS_0 d \tag{2.12}$$

We have already seen how to calculate f_u and f_d for different options. Solving (2.12) and (2.11) for x and y we will get:

$$y = \frac{f_u - f_d}{S_0(u - d)} \tag{2.13}$$

Substituting (2.13) to either (2.12) or (2.11) yield:

$$x = \frac{u f_d - d f_u}{e^{r\Delta t}(u - d)}$$

We already know that in two steps binomial tree the following equation holds:

$$f_0 = e^{-r\Delta t}[p f_u + (1 - p) f_d] \tag{2.14}$$

Moreover, if we substitute the value of x and y into he value process formula for our portfolio h at time zero we will obtain:

$$V(0, h) = f_0 = x + yS_0 = \frac{u f_d - d f_u}{e^{r\Delta t}(u - d)} + \frac{f_u - f_d}{S_0(u - d)} S_0$$

$$= f_0 = e^{-r\Delta t} \left[\left(\frac{e^{r\Delta t} - d}{u - d} \right) f_u + \left(\frac{u - e^{r\Delta t}}{u - d} \right) f_d \right] \tag{2.15}$$

comparing (2.14) and (2.15) shows the value of p and $1 - p$. Here we had a two steps binomial tree so $\Delta t = t_1 - t_0 = T - 0 = T$, but for a binomial tree with more than two steps it is better to denote the change in time for each step by Δt. The risk-neutral probability measure will be [4]:

$$p^* = \frac{e^{r\Delta t} - d}{u - d} \qquad 1 - p^* = \frac{u - e^{r\Delta t}}{u - d} \qquad d \le r \le u \tag{2.16}$$

Remark 2.5.1. *The condition $d \le r \le u$ will guarantee that our portfolio is free of arbitrage, our neutral probabilities will lie between zero and one, and we will not obtain zero in the denominator. It is easy to see that the sum of two fractions will be exactly one, and this is what we expected. Additionally, this formula tells us, in a risk-neutral world, the expected return on a stock must be equal to the risk-free interest rate [4].*

2.6   Volatility with u and d factors

In practice, the volatility of a financial security can be estimated by the historical market data. Thus, it is logical to calculate the u and d factors which are related to such volatility [7].

These factors are calculated in different ways[3], but we use the result proposed by Cox, Ross and Rubinstein in 1979 [4] which are as follows:

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}} \tag{2.17}$$

Calculating u and d factors

To calculate up and down factors we can consider two significantly different ways. One approach is having our stochastic process in continuous time and the second approach is considering our stochastic process with jump diffusion. In the first case, the length of one time intervals plays a vital role for our process. But in second case the movement of the random variable will be more smooth, and it can have sudden discontinuous jumps or changes [4]. We will go through the first approach. We have seen this approach and its result in [4],[7] but there is no full detailed derivation of up and down factors in neither references [4],[7]. We will start by following a corollary.

Corollary 2.6.1. *The up and down factors in discrete time are given by [4]:*

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}}.$$

Proof. To begin with, let's consider a two-step binomial tree. We know that the possible stock prices at time $t = T$ are :

$$S_T = \begin{cases} S_0 u & \text{,if stock goes up with probability p} \\ S_0 d & \text{,if stock goes down with probability 1-p} \end{cases}$$

Using Definition 2.2.3 the expected value will be:

$$E[S_T] = p^* S_0 u + (1 - p^*) S_0 d$$

Recall equations (2.8) and (2.10) and consider the fact that in a risk neutral world the drift coefficient is equal to the risk-free interest rate, i.e., $\mu = r$ (See [4],[7]).

$$E[S_T] = p^* S_0 u + (1 - p^*) S_0 d = S_0 e^{\mu \Delta t}$$

Dividing by S_0 we will get the first equation to calculate u and d.

$$E[S_T/S_0] = p^* u + (1 - p^*) d = e^{\mu \Delta t} \tag{2.18}$$

Using Theorem 2.2.2 and (2.6), the variance will be

$$V[S_T] = E[S_T^2] - (E[S_T])^2$$

---

[3]It is possible to calculate u and d factors with normal distribution, log-normal distribution, mixed normal/log-normal distribution, the Cox-Ross-Rubenstein model, the Second order Cox-Ross-Rubenstein, the Jarrow-Rudd model, the Tian model, the Trigeorgis model, ...[17]

$$V[S_T/S_0] = \frac{1}{S_0^2}V[S_T] = p^*u^2 + (1-p^*)d^2 - e^{2\mu\Delta t}$$

In a small time interval Δt, the variance must be equal to $\sigma^2 \Delta t$, so we will get the second equation [7]

$$p^*u^2 + (1-p^*)d^2 - e^{2\mu\Delta t} = \sigma^2 \Delta t \qquad (2.19)$$

Substituting the values of p^* and $(1-p^*)$ from (2.16) to (2.19) we will get:

$$\begin{aligned}
\sigma^2 \Delta t &= \frac{e^{\mu\Delta t} - d}{u-d}u^2 + \frac{u - e^{\mu\Delta t}}{u-d}d^2 - e^{2\mu\Delta t} \\
&= \frac{e^{\mu\Delta t}(u^2 - d^2) - ud(u-d)}{u-d} - e^{2\mu\Delta t} \\
&= \frac{e^{\mu\Delta t}(u-d)(u+d) - ud(u-d)}{u-d} - e^{2\mu\Delta t} \\
&= e^{\mu\Delta t}(u+d) - ud - e^{2\mu\Delta t} \qquad (2.20)
\end{aligned}$$

Now, we have two equations and three unknowns. To solve this we can consider Cox, Ross and Rubinstein approach where they consider a recombining tree and they put $ud = 1$ [4]. By letting $ud = 1$ we will have two equations for the expected value and variance and two unknowns, u and d. Solving (2.20) will give us the result for u and d in formula (2.17).
Let's try to do a little algebra and see how it is possible to solve this. To begin with, we know that the Maclaurin expansion for exponential functions is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + ... \qquad (2.21)$$

using (2.21) and ignoring the terms of higher order than Δt [7], we can introduce the following equations:

$$\begin{cases}
e^{\sigma^2 \Delta t} = 1 + \sigma^2 \Delta t \\
e^{\mu\Delta t} = 1 + \mu\Delta t \\
e^{2\mu\Delta t} = 1 + 2\mu\Delta t
\end{cases}$$

Substituting $d = 1/u$ and solving (2.20) for u we will get:

$$u^2 - \left(\frac{1 + \sigma^2 \Delta t + e^{2\mu\Delta t}}{e^{\mu\Delta t}}\right)u + 1 = 0$$

Let's solve this quadratic equation, we introduce the notation b':

$$-b' = \frac{1 + \sigma^2 \Delta t + e^{2\mu\Delta t}}{e^{\mu\Delta t}}$$

and solve the equation

$$u_{1,2} = \frac{-b' \pm \sqrt{b'^2 - 4}}{2}$$

Now let us calculate b':

$$-b' = \frac{1 + \sigma^2 \Delta t + e^{2\mu \Delta t}}{e^{\mu \Delta t}} = \frac{1 + \sigma^2 \Delta t}{e^{\mu \Delta t}} + e^{\mu \Delta t} = \frac{e^{\sigma^2 \Delta t}}{e^{\mu \Delta t}} + e^{\mu \Delta t}$$

$$= e^{(\sigma^2 - \mu)\Delta t} + e^{\mu \Delta t} = [1 + (\sigma^2 - \mu)\Delta t] + (1 + \mu \Delta t) = 2 + \sigma^2 \Delta t$$

It follows:

$$b'^2 - 4 = (2 + \sigma^2 \Delta t)^2 - 4 = 4 + 4\sigma^2 \Delta t + \sigma^4 (\Delta t)^2 - 4 = 4\sigma^2 \Delta t$$

so

$$u_{1,2} = \frac{2 + \sigma^2 \Delta t \pm 2\sigma \sqrt{\Delta t}}{2} = 1 \pm \sigma \sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t$$

$$\begin{cases} u_1 = 1 + \sigma \sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t = e^{\sigma \sqrt{\Delta t}} \\ u_2 = 1 - \sigma \sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta = e^{-\sigma \sqrt{\Delta t}} \end{cases}$$

Similarly, if we solve (2.20) for d:

$$\begin{cases} d_1 = 1 - \sigma \sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t = e^{-\sigma \sqrt{\Delta t}} \\ d_2 = 1 + \sigma \sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t = e^{\sigma \sqrt{\Delta t}} \end{cases}$$

Since the condition $d \leq r \leq u$ must be fulfilled, we can only accept the answers which satisfy this condition. Thus $d = d_1$ and $u = u_1$. This result is the same as Cox-Ross-Rubinstein's (CRR) model [4], which is one of the most widely used models.

$\square$

Now we will review an important part of Cox, Ross and Rubinstein's paper [4]. Understanding their approach will help us to further study the binomial model. Generally, the price of a stock at the n-step binomial tree is determined by the following possible path on the binomial tree [4]:

$$S_T = u^m d^{n-m} S_0$$

If our random variable takes upward movements m times in n possible steps, then the random variable takes $n - m$ downward movements. Dividing both hand sides with S_0 and taking the logarithm yields [4]:

$$\ln\left(\frac{S_T}{S_0}\right) = \ln\left(u^m d^{n-m}\right) = m\ln u + (n-m)\ln d$$

$$= m\ln u + n\ln d - m\ln d = m\ln\left(\frac{u}{d}\right) + n\ln d$$

We calculate the expectation and consider the fact that we just have a random variable m here. The expectation of a constant is its value [4]

$$E\left[\ln\left(\frac{S_T}{S_0}\right)\right] = E\left[m\ln\left(\frac{u}{d}\right) + n\ln d\right]$$

$$= E\left[m\ln\left(\frac{u}{d}\right)\right] + E\left[n\ln d\right] = E[m]\ln\left(\frac{u}{d}\right) + n\ln d$$

Using Theorem 2.2.2 and (2.6), the variance of a random variable X can be calculated as $V[X] = E[X^2] - (E[X])^2$, so the variance will be:

$$V\left[\ln\left(\frac{S_T}{S_0}\right)\right] = E\left[\left(m\ln\left(\frac{u}{d}\right) + n\ln d\right)^2\right] - \left(E\left[m\ln\left(\frac{u}{d}\right) + n\ln d\right]\right)^2$$

$$= \left(\ln\left(\frac{u}{d}\right)\right)^2 (E[m^2] - (E[m])^2) = \left(\ln\left(\frac{u}{d}\right)\right)^2 V[m]$$

Since probability p^* corresponds to up movements, the mean and variance of m will be [4]

$$E[m] = np^* \quad , V[m] = np^*(1-p^*)$$

Thus the expected value and variance will be [4]:

$$E\left[\ln\left(\frac{S_T}{S_0}\right)\right] = np^*\ln\left(\frac{u}{d}\right) + n\ln(d) = \left[p^*\ln\left(\frac{u}{d}\right) + \ln(d)\right]n \equiv \hat{\mu}n$$

$$V\left[\ln\left(\frac{S_T}{S_0}\right)\right] = \left(\ln\left(\frac{u}{d}\right)\right)^2 np^*(1-p^*) \equiv \hat{\sigma}^2 n$$

We know that the length of each step is the length of time divided by steps in our binomial tree. So, $\Delta t = \frac{t}{n}$. If we have n big enough($n \to \infty$), we can choose u, d and p^* in a way that [4]:

$$\lim_{n\to\infty}\left[\left[p^*\ln\left(\frac{u}{d}\right) + \ln(d)\right]n\right] = \mu t$$

$$\lim_{n\to\infty}\left[\left(\ln\left(\frac{u}{d}\right)\right)^2 np^*(1-p^*)\right] = \sigma^2 t$$

Cox, Ross and Rubinstein showed that the possible values to satisfy these conditions are [4]

$$u = e^{\sigma\sqrt{\Delta t}} \quad , d = e^{-\sigma\sqrt{\Delta t}} \quad , p^* = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\Delta t}$$

and for any n we will have [4]:

$$\hat{\mu}n = \mu t \quad , \hat{\sigma}^2 n = \sigma^2 t - \mu^2 t \Delta t$$

It is easy to see that, if $n \to \infty$, $\hat{\sigma}^2 n \to \sigma^2 t$.

Remark 2.6.1. *Cox, Ross and Rubinstein continued their paper by studying the convergence of their model to the Black-Scholes pricing formula. We will not discuss it now because we need to increase our knowledge about convergence of the binomial to normal distribution. However, in the next section we will study the convergence of the binomial model to normal distribution.*

2.7   Random Walks

RANDOM WALK IN THE BINOMIAL MODEL

For this part we used material from lecture 4 of *Stochastic processes* by Anatoliy Malyarenko [14] and Chapter 6, from Kijima [12]. The lecture notes are based the book and we found it advantageous to study both since they complement each other very well.

Let X_1, X_2 ... X_n, $n \in Z_+$ be random variables which are independent and identically distributed. They represent upward and downward movements which for now are of step size 1. X_n is either $u = 1$ or $d = -1$.

The starting position X_0 equals to zero. Adding the subsequent values of n variables to X_0 gives the position at X_n, in general

$$X_n = X_0 + \sum_{j=1}^{n} X_j \quad j = 1, 2, ..., n$$

which can be expressed as the **Partial sum process**

$$W_n = \begin{cases} 0, & n = 0 \\ X_1 + X_2 + ... + X_n, & n \geq 1 \end{cases}$$

Therefore $W_{n+1} = W_n + X_{n+1}$. Moreover $W_{n+1} - W_n$ is independent of the previous step W_n, i.e., the increments are independent. Furthermore $W_0 = 0$.

In the binomial tree model $P(X_1 = u) = p$ and $P(X_1 = d) = 1 - p$. The same applies for the branches that follow, so $P(X_j = u) = p$ and $P(X_j = d) = 1 - p$.

If we have the special case of a symmetric random walk with n steps $p = (1 - p) = \frac{1}{2}$, there are 2^n possible events, each with probability $\dfrac{1}{2^n}$. Otherwise the probabilities have different distributions. It does not matter in which order u and d occur and n equals the number of k

upward steps + the number of $(n-k)$ downward steps. Thus using (2.3) the partial sum after n steps equals $W_n = ku + (n-k)d$ and the probability distribution of any partial sum is

$$P\{W_n = ku + (n-k)d\}$$
$$= b_k(n,p)$$
$$= \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, ..., n$$

Multiplication by the binomial coefficient is necessary because some distributions can be obtained in different combinations.

Maintaining our precondition that $u = 1$ and $d = -1$, we can establish the sample space for W_n as $\{-n, -n+1, ..., n-1, n\}$ and the union of the sample spaces is called state space (Kijima, page 96, [12])

$$Z \equiv \{0, \pm 1, \pm 2, ...\}$$

It is clear that the probability to end up at a certain value for W_n has the condition to have W_{n-1} as its previous value for the partial sum. With our assumption of $X_n = \pm 1$, we have the following possibilities: If, for example, $W_{n-1} = 3$ then $W_n = 2$ with probability p, or 4 with probability $1-p$. All other values for W_n have zero probability. To come from state i at time n to state j at time $(n+1)$ has with other words the conditional probability

$$u_{ij} = P\{W_n = j \mid W_{n+1} = i\}$$

Because this expresses a transition from one state to another, we also call it the *One-Step transition probability*. Expressed in mathematical language it is

$$u_{ij}(n, n+1) = \begin{cases} p, & j = j+1 \\ 1-p, & j = j-1 \end{cases}$$

More general, we can say that the probability to end up in state j does not depend on time n. Moreover it only depends on the difference $j - i$. This means that we deal with time-homogeneity and spatial homogeneity. Therefore we simply can formulate a *transition probability*

$$u_j(n) = P\{W_n = j \mid W_0 = 0\}$$

We can show that the transition probability solves the following boundary value problem

$$u_j(n+1) = pu_{j-1}(n) + (1-p)u_{j+1}(n) \tag{2.22}$$

$$u_j(0) = \delta_{j0} \tag{2.23}$$

21

This problem formulation states in (2.22) that at the next time step the transition probability to end up in state j is the sum of the probabilities right now to be in state $(j-1)$ or $(j+1)$ respectively (where the state is a partial sum). Moreover we have the condition in (2.23) that the probability to be in state 0 at the initial point equals 1 and to be at any other state right from the beginning equals zero.

Proof. We can prove this using $W_{n+1} = W_n + X_{n+1}$, recalling that the elements of the right hand side are independent. Thus

$$\begin{aligned} u_j(n+1) &= P\{W_{n+1} = j \mid W_0 = 0\} \\ &= P\{X_{n+1} = 1, W_n = j-1 \mid W_0 = 0\} + P\{X_{n+1} = -1, W_n = j-1 \mid W_0 = 0\} \end{aligned}$$

From probability theory we have $P(A) \cap P(B) = P(A)P(B)$, so the above

$$\begin{aligned} &= P\{X_{n+1} = 1\}P\{W_n = j-1 \mid W_0 = 0\} + P\{X_{n+1} = -1\}P\{W_n = j+1 \mid W_0 = 0\} \\ &= pu_{j-1}(n) + (1-p)u_{j+1}(n) \end{aligned}$$

$\square$

This is also called **forward equation**.

Similarly we can formulate the **backward equation** as
$$u_{ij}(n+1) = pu_{i+1,j}(n) + (1-p)u_{i-1,j}(n)$$
under the initial condition
$$u_{ij}(0) = \delta_{ij}$$

Proof. Since $W_1 = W_0 + X_1$, we obtain
$$\begin{aligned} u_{ij}(n+1) &= P\{W_{n+1} = j \mid W_0 = i\} \\ &= P\{W_{n+1} = j, X_1 = 1 \mid W_0 = i\} + P\{W_{n+1} = j, X_1 = -1 \mid W_0 = i\} \\ &= pP\{W_{n+1} = j \mid W_1 = i+1\} + (1-p)P\{W_{n+1} = j \mid W_1 = i-1\} \end{aligned}$$

Due to time-homogeneity

$$\begin{aligned} P\{W_{n+1} = j \mid W_1 = i+1\} &= P\{W_n = j \mid W_0 = i+1\}, \\ P\{W_{n+1} = j \mid W_1 = i-1\} &= P\{W_n = j \mid W_0 = i-1\}, \end{aligned}$$

thus
$$u_{ij}(n+1) = pu_{i+1,j}(n) + (1-p)u_{i-1,j}(n)$$

$\square$

This shows that the transition probability from one state (i) to another (j) under conditional probability, is the expectation of the outcome of state (j).

2.8 Pricing the option

Now that we have derived the forward and backward formulas we understand the general formula for option pricing, expressed in Hull, p. 412 [7]. Assuming n subintervals of length Δt, the option value formula is formulated like this:

$$f(t) = e^{-r\Delta t} \left[p f_u(t+1) + (1-p) f_d(t+1) \right], \qquad 0 \le t \le n-1. \qquad (2.24)$$

The price of an option f equals the discounted expected future option value under an equivalent martingale measure, i.e., risk neutrality. Intuitively we understand that we have to take the time value of money into consideration and multiply by the discount factor $e^{-r\Delta t}$. Derivations in the Section 2.5 and 2.7, proved what (2.24) states. This reflects at the same time that we determine the option value using transition probabilities based on current information (the condition). Now we have everything we need to go through the computations that are valid for binomial trees.

In a binomial tree, at each time step t_i we have $i+1$ numbers of nodes. Figure 2.3 shows what is meant by that. At t_0 we have one node (the very first one) where $i = 0$ and the number of nodes $j = 0 + 1 = 1$, at t_1 we have two nodes (u and d), at t_2 we have three nodes (recall that $ud = du$), and so on. If we have n time intervals of length Δt and i is the index of t, we can express that we for example have three nodes for $i = 2$. The nodes are denoted with j for $j = 0, 1, ..., n$. In that way we can express all the nodes as a pair (i, j), and the stock price at every node is $S_0 u^j d^{i-j}$. For an American put, the value of the option at maturity is therefore

$$f_{i,j} = \max(K - S_0 u^j d^{n-j}, 0).$$

From an intermediate node (i, j) at time $i\Delta t$ we have the probability p of making an upward movement leading to node $(i+1, j+1)$ at time $(i+1)\Delta t$. Correspondingly $(1-p)$ is the probability of making a downward movement to $(i+1, j)$. The value with risk neutral valuation can thus at any node be written as

$$f_{i,j} = e^{-r\Delta t} \left[p f_{i+1,j+1} + (1-p) f_{i+1,j} \right]$$

This is exactly what (2.22) states and what was proved in the Section 2.7. The difference is that we here specify the nodes. For the case of options with early exercise we need to take the intrinsic value for comparison into the computations, yielding

$$f_{i,j} = \max\{ K - S_0 u^j d^{i-j}, e^{-r\Delta t} \left[p f_{i+1,j+1} + (1-p) f_{i+1,j} \right] \}$$

Due to backward induction, the fair option value captures possible early exercise during the life of the option and as Δt becomes smaller and it's limit approaches zero, the number of time steps n increases. As n increases, the calculated price of the option converges to the exact price of the option.
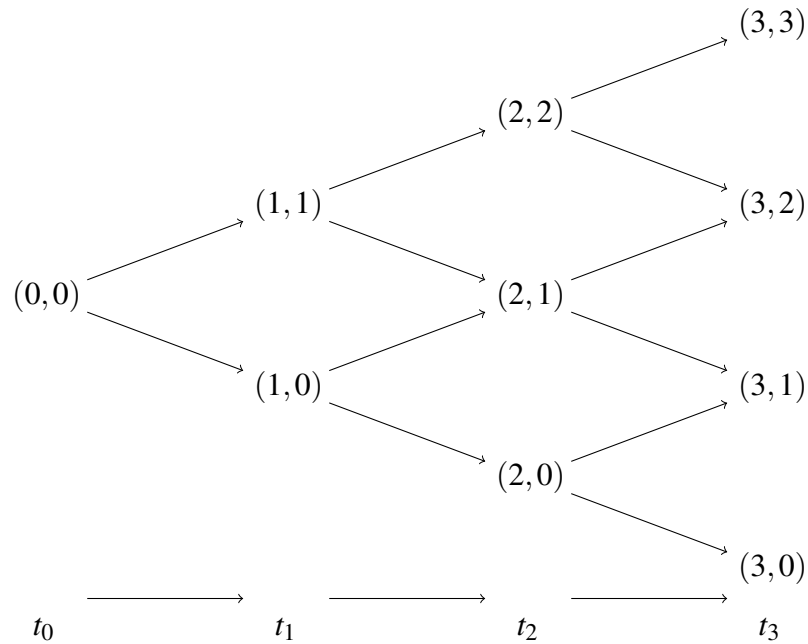
Figure 2.3: **Picture of notation in a binomial tree**

2.9 Examples

Long European Call   We can calculate how a European option should be priced at t_0 in the following way (example from Hull, page 243 [7]).

Let's consider an asset with initial price $S_0 = 20$. The strike price $K = 21$ and the time to maturity is six months. The price of the underlying security will either go up or down by 10%. One time step equals three months, thus we have a two-step tree which is depicted[4] in Figure 2.4.

The pay-off from the call option at maturity is either 3.2 (node D), or zero (nodes E and F). This is calculated applying the formula shown earlier, pay-off long call $= \max\{(S_t - K, 0)\}$, which gives $24.2 - 21 = 3.2$. At nodes E and F the strike price exceeds the spot price, making the call worthless.

Now we work backwards through the tree to get the (hypothetical) price for the option at t_1, i.e., nodes B and C. For that we have to take new parameters into consideration. At first we will calculate the probabilities for up and down movements using (2.7). Let's say that the risk free interest rate $r = 0.12$. We already know that $\Delta t = 0.25$, $u = 1.1$ and $d = 0.9$. Therefore

$$p = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

---

[4]In the second rows we show the asset prices and in the third rows the payoffs.
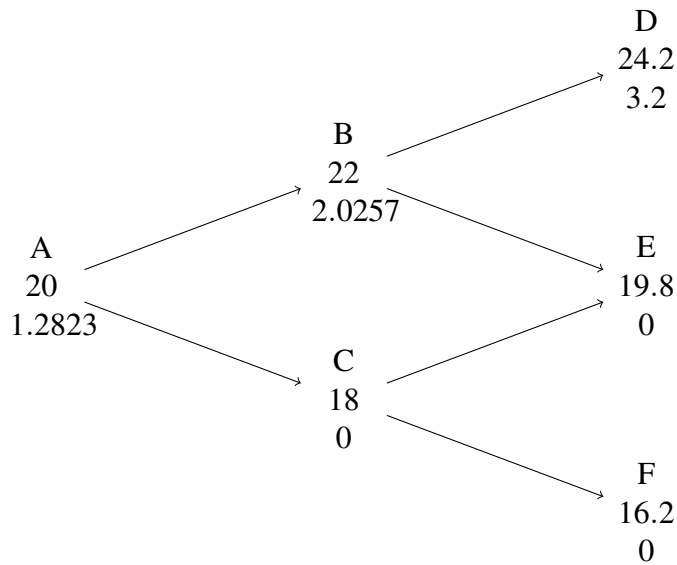
Figure 2.4: **Model for European Call example**

and

$$(1-p) = 1 - 0.6523 = 0.3477$$

Now we can calculate the expected pay-off at node B by discounting the expected pay-off at this point. We calculate the call at node B applying

$$e^{-r\Delta t}(pf_{uu} + (1-p)f_{ud})$$

yielding

$$e^{-0.12 \times 0.25}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$$

Likewise we will obtain the result for the option value at t_0:

$$e^{-0.12 \times 0.25}(0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$$

An easier and faster way to calculate the price of the option is to use the formula

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}] \tag{2.25}$$

which is simply the squared version of (2.24).

Plugging in our variables yields

$$f = e^{-2 \times 0.12 \times 0.25} \left[0.6523^2(3.2) + 2(0.6523)(0.3477)(0) + (0.3477)(2)(0) \right] = 1.2823$$

This is the exact same answer that we obtained using the tree-model for American options. The point of these extra calculations is to show that the value of an option always is the result of iteratively working backwards. Since European options only can be exercised at maturity and not earlier, it is not necessary to do all those intermediate steps and the value of 1.2823 should be computed much faster by (2.25).

Long American Put   To show how the value of an American put is computed, we use the following parameters from Hull, page 247 [7], which is shown in[5] Figure 2.5.

$S_0 = 50$, $K = 52$, $r = 0.05$, $u = 1.2$, $d = 0.8$, $\Delta t = 1$ and $T = 2$ years

The probability of an upward movement equals

$$p = \frac{e^{0.05 \times 1} - 0.8}{1.2 - 0.8} = 0.6282$$

Thus the probability of a downward movement is

$$(1 - p) = 0.3718$$

The corresponding tree shows the price of the underlying asset at each node and the value of the option at maturity. Recall that the pay-off for put options equals $\max\{(K - S_t), 0\}$.

We can see that holding the option to maturity could give a pay-off of either 0, 4 or 20. As in the previous example we can not know how the underlying asset is priced at maturity, but we know that we have the possibility to exercise the option early. Thus we work backwards through the tree to evaluate the option price at each time step. Then we compare the binomial value with the exercise value. The one that is greater tells how to proceed with the option. If the binomial value is greater, we continue to hold the option. At t_1 the option is either out of the money (node B), or it generates a pay-off of 12. We should clearly not exercise if we end up at B. Since the chance of a positive pay-off due to the end of the option is only 37.18%, the value of the put is expected to be low. Calculation yields

$$e^{-0.05 \times 1}(0.6282 \times 0 + 0.3718 \times 4) = 1.4147$$

The binomial value of 1.4147 is greater than the exercise value of zero. The option should therefore be held.

---

[5]In the second rows we show the asset prices and in the third rows the payoffs.
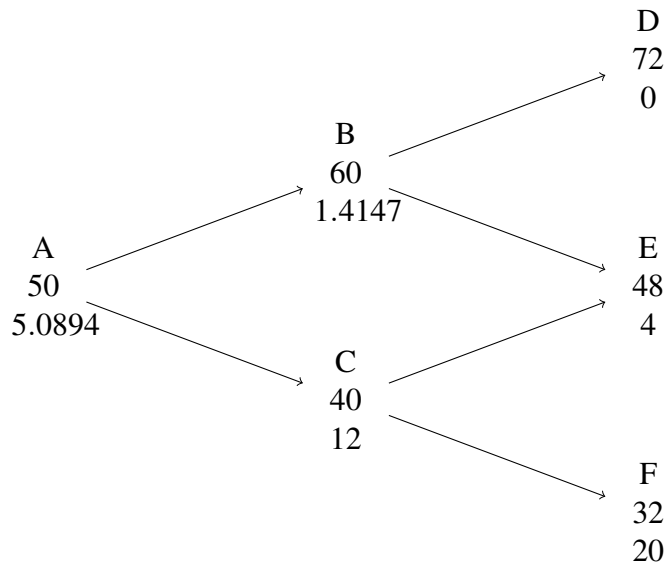
Figure 2.5: **Model for American Put example**

If we ended up at node C, exercising would pay 12, holding the put could increase or decrease pay-off until maturity. The computation of the binomial price gives a value of 9.3646 (for simplicity we skip the actual calculation). This is lower than the exercise price, thus exercising at t_1 is recommended and the value of the option is 12.

Conducting the corresponding computations for node A, we obtain the binomial price of 5.0894 which is compared to early exercise value 2. The binomial price is greater, thus it should not be exercised.

Chapter 3

Convergence of binomial model to Geometric Brownian motion

3.1 Introduction and Background

As we saw in the previous section, in Cox, Ross and Rubinstein's approach, if $n \to \infty$, the behavior of he binomial models can be approximated as a stochastic process in continuous time. Black and Scholes 1973 [1] and Merton 1973 [15] assumed that the dynamic of a risky security price follows a Geometric Brownian Motion. Following the Cox, Ross and Rubinstein's approach it is possible to see that the sequence of the binomial models also converges to a Geometric Brownian Motion [12],[14]. To begin with, we can express some definitions:

Definition 3.1.1. *A stochastic process (Wiener Process) $W(t)$, $0 \le t \le T$, is called a standard Brownian Motion if [12],[6],[14]*
1. $W(0) = 0$.
2. $W(t)$ is continuous on $[0,T]$ with probability 1.
3. $W(t)$ has independent increments.
4. the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$.

Theorem 3.1.1. *$W(t) - W(s)$ is a normal random variable. A Brownian Motion with drift $\mu \in \mathbb{R}$ and $\sigma > 0$ coefficients is*
$$G(t) = \mu t + \sigma W(t)$$
Here μ and σ^2 may be time-dependent. [12],[6],[14].

Definition 3.1.2. *Let $G(t)$ be a Brownian motion with drift coefficient μ and diffusion coefficient σ and $S(0)$ be a positive real number. Then the process*
$$S(t) = S(0)e^{G(t)} = S(0)e^{\mu t + \sigma W(t)} \tag{3.1}$$
is called Geometric Brownian Motion [12],[6],[14].

From now on we denote $X = S_T/S_0$ and $Y = \ln(S_T/S_0)$. Moreover, Y is a random variable which is normally distributed, and X is a random variable which is log-normally distributed. Now we will continue with some investigations on different results for the binomial models. Recalling (2.18) and (2.19) for the random variable X in a one-step binomial tree, we will have[1]:

$$E[X] = E[S_T/S_0] = p^*u + (1-p^*)d$$
$$V[X] = V[(S_T/S_0)] = p^*u^2 + (1-p^*)d^2 - (E[X])^2$$

Substituting the value of $(E[X])^2$ and simplifying $V[X]$ will yield:

$$\begin{aligned}
V[X] &= p^*u^2 + (1-p^*)d^2 - (p^*u + (1-p^*)d)^2 \\
&= p^*u^2 + d^2 - p^*d^2 - p^{*2}u^2 - 2p^*(1-p^*)ud - (1-p^*)^2d^2 \\
&= p^*u^2 - p^{*2}u^2 + p^*d^2 - p^{*2}d^2 - 2p^*ud + 2p^{*2}ud \\
&= p^*(1-p^*)(u-d)^2
\end{aligned}$$

To find the expected value and variance of the random variable Y we will do the following steps. Firstly, we know that the possible stock prices at a one step binomial tree are:

$$S_T = \begin{cases} S_0u & \text{,if stock goes up with probability p} \\ S_0d & \text{,if stock goes down with probability 1-p} \end{cases}$$

Since S_0 is constant we can divide both hand sides with S_0. Additionally, for obtaining random variable Y we then can take the natural logarithm from both hand sides. Doing this will give us:

$$Y = \ln\left(\frac{S_T}{S_0}\right) = \begin{cases} \ln u & \text{,if stock goes up with probability p} \\ \ln d & \text{,if stock goes down with probability 1-p} \end{cases}$$

Finally, considering Definition 2.2.3 and calculating the expected value with (2.5) will give us:

$$E[Y] = E[\ln(S_T/S_0)] = p^*\ln u + (1-p^*)\ln d$$

Considering Theorem 2.2.2 and calculating the variance with (2.6) will yield:

$$V[Y] = V[\ln(S_T/S_0)] = p^*[\ln u]^2 + (1-p^*)[\ln d]^2 - (E[Y])^2$$

Substituting the value of $(E[Y])^2$ and simplifying $V[Y]$ will yield:

$$\begin{aligned}
V[Y] &= p^*[\ln u]^2 + (1-p^*)[\ln d]^2 - [p^*\ln u + (1-p^*)\ln d]^2 \\
&= p^*(\ln u)^2 + (\ln d)^2 - p^*(\ln d)^2 - p^{*2}(\ln u)^2 - 2p^*(1-p^*)\ln u \ln d - (1-p^*)^2(\ln d)^2 \\
&= p^*(\ln u)^2 - p^{*2}(\ln u)^2 + p^*(\ln d)^2 - p^{*2}(\ln d)^2 - 2p^*\ln u \ln d + 2p^{*2}\ln u \ln d \\
&= p^*(1-p^*)\left[(\ln u)^2 - 2\ln u \ln d + (\ln u)^2\right] \\
&= p^*(1-p^*)[\ln u - \ln d]^2
\end{aligned}$$

---

[1]A variable has log-normal distribution if the natural logarithm of the variable is normally distributed [7].

Remark 3.1.1. *Now we have a system of equations for both normal and log-normal random variables whit some unknown parameters p, u and d in a binomial lattice. We can remember that Cox, Ross and Rubinstein also ended up with two equations for expected value and variance and three unknowns p, u and d. They solved this system of equations after calculating p and they let ud = 1 to obtain an equal number of equations and unknowns.*

Now let us consider the sequence of binomial models and its convergence to the Geometric Brownian Motion.

3.2 The sequence of the binomial models and its convergence to Geometric Brownian Motion

In this part we will investigate the sequence of the binomial models and its convergence to Geometric Brownian Motion. To begin with, we can expand the sequence of the random variable Y as follows [12],[14]:

$$E[Y] = E\left[\sum_{k=1}^{t} Y_{n,k}\right] = E\left[\ln \frac{S_{n,t}}{S_{n,0}}\right] = E\left[Y_{n,1} + Y_{n,2} + \dots + Y_{n,t}\right], \quad 1 \le t \le n \qquad (3.2)$$

We have already calculated the expected value for Y, so the expected value at each time will be:

$$E\left[Y_{n,t}\right] = p\ln u_n + (1-p)\ln d_n$$

We have already seen in the CRR model that as $n \to \infty$ the expected value and variance of our process μT and $\sigma^2 T$ [4].

Moreover, we want the binomial model to converge to the Geometric Brownian Motion. So we will have:

$$Y = \mu t + \sigma W(t) \qquad\qquad 0 \le t \le T$$
$$E[Y] = \mu T \qquad\qquad V[Y] = \sigma^2 T$$

Additionally, we know that in an n step binomial tree we have one random variable; let us call it m, which can take a certain number of upward movements. Then the number of downward movements will be $n - m$ [4]. So the possible value for upward movements is $u_n = u^m$ and the possible value for downward movements is $d_n = d^{n-m}$. For simplicity we will denote $x_n = \ln u_n$ and $y_n = \ln d_n$. Using the formulas for variance and expected value in the binomial model we will have [4],[12],[14]

$$E[Y] = n\left[px_n + (1-p)y_n\right] = \mu T$$
$$V[Y] = np(1-p)(x_n - y_n)^2 = \sigma^2 T$$

considering the fact that $d_n < u_n$, we know that $y_n < x_n$. We can re-write the equation as follows and solve the system of two equations with two unknowns.

$$\begin{cases} px_n + (1-p)y_n = \mu T/n \\ \\ x_n - y_n = \sigma\sqrt{\frac{T}{np(1-p)}} \end{cases}$$

Solving this system of equations will give:

$$\begin{cases} x_n = \frac{\mu T}{n} + \sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}} \\ \\ y_n = \frac{\mu T}{n} - \sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}} \end{cases} \tag{3.3}$$

Recall (3.2). Since our sequence is a sequence of independent identically random variables, we will have $nE[Y_{n,1}] = \mu T$ and $nV[Y_{n,1}] = \sigma^2 T$. Now we can apply the central limit theorem [12],[14]

$$\lim_{n \to \infty} P\left\{ \frac{Y_{n,1} + Y_{n,2} + \dots + Y_{n,n} - nE[Y_{n,1}]}{\sqrt{nV[Y_{n,1}]}} \le x \right\} = p\left\{ \frac{\ln(S_T/S_0) - \mu T}{\sigma\sqrt{T}} \le x \right\} = \Phi(x)$$

This proves that binomial models at time T, follow the normal distribution with mean μT and $\sigma^2 T$.

3.3 The sequence of binomial models and its convergence to Black-Scholes model under risk-neutral probability

We have already shown that binomial models at time T converge to the normal distribution with mean μT and variance $\sigma^2 T$. Moreover, recall that Black and Scholes 1973 and Merton 1973 considered that the risky stock follows a Geometric Brownian Motion with drift coefficient μ and diffusion coefficient σ. Black and Scholes proved that in their model, risky stocks are following the Geometric Brownian Motion with mean $\mu = \left(r - \frac{\sigma^2}{2} \right) T$ and variance $\sigma^2 T$ [1]. We will follow Black and Scholes approach and we will derive their pricing formula in the appendix of this thesis. Since we are talking about pricing options via lattice approaches and our random variable S_T is discrete, we would like to investigate if pricing options using the binomial models converges to the Black-Scholes formula where S_T is a continuous random variable. So we will investigate the convergence of the binomial model to the Black-Scholes model under risk neutral probability measure. First, for risk neutral probability measure we have [12],[14]

$$p_n^* = \frac{e^{r\frac{T}{n}} - d_n}{u_n - d_n}, \qquad 1 - p_n^* = \frac{u_n - e^{r\frac{T}{n}}}{u_n - d_n} \tag{3.4}$$

From (3.3) we can obtain [2]:

$$
\begin{cases}
x_n = \ln u_n \Rightarrow u_n = e^{x_n} = \exp\left\{\frac{\mu T}{n} + \sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} \\
y_n = \ln d_n \Rightarrow d_n = e^{y_n} = \exp\left\{\frac{\mu T}{n} - \sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}
\end{cases}
\tag{3.5}
$$

Substituting (3.5) in (3.4), we will obtain:

$$
p_n^* = \frac{e^{r\frac{T}{n}} - d}{u - d} = \frac{\exp\left\{\frac{rT}{n}\right\} - \exp\left\{\frac{\mu T}{n} - \sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}{\exp\left\{\frac{\mu T}{n} + \sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{\frac{\mu T}{n} - \sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}
$$

$$
= \frac{\exp\left\{(r-\mu)\frac{T}{n}\right\} - \exp\left\{-\sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}{\exp\left\{\sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{-\sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}
$$

$$
1 - p_n^* = \frac{u - e^{r\frac{T}{n}}}{u - d} = \frac{\exp\left\{\frac{\mu T}{n} + \sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{\frac{rT}{n}\right\}}{\exp\left\{\frac{\mu T}{n} + \sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{\frac{\mu T}{n} - \sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}
$$

$$
= \frac{\exp\left\{\sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{(r-\mu)\frac{T}{n}\right\}}{\exp\left\{\sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{-\sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}
$$

It can be shown that $\lim_{n\to\infty} p_n^* = p$ and $\lim_{n\to\infty}(1 - p_n^*) = 1 - p$ [12],[14].

Using the last result we can calculate the variance of binomial model as $n \to \infty$ [12],[14]:

$$
\lim_{n\to\infty} V^*[Y] = \lim_{n\to\infty} np^*(1 - p^*)(u_n - \ln d_n)^2 = \lim_{n\to\infty} np(1 - p)(x_n - y_n)^2
$$

---

[2]Rendleman and Bartter 1979 got the same result in [16].

32

Substituting x_n and y_n from (3.3) we will calculate [12],[14]:

$$
\lim_{n\to\infty} V^*[Y] = \lim_{n\to\infty} np(1-p)\left[\left(\frac{\mu T}{n} + \sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right) - \left(\frac{\mu T}{n} - \sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right)\right]^2
$$

$$
= \lim_{n\to\infty} np(1-p)\left[\sigma\sqrt{\frac{T}{n}}\left(\sqrt{\frac{1-p}{p}} + \sqrt{\frac{p}{1-p}}\right)\right]^2
$$

$$
= \lim_{n\to\infty} np(1-p)\frac{\sigma^2 T}{n}\left(\frac{1-p}{p} + 2\sqrt{\frac{1-p}{p}} \times \sqrt{\frac{p}{1-p}} + \frac{p}{1-p}\right)
$$

$$
= \lim_{n\to\infty} p(1-p)\sigma^2 T\left(\frac{(1-p)^2 + p^2}{p(1-p)} + 2\right)
$$

$$
= p(1-p)\sigma^2 T\left(\frac{1}{p(1-p)} - 2\frac{p(1-p)}{p(1-p)} + 2\right) = p(1-p)\sigma^2 T\left(\frac{1}{p(1-p)}\right)
$$

$$
= \sigma^2 T
$$

Secondly, for the expected value we have:

$$
\lim_{n\to\infty} E^*[Y] = \lim_{n\to\infty} n[p^*x_n + (1-p^*)y_n]
$$

$$
= \lim_{n\to\infty} n\left[\left(\frac{\exp\left\{(r-\mu)\frac{T}{n}\right\} - \exp\left\{-\sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}{\exp\left\{\sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{-\sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}\right) \times \left(\frac{\mu T}{n} + \sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right)\right.
$$

$$
\left. + \left(\frac{\exp\left\{\sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{(r-\mu)\frac{T}{n}\right\}}{\exp\left\{\sigma\sqrt{\frac{1-p}{p}}\sqrt{\frac{T}{n}}\right\} - \exp\left\{-\sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right\}}\right) \times \left(\frac{\mu T}{n} - \sigma\sqrt{\frac{p}{1-p}}\sqrt{\frac{T}{n}}\right)\right]
$$

$$
= \left(r - \frac{\sigma^2}{2}\right)T
$$

To solve the last equation the Maclaurin expansion was used. Furthermore, applying the central limit theorem we will obtain:

$$
\lim_{n\to\infty} P^*\left\{\frac{Y - n\mu_n]}{\sigma_n\sqrt{n}} \leq x\right\} = p^*\left\{\frac{\ln(S_T/S_0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \leq x\right\} = \Phi(x)
$$

which means, under risk-neutral probability measure, our stochastic process (binomial models) at time T converges to normal distribution with mean $(r - \frac{\sigma^2}{2})T$ and variance $\sigma^2 T$.

3.4 Mean and variance of a random variable which is log-normally distributed

As we have shown in the previous part, binomial models at time T converges to normal distribution. In lots of scientific fields as well as finance, it is common to calculate and derive the expectation and variance formulas for a random variable which is log-normally distributed. So we will try to show how the expected value and variance of a random variable can be derived from a normal distribution. To begin with we write some definitions [21].

Definition 3.4.1. *A random variable U is said to have a normal probability distribution if and only if, for $\sigma > 0$ and $-\infty < \mu < \infty$, the density function of U is:*

$$f(u) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(u-\mu)^2/(2\sigma^2)}, \quad -\infty < u < \infty$$

and the following theorem tells us [21]:

Theorem 3.4.1. *If U is a normally distributed random variable with parameter μ and σ, then:*

$$E[U] = \mu \quad \text{and} \quad V[U] = \sigma^2$$

It is possible to transform a normal random variable U to a standard normal random variable Z by [21]:

$$Z = \frac{U - \mu}{\sigma}$$

Applying Definition 3.4.1 and Theorem 3.4.1 to our random variable Y, we will have:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(y-\mu_y)^2/(2\sigma_y^2)}, \quad -\infty < y < \infty$$

where we have already calculated the mean and variance of binomial models at time T and its convergence to normal distribution:

$$E[Y] = \mu_y = \mu T \quad \text{and} \quad V[Y] = \sigma_y^2 = \sigma^2 T$$

Now, we can continue with another definition [21]:

Definition 3.4.2. *If a random variable Y is normally distributed with mean μ_y and variance σ_y^2 and $X = e^Y$ [equivalently, $Y = \ln X$], then X is said to have a log-normal distribution. Then the density function for X is:*

$$f(x) = \begin{cases} \left(\frac{1}{x\sigma\sqrt{2\pi}}\right)e^{-(\ln x - \mu_y)^2/(2\sigma_y^2)}, & x > 0 \\ 0, & \textit{elsewhere.} \end{cases} \tag{3.6}$$

Corollary 3.4.1. *If Y is normally distributed with mean μ_y and variance σ_y^2. Then the expected value and variance of the log-normal distribution for a random variable X, where $X = e^Y$ [equivalently, $Y = \ln X$], are given by*

$$E[X] = e^{(\mu_y + \sigma^2/2)} \quad and \quad V[X] = (E[X])^2 (e^{\sigma_y^2} - 1)$$

Proof. We know that the expected value of a continuous random variable is:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

where $f(x)$ is the density function of the random variable x. Substituting (3.6), we will get:

$$E[X] = \int_{-\infty}^{\infty} x \left(\frac{1}{x \sigma_y \sqrt{2\pi}} \right) e^{-(\ln x - \mu_y)^2/(2\sigma_y^2)} dx$$

Now, we can use the property of the moment generating function and calculate $E[e^Y]$ instead [21]. Then we will have $x = e^y \Rightarrow dx = e^y dy$. Substituting we will obtain:

$$E[X] = E[e^Y] = \int_{-\infty}^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} e^{-(y-\mu_y)^2/(2\sigma_y^2)} (e^y dy)$$

Again, for simplicity we can change the variable $z = y - \mu_y \Rightarrow dz = dy$ and $y = z + \mu \Rightarrow e^y = e^{\mu_y + z}$. So the expected value will be:

$$E[X] = E[e^Y] = \int_{-\infty}^{\infty} e^{\mu_y + z} \frac{1}{\sigma_y \sqrt{2\pi}} e^{-z^2/(2\sigma_y^2)} dz = e^{\mu_y} \int_{-\infty}^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} e^{z - z^2/(2\sigma_y^2)} dz$$

Then we can do as follow:

$$\frac{z - z^2}{(2\sigma_y^2)} = - \left[\left(\frac{z}{\sqrt{2}\sigma_y} \right)^2 - z + \left(\frac{\sqrt{2}\sigma_y}{2} \right)^2 - \left(\frac{\sqrt{2}\sigma_y}{2} \right)^2 \right]$$

$$= - \left[\left(\frac{z}{\sqrt{2}\sigma_y} - \frac{\sqrt{2}\sigma_y}{2} \right)^2 - \frac{\sigma_y^2}{2} \right] = - \left[\left(\frac{z - \sigma_y^2}{\sqrt{2}\sigma_y} \right)^2 - \frac{\sigma_y^2}{2} \right]$$

$$= \frac{-(z - \sigma_y^2)^2}{2\sigma_y^2} + \frac{\sigma_y^2}{2}$$

Now, let us denote $w = (z - \sigma_y^2) \Rightarrow dz = dw$. Substituting w and dw, our integral will change as follows:

$$E[X] = E[e^Y] = \frac{e^{(\mu_y + \sigma_y^2/2)}}{\sigma_y \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-w^2/(2\sigma_y^2)} dw$$

Solving this integral is complicated, but we can use a typical trick and calculate the square value of the expected value and then the positive square root of the result will be the answer. So, we will have:

$$(E[X])^2 = (E[e^Y])^2 = \frac{e^{(2\mu_y + \sigma_y^2)}}{2\pi\sigma_y^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(v^2 + w^2)/(2\sigma_y^2)} dv dw$$

To solve this integral we can use polar form and change variables $v = r\cos\theta$, $w = r\sin\theta$, $dvdw = det(J)drd\theta$ and $det(J) = r$. Where $det(J)$ is the Jacobian. So we will have:

$$v^2 + w^2 = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2(\cos^2 + \sin^2) = r^2$$

and the integral will be:

$$(E[X])^2 = (E[e^Y])^2 = \frac{e^{(2\mu_y + \sigma_y^2)}}{2\pi\sigma_y^2} \int_0^{\infty} \int_0^{2\pi} re^{-r^2/(2\sigma_y^2)} drd\theta$$

$$= \frac{e^{(2\mu_y + \sigma_y^2)}}{\sigma_y^2} \int_0^{\infty} re^{-r^2/(2\sigma_y^2)} dr$$

$$= -e^{(2\mu_y + \sigma_y^2)} \int_0^{\infty} \frac{-2r}{2\sigma_y^2} e^{-r^2/(2\sigma_y^2)} dr$$

$$= -e^{(2\mu_y + \sigma_y^2)} \left[e^{-r^2/(2\sigma_y^2)} \right]_0^{\infty}$$

$$= -e^{(2\mu_y + \sigma_y^2)} \left(\lim_{r\to\infty} e^{-r^2/(2\sigma_y^2)} - e^0 \right)$$

$$= -e^{(2\mu_y + \sigma_y^2)} (0 - 1) = e^{(2\mu_y + \sigma_y^2)}$$

taking square root of the last result, will yield:

$$E[X] = E[e^Y] = \left[e^{(2\mu_y + \sigma_y^2)} \right]^{1/2} = e^{(\mu_y + \sigma_y^2/2)}$$

Substituting the value for $\sigma_y^2 = \sigma^2 T$ and $\mu_y = \mu T$, we will obtain the mean of the log-normal distribution.

$$E[X] = e^{(\mu + \frac{1}{2}\sigma^2)T}$$

To calculate the variance we have:

$$V[X] = E[(X)^2] - (E[X])^2 \tag{3.7}$$

We have already known the result of $(E[X])^2$. To calculate the $E[X^2]$ we can use the property of the moment generating function [21] and we will have:

$$E[X^2] = E[e^{2Y}] = \int_{-\infty}^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} e^{-(y-\mu_y)^2/(2\sigma_y^2)} (e^{2y} dy)$$

Again, for simplicity we can change the variable $u = y - \mu_y \Rightarrow du = dy$ and $y = u + \mu \Rightarrow e^{2y} = e^{2(\mu_y + z)}$. So, the expected value will be:

$$E[X^2] = E[e^{2Y}] = \int_{-\infty}^{\infty} e^{2(\mu_y + u)} \frac{1}{\sigma_y \sqrt{2\pi}} e^{-u^2/(2\sigma_y^2)} du$$

$$= e^{2\mu_y} \int_{-\infty}^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} e^{2u - u^2/(2\sigma_y^2)} du$$

$$= e^{2\mu_y} \int_{-\infty}^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} e^{\frac{-(u^2 - 4u\sigma_y^2)}{2\sigma_y^2}} du$$

$$= e^{2\mu_y} \int_{-\infty}^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} e^{\frac{-(u^2 - 4u\sigma_y^2 + (2\sigma_y^2)^2 - (2\sigma_y^2)^2)}{2\sigma_y^2}} du$$

$$= e^{2(\mu_y + \sigma_y^2)} \int_{-\infty}^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} e^{\frac{-(u - 2\sigma_y^2)^2}{2\sigma_y^2}} du$$

$$= e^{2(\mu_y + \sigma_y^2)}$$

Substituting the value for $E[X^2] = e^{2(\mu_y + \sigma_y^2)}$ in (3.7), we will get:

$$V[X] = e^{2(\mu_y + \sigma_y^2)} - e^{(2\mu_y + \sigma_y^2)} = e^{2\mu_y + \sigma_y^2}\left(e^{\sigma_y^2} - 1\right)$$

And finally, substituting $\sigma_y^2 = \sigma^2 T$ and $\mu_y = \mu T$, we will get:

$$V[X] = e^{(2\mu + \sigma^2)T}\left(e^{\sigma^2 T} - 1\right)$$

$\square$

Remark 3.4.1. *Calculating such integrals is usual for continuous random variables. So, having proper skills to calculate ordinary and stochastic integrals is vital for a financial analyzer.*

Remark 3.4.2. *Often in literature we can see that the authors do not calculate the integrals; they jumped to the results. Considering the calculation above we can explain it. As we saw a density function for a normal random variable $X \sim N[\mu, \sigma]$ is given by $\phi(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, then considering the calculations above, we know that the distribution function of a normal random variable has the value of $\Phi(X) = \int_{-\infty}^{X} \phi(x) dx$. As an example, for a standard normal random variable Z, where $\mu = 0$ and variance is $\sigma = 1$ or equivalently $Z \sim N[0, 1]$, we can directly say that the density function of a standard normal random variable Z is $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and $\Phi(Z) = \int_{-\infty}^{Z} \phi(z) dz$. So, if we can make the form of our integral like an integral of a density function for the standard normal random variable, then we can find the probability (area under normal curve) from the standard normal probability table. Additionally, if we have a normal random variable instead of standard normal variable, we can transform the normal variable to standard normal random variable and again use the table to find the probability. Finally, it is easy to say the area under the any density function is one or equivalently $\int_{-\infty}^{\infty} \phi(z) dz = 1$ [21].*

Chapter 4

Different approaches on Binomial Models

In the previous chapter we calculated the expected value and variance of the normal and log-normal distribution for normal and log-normal random variables. Then we calculated the expected value and variance of the binomial models which converges to a Geometric Brownian Motion. Moreover, we have seen that the sequence of binomial models at time T converges to the Geometric Brownian Motion under risk-neutral probability. So if we want to estimate binomial models with Log-normal distribution and at the same time we want that our model converges to a Geometric Brownian Motion, we can substitute the expected value[1] $\mu_y = \left(r - \frac{\sigma^2}{2} \right) T$ and variance $\sigma_y^2 = \sigma^2 T$ of the Geometric Brownian Motion to our expected value and variance formulas which we have obtained for log-normal distribution [17]. Thus we will have:

$$\lim_{n \to \infty} E^*[X] = \lim_{n \to \infty} n[pu + (1 - p)d]$$
$$= e^{(\mu_y + \frac{1}{2}\sigma_y^2)} = e^{(r - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2)T} = e^{rT}$$
$$\lim_{n \to \infty} V^*[X] = \lim_{n \to \infty} np(1 - p)(u - d)^2$$
$$= e^{(2\mu_y + \sigma_y^2)} \left(e^{\sigma_y^2} - 1 \right) = e^{(2r - \sigma^2 + \sigma^2)T} \left(e^{\sigma^2 T} - 1 \right)$$
$$\lim_{n \to \infty} V^*[X] = e^{2rT} \left(e^{\sigma^2 T} - 1 \right)$$
$$X \sim LN \left[rT, \sigma^2 T \right]$$

Again, we will have two equations for expected value and variance and three unknowns u, d and p. We can choose different value for either p, u or d to obtain our third equation and make a survey of different binomial models where we want results to converge to the Geometric

---

[1] We know that in risk neutral probability measure, or equivalent Martingale probability measure, the drift coefficient is equal to the risk-free interest rate.

Brownian Motion [17]. Additionally, our calculation was for n step binomial models. Considering this fact, we can say that in each step we will have $\Delta t = T/n$, and we can calculate the expected value and variance for each step in the binomial tree. So for $X = \frac{S_{i+1}}{S_i}$ which is log-normally distributed we will have [17]:

$$X = \frac{S_{i+1}}{S_i}$$
$$E[X] = pu + (1-p)d = e^{r\Delta t}$$
$$V[X] = p(1-p)(u-d)^2 = e^{2r\Delta t}\left(e^{\sigma^2\Delta t} - 1\right)$$
$$X \sim LN\left[r\Delta t, \sigma^2\Delta t\right]$$

For $Y = \ln\left(\frac{S_{i+1}}{S_i}\right)$ which is normally distributed, we will have:

$$Y = \ln\left(\frac{S_{i+1}}{S_i}\right)$$
$$E[Y] = p\ln u + (1-p)\ln d = (r - \frac{\sigma^2}{2})\Delta t$$
$$V[Y] = p(1-p)\left[\ln u - \ln d\right]^2 = \sigma^2\Delta t$$
$$Y \sim N\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2\Delta t\right]$$

Now we have the system of two equations for both normal and log-normal random variables whit three unknown parameters p, u and d in a binomial lattice. IN next two sections we will introduce the third equation when the stock price is normally and log-normally distributed by $p = 1/2$ and $ud = 1$ and we will try to calculate p, u and d. Then we will make a survey on some well-known and famous binomial models[2][17].

4.1 Random variable $Y = \ln\left(\frac{S_{i+1}}{S_i}\right)$ is normally distributed

When the random variable $Y = \ln\left(\frac{S_{i+1}}{S_i}\right)$ is normally distributed, we will have two equations for expected value and variance and three unknowns p, u and d. Here we will introduce one extra equation to increase our equations to three and then we will solve a system of three equations and three unknowns.

---

[2]We had already written a preliminary version of our thesis when we found an interesting paper "Two-State Option Pricing: Binomial Models Revisited" by Jabbour, Kramin and Young [8]. We would suggest that the reader looks at that paper as well.

4.1.1 Introducing the third equation by $p = 1/2$

Writing the equations for the expected value and variance of the random variable Y, which is normally distributed, we will have two equations and three unknowns:

$$E[Y] = p \ln u + (1 - p) \ln d = (r - \frac{\sigma^2}{2}) \Delta t$$
$$V[Y] = p(1 - p)[\ln u - \ln d]^2 = \sigma^2 \Delta t$$

Substituting $p = 1/2$ we will get the system of two equations with two unknowns.

$$\begin{cases} (\ln u - \ln d)^2 = 4\sigma^2 \Delta t \\ \ln u + \ln d = (2r - \sigma^2) \Delta t \end{cases}$$

We denote $x = \ln u$ and $y = \ln d$. Moreover, since $u > d \Rightarrow x > y$, we will have:

$$\begin{cases} x - y = 2\sigma \sqrt{\Delta t} \\ x + y = (2r - \sigma^2) \Delta t \end{cases}$$

we will obtain:

$$x = (r - \frac{\sigma^2}{2}) \Delta t + \sigma \sqrt{\Delta t} \Rightarrow u = e^{(r - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}}$$
$$y = (r - \frac{\sigma^2}{2}) \Delta t - \sigma \sqrt{\Delta t} \Rightarrow d = e^{(r - \frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t}}$$

Remark 4.1.1. *The value for p is different in the Jarrow-Rudd model, but as we will see later, the values for the u and d factors in our approach are exactly the same as the values for u and d factors which have been derived by Jarrow and Rudd. However, Jarrow and Rudd have shown that p = 1/2 as $\Delta t \to 0$ [10],[9].*

Remark 4.1.2. *We can consider the formula which was obtained by Rendleman and Bartter [16], which is as follows*

$$\begin{cases} u = \exp\left\{ \frac{\mu T}{n} + \sigma \sqrt{\frac{1-p}{p}} \sqrt{\frac{T}{n}} \right\} \\ d = \exp\left\{ \frac{\mu T}{n} - \sigma \sqrt{\frac{p}{1-p}} \sqrt{\frac{T}{n}} \right\} \end{cases}$$

Where p is unknown. Susbtituting p = 1/2 we will obtain exactly the same result for u and d in our calculations.

Remark 4.1.3. *In this model $ud = e^{(2r - \sigma^2)\Delta t}$, whereas in the CRR model $ud = 1$.*

4.1.2 Introducing the third equation by $ud = 1$

Again, we have two equations and three unknowns.

$$E[Y] = p \ln u + (1 - p) \ln d = (r - \frac{\sigma^2}{2}) \Delta t$$
$$V[Y] = p(1 - p)[\ln u - \ln d]^2 = \sigma^2 \Delta t$$

Substituting $d = 1/u$ we will get a system of two equations with two unknowns.

$$\begin{cases} p(1-p)(\ln u - \ln u^{-1})^2 = p(1-p)(\ln u + \ln u)^2 = \sigma^2 \Delta t \\ p \ln u + (1-p)\ln u^{-1} = p \ln u - (1-p)\ln u = (r - \frac{\sigma^2}{2})\Delta t \end{cases}$$

We denote $x = \ln u$ and $y = \ln d$. Moreover, since $u > d \Rightarrow x > y$, we will have:

$$\begin{cases} 4p(1-p)x^2 = \sigma^2 \Delta t \\ px - (1-p)x = x[p - (1-p)] = (r - \frac{\sigma^2}{2})\Delta t \end{cases}$$

we will obtain:

$$x = \frac{(r - \frac{\sigma^2}{2})\Delta t}{2p - 1}$$

$$p - p^2 = \frac{\sigma^2 \Delta t}{4x^2}$$

Substituting and denoting $a = (r - \frac{\sigma^2}{2})^2 \Delta t$ we can solve the equations above for p:

$$p^2 - p + \frac{\sigma^2 \Delta t}{4 \left(\frac{(r - \frac{\sigma^2}{2})\Delta t}{2p - 1} \right)^2} = 0$$

$$p^2 - p + \frac{\sigma^2 (4p^2 - 4p + 1)}{4(r - \frac{\sigma^2}{2})^2 \Delta t} = 0 \Rightarrow ap^2 - ap + \sigma^2 p^2 - \sigma^2 p + \frac{\sigma^2}{4} = 0$$

$$p^2(a + \sigma^2) - p(a + \sigma^2) + \sigma^2 = 0$$

$$p^2 - p + \frac{\sigma^2}{4(a + \sigma^2)} = 0$$

then

$$p_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4\sigma^2/4(a + \sigma^2)}}{2}$$

$$p = \frac{1}{2} \pm \frac{1}{2} \times \sqrt{\frac{a + \sigma^2 - \sigma^2}{a + \sigma^2}} = \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{a}}{\sqrt{a + \sigma^2}}$$

$$p_{1,2} = \frac{1}{2} \pm \frac{1}{2} \times \frac{\sqrt{\Delta t}(r - \frac{\sigma^2}{2})}{\sqrt{\Delta t}\sqrt{\frac{\sigma^2}{\Delta t} + (r - \frac{\sigma^2}{2})^2}}$$

$$p = \frac{1}{2} + \frac{1}{2} \times \frac{(r - \frac{\sigma^2}{2})}{\sqrt{\frac{\sigma^2}{\Delta t} + (r - \frac{\sigma^2}{2})^2}}$$

41

To calculate x we can use the variance formula which is much more convenient and will help us to avoid the quadratic calculation.

$$V[Y] = p[\ln(u)]^2 + (1-p)[\ln(d)]^2 - (E[Y])^2 = \sigma^2 \Delta t$$

$$p[\ln(u)]^2 + (1-p)[\ln(d)]^2 - \left[(r - \frac{\sigma^2}{2})\Delta t\right]^2 = \sigma^2 \Delta t$$

$$p[\ln(u)]^2 + (1-p)[\ln(u^{-1})]^2 - \left[(r - \frac{\sigma^2}{2})\Delta t\right]^2 = \sigma^2 \Delta t$$

$$p[\ln(u)]^2 + (1-p)[-ln(u)]^2 - \left[(r - \frac{\sigma^2}{2})\Delta t\right]^2 = \sigma^2 \Delta t$$

$$px^2 + (1-p)(-x)^2 - \left[(r - \frac{\sigma^2}{2})\Delta t\right]^2 = \sigma^2 \Delta t$$

$$x = \sqrt{\sigma^2 \Delta t + \left[(r - \frac{\sigma^2}{2})\Delta t\right]^2}$$

$$u = e^{\sqrt{\sigma^2 \Delta t + \left[(r - \frac{\sigma^2}{2})\Delta t\right]^2}}$$

$$d = e^{-\sqrt{\sigma^2 \Delta t + \left[(r - \frac{\sigma^2}{2})\Delta t\right]^2}}$$

Remark 4.1.4. *Our calculation to derive p raise a controversial question. Can we obtain two answers for p which both lie between zero and one? Perhaps yes. Because, the term $\frac{1}{2} \times \frac{(r - \frac{\sigma^2}{2})}{\sqrt{\frac{\sigma^2}{\Delta t} + (r - \frac{\sigma^2}{2})^2}}$ must lie between $-1/2$ and $1/2$, since, the probability p must lie between zero and one. So perhaps we can say:*
$p_{1,2} = \frac{1}{2} \pm \frac{1}{2} \times \frac{(r - \frac{\sigma^2}{2})}{\sqrt{\frac{\sigma^2}{\Delta t} + (r - \frac{\sigma^2}{2})^2}}$
So, if we choose a negative sign, then p can be close to zero and if we choose a plus sign, p can be close to one, which is what we expected.

Remark 4.1.5. *In this model we have the same arguments as in all binomial models which we have considered. If $n \to \infty$ then $\Delta t \to 0$ then $\lim_{n\to\infty} p_n = 1/2$.*

4.2 Random variable $X = \frac{S_{i+1}}{S_i}$ is log-normally distributed

When the random variable $X = \frac{S_{i+1}}{S_i}$ is log-normally distributed, we will have two equations for expected value and variance and three unknowns p, u and d. Here we will introduce one extra equation to increase our equations to three and then we will solve a system of three equations and three unknowns.

4.2.1 Introducing the third equation by $p = 1/2$

Writing a system of two equations and three unknowns, we will have:

$$E[X] = pu + (1-p)d = e^{r\Delta t}$$
$$V[X] = p(1-p)(u-d)^2 = e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right)$$

Substituting $p = 1/2$ we will obtain:

$$u + d = 2e^{r\Delta t} \Rightarrow u = 2e^{r\Delta t} - d$$
$$(u-d)^2 = 4e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right)$$

substituting u:

$$(2e^{r\Delta t} - d - d)^2 = (2e^{r\Delta t} - 2d)^2$$
$$= (4e^{2r\Delta t} - 8de^{r\Delta t} + 4d^2) = 4e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right)$$
$$\Rightarrow d^2 - 2e^{r\Delta t}d - e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 2\right) = 0$$
$$d_{1,2} = \frac{2e^{r\Delta t} \pm \sqrt{(-2e^{r\Delta t})^2 + 4e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 2\right)}}{2}$$
$$d_{1,2} = e^{r\Delta t} \pm \sqrt{e^{2r\Delta t} + e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 2\right)}$$
$$d_{1,2} = e^{r\Delta t} \pm e^{r\Delta t}\sqrt{e^{\sigma^2 \Delta t} - 1}$$

since $d < u$ then:

$$u = e^{r\Delta t}\left(1 + \sqrt{e^{\sigma^2 \Delta t} - 1}\right)$$
$$d = e^{r\Delta t}\left(1 - \sqrt{e^{\sigma^2 \Delta t} - 1}\right)$$

Remark 4.2.1. *In this model, we will have:*

$$ud = e^{2r\Delta t}\left(1^2 - \left[\sqrt{e^{\sigma^2 \Delta t} - 1}\right]^2\right) = e^{2r\Delta t}\left(2 - e^{\sigma^2 \Delta t}\right)$$

4.2.2 Introducing the third equation by $ud = 1$

Solving the following equation for p we will have:

$$E[X] = pu + (1-p)d = e^{r\Delta t}$$
$$\Rightarrow pu + d - pd = p(u-d) + d = e^{r\Delta t}$$
$$\Rightarrow p = \frac{e^{r\Delta t} - d}{u - d} = \frac{ue^{r\Delta t} - 1}{u^2 - 1}$$

and solving the variance equation for u we will get:

$$V[X] = p(1-p)(u-d)^2 = e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right)$$
$$\Rightarrow \frac{ue^{r\Delta t} - 1}{u^2 - 1} \times \frac{u(u - e^{r\Delta t})}{u^2 - 1} \times \left(\frac{u^2 - 1}{u}\right)^2 = e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right)$$
$$\Rightarrow \frac{u^2 e^{r\Delta t} - ue^{2r\Delta t} - u + e^{r\Delta t}}{u} = e^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right)$$
$$\Rightarrow u^2 e^{r\Delta t} - ue^{2r\Delta t} - u + e^{r\Delta t} - ue^{2r\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right) = 0$$
$$\Rightarrow u^2\left(e^{r\Delta t}\right) - u\left(e^{(2r+\frac{\sigma^2}{2})\Delta t} + 1\right) + e^{r\Delta t} = 0$$
$$\Rightarrow u_{1,2} = \frac{\left(e^{(2r+\frac{\sigma^2}{2})\Delta t} + 1\right) \pm \sqrt{\left(e^{(2r+\frac{\sigma^2}{2})\Delta t} + 1\right)^2 - 4e^{2r\Delta t}}}{2e^{r\Delta t}}$$
$$u = \frac{1}{2}e^{-r\Delta t}\left[e^{(2r+\sigma^2)\Delta t} + 1\right] + \sqrt{\frac{1}{4}e^{-2r\Delta t}\left[e^{(2r+\sigma^2)\Delta t} + 1\right]^2 - 1}$$
$$d = \frac{1}{2}e^{-r\Delta t}\left[e^{(2r+\sigma^2)\Delta t} + 1\right] - \sqrt{\frac{1}{4}e^{-2r\Delta t}\left[e^{(2r+\sigma^2)\Delta t} + 1\right]^2 - 1}$$

Remark 4.2.2. *As we can see in this model, the value of p is equal to the risk-neutral probability measure in the CRR model, but the u and d factors are different. The reason is, in our equation for variance we have a different value compared to the CRR model because from the beginning, we constructed our model in such a way that our binomial model converges to the Geometric Brownian Motion.*

4.3 The Jarrow-Rudd model

This model was proposed by Jarrow and Rudd [10]. However, we used additional literature to study this model [9],[17]. We have already calculated the expected value and variance of a random variable which is log-normally distributed. Using the previous result we have [9]:

$$E[X] = E\left[\frac{S_T}{S_0}\right] = \exp\left\{\mu T + \frac{\sigma^2}{2}T\right\}$$

Since S_0 is constant, the equation above can be rewritten as:

$$E[S_T] = S_0 \exp\left\{\mu T + \frac{\sigma^2}{2}T\right\}$$

Moreover, we know that the option pricing follows a martingale process. Thus under the equivalent martingale probability measure we have [9]:

$$E^{p^*}[S_T|S_0] = S_0 \exp\left\{\hat{\mu} T + \frac{\sigma^2}{2}T\right\} \tag{4.1}$$

and considering the fact that in the risk-neutral world the expected return on a stock must be equal to the risk-free interest rate, we can write the next equation as follows [9]:

$$S_0 = e^{-rT}E^{p^*}[S_T|S_0] \tag{4.2}$$

Substituting (4.1) into (4.2) we will obtain $\hat{\mu} = r - \frac{\sigma^2}{2}$, which means in the risk-neutral world that the drift coefficient must be equal to $r - \frac{\sigma^2}{2}$.

Now we can approximate this process in the discrete binomial model. It follows [9]:

$$S_{t+1} = S_t \begin{cases} u = e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}} & \text{with probability } p^* \\ d = e^{(r-\frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t}} & \text{with probability } 1 - p^* \end{cases}$$

where $p^* = \frac{e^{rT}-d}{u-d}$, and $Y = \frac{S_{t+1}}{S_t} \sim N\left[\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2\Delta t\right]\right.$. If we substitute the value of u and d in our probability equation we will obtain [9]:

$$p^* = \frac{e^{\frac{\sigma^2}{2}\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

Moreover, if we calculate the limit, we will obtain [9]:

$$\lim_{\Delta t \to 0} p^* = \lim_{\Delta t \to 0} \frac{e^{\frac{\sigma^2}{2}\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \frac{1}{2}$$

Remark 4.3.1. *As we can see, Jarrow and Rudd used a significantly different approach from Black and Scholes. However, they find the same result for the drift and diffusion coefficient under risk-neutral probability measure.*

Remark 4.3.2. *In the Jarrow-Rudd model $ud = e^{(2r-\sigma^2)\Delta t}$, whereas in the CRR model $ud = 1$.*

4.4 The Tian model

Tian introduce his binomial model in discrete time with following p, u and d factors:

$$p = \frac{M-d}{u-d}, \quad q = 1-p = \frac{u-M}{u-d}$$

$$u = \frac{MV}{2}\left[V+1+\sqrt{V^2+2V+3}\right]$$

$$d = \frac{MV}{2}\left[V+1-\sqrt{V^2+2V+3}\right]$$

Now, let's see how he derived equations above and what M and V stand for.

Tian constructed his model assuming that in a risk neutral world, the stock price is following a stochastic process which is given by following the stochastic differential equation [19]:

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW$$

Then he considered the logarithmic transformation of the process above. To do so, we denote $F = \ln(S(t))$ and then we will apply Itô's formula to the SDE above, so we will have [19]:

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2$$

$$dF = 0 + \frac{1}{S}dS - \frac{1}{2S^2}(dS)^2$$

$$dF = \frac{1}{S}\left(rSdt + \sigma SdW\right) - \frac{1}{2S^2}\left(\sigma^2 S^2(dW)^2\right)$$

$$dF = (rdt + \sigma dW) - \frac{1}{2}\sigma^2 dt$$

$$dF = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW$$

$$d\ln S(t) = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW \tag{4.3}$$

Notice that $(dW)^2 = dt$ and $(dt)^2 = dtdW = 0$. Considering (4.3) and the discussion we had before, it is obvious that in (4.3) the one dimensional distribution is following a normal distribution with mean $(r - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$. Thus the mth non-central moment of the stock price can be interpreted by [19]:

$$E[S(t)^m|S_0] = S_0^m \exp\left\{\left(mr + (m-1)\frac{\sigma^2}{2}\right)t\right\}$$

This is a formula for a stochastic process in continuous time. To approximate this formula in discrete time with an n-step binomial model, we can say that $\Delta t = \frac{T}{n}$. We assumed that the price of a stock follows a binomial process which sometimes is called a two-jump process[3]. Now, as we have discussed before, we want our binomial model to converge to the log-normal distribution in continuous time. So, we have to choose the parameters u, d, p and q in a way that guarantees our objective. Using our knowledge and the proofs we did before, we know that in the binomial models, our random variable Y, is normally distributed. Moreover, using the result for the transformation of random variable X which is log-normally distributed, we can express the expected value and variance of binomial models at time T when our random variable is log-normally distributed. Thus we can write the following three equations [19]:

$$p+q = 1$$
$$pu + qd = e^{r\Delta t} = M$$
$$pu^2 + qd^2 = e^{(2r+\sigma^2)\Delta t} = M^2 V$$

Tian denotes $M = e^{r\Delta t}$ and $V = e^{\sigma^2 \Delta t}$. Substituting $q = 1 - p$ we have two equations and three unknowns. Cox, Ross and Rubinstein considered a recombining tree with $u = 1/d$, but Tian considered that the third moments of the discrete time process is also correct according to a continuous time process, so he obtained the forth equation [19]:

$$pu^3 + (1-p)d^3 = M^3 V^3$$

Solving this system of equations, Tian got [19]:

$$p = \frac{M-d}{u-d}, \quad q = 1-p = \frac{u-M}{u-d}$$
$$u = \frac{MV}{2}\left[V + 1 + \sqrt{V^2 + 2V + 3}\right]$$
$$d = \frac{MV}{2}\left[V + 1 - \sqrt{V^2 + 2V + 3}\right]$$

Remark 4.4.1. *By comparing and contrasting the CCR model and the Tian model, we can say that in the CRR model the variance is correct in the limit case and when $\Delta t \to 0$, whereas in the Tian model, both mean and variance are true for any given Δt [19].*

Remark 4.4.2. *Tian chose the correct third moment, but CCR chose $ud = 1$ for simplicity. In the Tian model we have $ud = (MV)^2$ [19].*

---

[3]See [19]

4.5 The Trigeorgis model

Trigeorgis began with the design of the log-transformed binomial model. Again, he considered the diffusion process as follows [20]:

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW$$

where α is expected value, σ is standard deviation and W is a Wiener process. Moreover, $X(t) \equiv \ln S(t)$ follows Brownian motion. So under risk-neutral probability we will have $\alpha = r$ and our process when, $S(t)$ is log-normally distributed, will be [20]:

$$dX = \ln\left(\frac{S(t+dt)}{S(t)}\right) = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW$$

Again, in this process, $\ln\left(\frac{S(t+dt)}{S(t)}\right)$ or equivalently dX is normally distributed. So, we have [20]:

$$E[dX] = \left(r - \frac{\sigma^2}{2}\right)dt, \quad V[dX] = \sigma^2 dt$$

In order to approximate our continuous process in discrete time in each sub-interval $\Delta t = T/n$, X will follow a Markov random walk which with risk-neutral probability p goes up by the amount $\Delta X = H \equiv u$ and with risk-neutral probability $1 - p$ goes down by the amount of $\Delta X = -H \equiv d$ [20]. It is easy to see that this binomial model is recombining. Thus calculating the expected value and variance for this Markov process we will obtain [20]

$$E[\Delta X] = pH + (1-p)(-H) = 2pH - H \equiv pu + (1-p)d$$
$$V[\Delta X] = p(1-p)(2H)^2 = 4pH^2 - 4p^2H^2 \equiv p(1-p)(u-d)$$
$$V[\Delta X] = H^2 - (H^2 - 4pH^2 + 4p^2H^2) = H^2 - (2pH - H)^2 = H^2 - (E[\Delta X])^2$$

We will obtain a system of two equations with two unknowns [20]:

$$2pH - H = 2p\Delta X - \Delta X = \Delta X(2p-1) = \left(r - \frac{\sigma^2}{2}\right)\Delta t$$
$$4pH^2 - 4p^2H^2 = 4p(\Delta X)^2(1-p) = \sigma^2 \Delta t$$

Solving the equations we will obtain [20]:

$$\Delta X = \sqrt{\sigma^2 \Delta t + \left(r - \frac{\sigma^2}{2}\right)^2 (\Delta t)^2}$$
$$p = \frac{1}{2}\left[1 + \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta t}{\Delta X}\right]$$

Remark 4.5.1. *It is easy to see that $\lim_{\Delta t \to 0} p = 1/2$ in Trigeorgis model as well.*

4.6 The Leisen-Reimer model

One of the latest binomial models was proposed by Leisen and Reimer in 1996 [13],[17]. They considered the Black-Scholes formula and the convergence of different binomial models to the Black-Scholes formula. Leisen and Reimer proposed and proved a theorem about the order of convergence in different binomial models. They showed that the previous models in discrete time such as CCR, JR, Tian and more, have convergence to the Black and Scholes results with order one. Their model however, has a second order of convergence (quadratic convergence) [13]. To begin with, we write the Black-Scholes formula [4]:

$$c(t, S_0) = S_0 N(d_1) - K e^{-r(T-t)} N(d_2), \quad d_{1,2} = \frac{\ln(S_0/K) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and the CRR option pricing formula[5] [4]:

$$c_n(t_0^n = 0, S_0) = S_0 \Phi(a; n, p_n') - K r_n^{-n} \Phi(a; n, p_n)$$

where [13],[4]:

$$p_n = \frac{r_n - d_n}{u_n - d_n} \quad p_n' = p_n \frac{u_n}{r_n} \quad a = \frac{\ln(K/S_0) - n \ln d_n}{\ln u_n - \ln d_n}$$

Then they consider the fact that in all approaches which they have followed in the binomial model, the probability p is approximated with the standard normal function $N(z)$, where the all input arguments are determined by some adjustment function $z = h(a; n, p)$ [13]. Leisen and Reimer denoted [6] a as the number of upward movements of the asset price to exceed strike price in a n step binomial tree (lattice) with martingale probability measure p [13] instead. Then they made an inverse transformation of the adjustment function where $h(a; n, p)$ specifies the distribution parameter $h^{-1}(z) = p$ to approximate $P = N(z)$ with $P \approx 1 - \Phi(a; n, p)$ [13]. Then they used the Peizer and Pratt derivation of the inversion formula to the Camp-Paulson method, one with[7] [$case : a + 1/2 = n - (a + 1/2), n = 2a + 1$] [13]:

$$h^{-1}(z) = \frac{1}{2} \mp \left[\frac{1}{4} - \frac{1}{4} \exp\left\{ -\left(\frac{z}{n + \frac{1}{3}} \right)^2 \left(n + \frac{1}{6} \right) \right\} \right]^{1/2}$$

After introducing the discussion above, Leisen and Reimer solved the system of equations in such way which guarantees the convergence of the binomial model. In other words, they needed to derive u_n, d_n and p_n. After a little algebra they obtained their new binomial model with the following parameters [13]:

$$p_n' = h^{-1}(d_1) \qquad p_n = h^{-1}(d_2)$$

$$u_n = r_n \frac{p_n'}{p_n} \qquad d_n = \frac{r_n - p_n u_n}{1 - p_n}$$

---

[4]To see the derivation of Black-Scholes option pricing formula, see Appendix A.

[5]To see the derivation see [4].

[6]This condition in the CRR model is given by $a \equiv$ smallest non-negative integer greater than $\frac{\ln(K/S_0) - n \ln d_n}{\ln u_n - \ln d_n}$.

[7]Two see the result for Camp-Paulson inversion and Peizer-Pratt method two, see [13].

Remark 4.6.1. *By having the parameter Z in the Leisen-Reimer model, we can get a unique distribution parameter p for each pair of (a, n). Thus in the continuous model the parameter $a(n)$ can be chosen freely, whereas in the discrete case $a(n)$ and $a'(n)$ can be chosen freely but they must be equal to determine p_n and $p'(n)$ in the system of equations above [13].*

Remark 4.6.2. *With inverse transformation, we have to have an odd number of steps in this model [13].*

Chapter 5

Trinomial Model

Except up and downward movements, we could consider no movement at all. We introduce a new variable m to describe a stable middle path. A three-step tree is depicted in Figure 5.1.

For a random walk with three possible directions our random variable X_j with $j \in Z_+$ behaves like this:

$$P(X_j = u) = p_u = p_1$$
$$P(X_j = m) = 1 - p_u - p_d = p_2$$
$$P(X_j = d) = p_d = p_3$$

where $p_1 + p_2 < 1$ and $\sum p_i = 1$ with $i = 1, 2, 3$.

If we have $k + l + (n - k - l) = n$ trials, event u will occur k times, event d will occur l times and event m will occur $(n - k - l)$ times, so the sequence of events gives an outcome of $u^k m^{n-k-l} d^l$. This is easier to understand if we look at Figure 5.1. Following a path consisting of $n = 3$ steps, gives a sequence of, for instance, uud which is $u^2 m^0 d^1$. In this example we reach node $(3, 4)$ which also can be reached in two more ways (udu and duu), three in total. Therefore, we say that the coefficient for a distribution with two up and one down movement is equal to three. Looking at node $(3, 3)$ you will see that there are six ways of combining one of each movements, (umd, udm, mud, mdu, dmu and dum) in order to get to this node. Here the coefficient is 6.

There are 3^n different paths which in this particular three step case is $3^3 = 27$. This is also the sum of the coefficients. If $p_1 = p_2 = p_3$, each of the paths would have probability $\frac{1}{3^n}$, but in cases where this is not true, the probability for each path depends on the respective values for p_1, p_2, p_3. If we knew the probability for each event, we can calculate the probability of each sequence of events. A particular path contains of a certain distribution of events, which means that the probability that event u occurs k times is p_1^k. In general the probability for obtaining

one particular sequence ω is

$$P(\omega) = p_1^k p_2^l p_3^{n-k-l}$$

As mentioned we could have similar sequences with the same amount of ups and downs, but in different orders and the total number of these is

$$\binom{n}{k} \binom{n-k}{l} = \frac{n!}{k!l!(n-k-l)!}$$

Thus the probability of one particular outcome, i.e., the **trinomial probability distribution** is the joint probability function for X_1, X_2, X_3

$$P(X_1 = k, X_2 = l, X_3 = (n-k-l)) = \binom{n}{k} \binom{n-k}{l} p_1^k p_2^l p_3^{n-k-l}$$

5.1 Properties of the trinomial distribution

We can think of the trinomial distribution as the composition of events that fall into one category and all other events that fall into another category. For the case of evolution of asset prices we say that upward movements (event X) occur with probability p and middle- and downward movement (event Y) with probability $q = 1 - p$. This makes it clear that we have marginal distributions for the two categories with binomial marginal probability distributions. It follows that

$$E(X) = np, \qquad V(X) = npq$$

5.1.1 A stretch to the multinomial distribution

The same thinking as for trinomial distribution applies to multinomial distribution. The joint probability function for $X_1, X_2, ...X_k$ is given by

$$p(x_1, x_2, ..., x_k) = \frac{n!}{x_1! x_2! ... x_k!} p_1^{x_1} p_2^{x_2} ... p_k^{x_k}$$

where

$$\sum_{i=1}^{k} p_i = 1, \qquad \sum_{i=1}^{k} x_i = n, \qquad p_i \geq 0, \quad x_i \geq 0$$

The expected value and variance of the marginal distributions are

$$E(X_i) = np_i, \qquad V(X_i) = np_iq_i$$

Trinomial and multinomial models are an extension of binomial models. There are more parameters involved, but the approach is the same, hence the similarity in the formulas. In fact, the binomial model is a special case of multinomial models. The easiest way to see that is to think about the marginal distribution. If we have two probabilities, we have two scenarios with their corresponding events; if we have three probabilities, we have one scenario with its corresponding event, and another scenario where this event not occurs. If we have n probabilities, we can again say that we have two scenarios; one with its event and another one where the first one not occurs (but all the others). This is well explained in Wackerly, et al, pages 278 and 279 [21] which is the basic material used up to now in this section. There are more explanations within the statements compared to the binomial model because it might be a little bit harder to cope with the formulas when a third variable is involved.
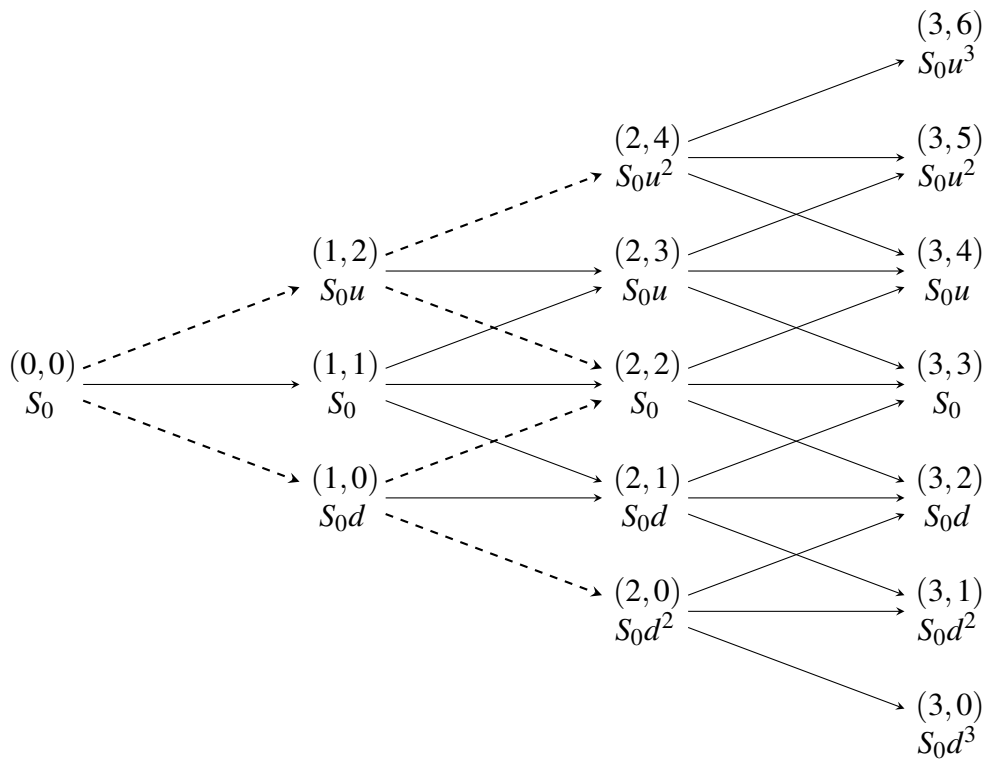

Figure 5.1: **Three-step trinomial tree**

5.2 Boyle's approach of deriving probabilities

In his paper [2], Boyle derived the values for the three probabilities and the values for u and d implying that the log normal continuous distribution of the evolution of the stock price

has the same properties as the discrete distribution, i.e., mean and variance of the respective distributions equal each other. As we have seen before, the probabilities for the movements sum up to 1. Therefore it is valid to substitute p_2 by $1 - p_1 - p_3$. In the continuation of the explanation of this approach we use the same notation as Boyle which is

S_u, S and S_d = price of the underlying asset after an upward, downward or horizontal movement respectively,
r = interest rate, continuously compounded,
σ^2 = variance of return,
h = length of one time interval, $d = \dfrac{1}{u}$.

The equation regarding equality between discrete and log normal continuous expected value is

$$SM = Se^{rh}$$
$$SM = p_1 S_u + (1 - p_1 - p_3)S + p_3 S_d$$

We rewrite this and obtain

$$p_1(u - 1) + p_3\left(\frac{1}{u} - 1\right) = M - 1 \tag{5.1}$$

The equation regarding equality between discrete and log normal continuous expected value is

$$S^2 V = S^2 V = S^2 M^2 exp[(\sigma^2 h) - 1)]$$
$$S^2 V = p_1(S^2 u^2 - S^2 M^2) + (1 - p_1 - p_3)(S^2 - S^2 M^2) + p_3\left(\frac{S^2}{u^2} - S^2 M^2\right)$$

which in rewritten form gives

$$p_1(u^2 - 1) + p_3\left(\frac{1}{u^2} - 1\right) = V + M^2 - 1. \tag{5.2}$$

Now we solve (5.1) and (5.2) for p_1 and p_3.

The result is:

$$p_1 = \frac{(V + M^2 - M)u - (M - 1)}{(u - 1)(u^2 - 1)}$$

and

$$p_3 = \frac{(V + M^2 - M)u^2 - u^3(M-1)}{(u-1)(u^2-1)}$$

$$p_2 = 1 - p_1 - p_3$$

What we can see directly from the equations is that u cannot attain the value 1 because it would lead to *zeros* in the denominators. This is totally in line with the nature of an upward movement. The factor has to be larger than one in order to make the value of the underlying asset grow. Moreover, since we have two constraints and are looking for five unknowns u, d, m, p_1 and p_3, we cannot find unique values for p_1 and p_3. Boyle used the Cox, Ross, Rubinstein value $u = e^{\sigma\sqrt{\Delta t}}$ in the calculations in his paper, but pointed out that this could give negative values for p_2. Therefore he found it reasonable to slightly modify the expression by including the variable λ so that

$$u = \lambda e^{\sigma\sqrt{\Delta t}}$$

where $\lambda > 1$, called stretch parameter.

Since it is not essential yet space consuming to show the actual calculations for p_1 and p_3, we will not do them here. The Boyle paper is in our opinion a reliable and well known source, so the result is trustworthy. For interested readers however, the authors of this thesis will provide the calculations.

5.3 The replicating portfolio in the trinomial model

To show that there is no unique risk neutral probability measure, we studied the lecture notes from lecture 5 of *Stochastic processes* by Anatoliy Malyarenko [14]. We changed the notation for the value of the option C to f because it is more suitable for the linearity of our paper.

A replicating portfolio is a portfolio that has the same value as the derivative for the underlying asset, independent of the price of the underlying.

For one single period the probabilities of the price change of an asset are as follows:

$$P\{S(1) = uS\} = p$$
$$P\{S(1) = mS\} = q$$
$$P\{S(1) = dS\} = 1 - p - q$$

where p and $q = (1-p)$ are real numbers which are greater than zero and their sum is less than one. Additionally $d < m < u$. Also, $R = r + 1$ and lies between d and u.

The value of the portfolio after one Δt is

$$V(1) = \begin{cases} wuS + RB, & S(1) = uS \\ wmS + RB, & S(1) = mS \\ wdS + RB, & S(1) = dS \end{cases}$$

where w is the weight attained to the underlying asset and B the money market account which has unit 1.

Assume that

$$f(1) = \begin{cases} f_u, & S(1) = uS \\ f_m, & S(1) = mS \\ f_d, & S(1) = dS \end{cases}$$

By definition of a replicating portfolio the following system of equations has to be satisfied:

$$wuS + RB = f_u \tag{5.3}$$

$$wmS + RB = f_m \tag{5.4}$$

$$wdS + RB = f_d \tag{5.5}$$

If we subtract (5.5) from (5.3) and divide by $S(u-d)$ we get

$$w = \frac{f_u - f_d}{S(u-d)}$$

We also subtract (5.4) from (5.3) and (5.5) from (5.4). We obtain

$$w = \frac{f_u - f_m}{S(u-m)}$$

$$w = \frac{f_m - f_d}{S(m-d)}$$

Obviously all three equations have equal left hand sides which must mean that the right hand sides are equal as well, thus, after multiplication by S we have

$$\frac{f_u - f_d}{(u-d)} = \frac{f_u - f_m}{(u-m)} = \frac{f_m - f_d}{(m-d)} \tag{5.6}$$

56

If the condition described in the last equation is fulfilled, the contingent claim is attainable. If this really is so, the simultaneous equations for f_u, f_m and f_d give us the values for the respective weights as

$$w = \frac{f_u - f_m}{S(u - m)}$$

$$B = \frac{u f_m - m f_u}{R(u - m)}$$

We already know from the Asset Pricing Theorem (Kijima page 87 [12]) that the fair price of our option should be

$$f = \frac{f_u - f_m}{(u - m)} + \frac{u f_m - m f_u}{R(u - m)}$$

If there are no arbitrage opportunities, we have a risk neutral probability measure P^* such that

$$P^*\{S(1) = uS\} = p^*$$
$$P^*\{S(1) = mS\} = q^*$$
$$P^*\{S(1) = dS\} = 1 - p^* - q^*$$

For the underlying asset price to be a martingale under P^* requires that

$$(u - d)p^* + (m - d)q^* = R - d$$

Similarly for the value of the contingent claim to be a martingale under P^* is

$$(f_u - f_d)p^* + (f_m - f_d)q^* = fR - f_d$$

The fair price of an option is

$$
\begin{aligned}
f &= R^{-1}[p^* f_u + q^* f_m + (1 - p^* - q^*)f_d] \\
&= \frac{f_u - f_m}{u - m} + \frac{u f_m - m f_u}{R(u - m)} + \left[\frac{m - d}{u - m}(f_u - f_m) - (f_m - f_d) \right] q^*
\end{aligned}
$$

The very last term of the last equation vanishes if condition (5.6) holds.

There is no unique probability distribution $(p^*, q^*, 1 - p^* - q^*)$ that satisfies the above equation, thus the market is incomplete because not every contingent claim is attainable.

57

5.4 Trinomial probability under lognormal transformation

The Kamrad and Ritchken paper [11] explained this topic very well, so we used it as basic material for this section.

For an underlying asset that follows a Geometric Brownian Motion we have parameters r, riskless interest rate, and σ, instantaneous volatility (standard deviation), and drift

$$\mu = r - \frac{\sigma^2}{2}.$$

Since we want to calculate returns implementing continuous compounding, we would like to explore the natural logarithm of the return over one time interval as

$$\ln\{S(t + \Delta t)\} = \ln\{S(t)\} + \zeta(t)$$

This tells us that we can express our return as a sum consisting of two components: the logarithm of the price of the underlying at time t, and a random part $\zeta(t)$. The latter is a normal random variable with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$.

Let $\zeta^a(t)$ be the approximating function over the interval $[t, t + \Delta t]$ which is distributed in the following way:

$$\zeta(t) = \begin{cases} v & \text{with } p_1 \\ 0 & \text{with } p_2 \\ -v & \text{with } p_3 \end{cases}$$

$$\sum_{i=1}^{3} p_i = 1$$

Notice that Kamrad and Ritchken were interested in the logarithm of S and emphasized it by the use of v as variable for an upward movement (in contrast to u which we have seen earlier) and we follow this approach. $-v$ is of course a downward movement. Since $u = e^{\sigma\sqrt{\Delta t}}$ we set $v = \lambda \sigma \sqrt{\Delta t}$. The λ is needed in order to assure that p_2 does not attain negative values, which was suggested in Boyle [2].

The approximating distribution should have the same mean and variance as the random variable $\zeta(t)$ so we can set the two formulas

$$E\left[\zeta^a(t)\right] = vp_1 + 0p_2 + (-v)p_3 = v(p_1 - p_3) = \mu \Delta(t) \tag{5.7}$$

and

$$V[\zeta^a(t)] = v^2 p_1 + 0^2 p_2 + (-v^2) p_3 = v_2(p_1 + p_3 = \sigma^2 \Delta t + o(\Delta t) \qquad (5.8)$$

$o(\Delta t)$ stands for terms of higher order of Δt which can be ignored if Δt is very small.

For (5.7) we obtain

$$p_1 = \frac{\mu \Delta t + v p_3}{v}$$

which we plug into (5.8) so that we obtain

$$\frac{v^2 \mu \Delta t + v^3 p_3}{v} + v^2 p_3 = \sigma^2 \Delta t$$

Solving for p_3 and substituting v with $\lambda \sigma \sqrt{\Delta t}$ yields

$$p_3 = \frac{\sigma^2 \Delta t}{2\lambda^2 \sigma^2 \Delta t} - \frac{\lambda \sigma \sqrt{\Delta t} \mu \Delta t}{2\lambda^2 \sigma^2 \Delta t} = \frac{1}{2\lambda^2} - \frac{\mu \sqrt{\Delta t}}{2\lambda \sigma}$$

Solving for p_1 and p_2 as well, we obtain the probabily measures under lognormal transformation

$$p_1 = \frac{1}{2\lambda^2} + \frac{\mu \sqrt{\Delta t}}{2\lambda \sigma}$$

$$p_2 = 1 - \frac{1}{\lambda^2}$$

$$p_3 = \frac{1}{2\lambda^2} - \frac{\mu \sqrt{\Delta t}}{2\lambda \sigma}$$

5.5   Explicit finite difference approach

The original paper that shows the explicit finite difference method is the Brennan and Schwartz paper [3]. It is very hard to understand since both the notations and the style in the paper differ from other material researched for our paper. Therefore we also used Hull, p. 435-441 [7]. Moreover we made our own inferences and used pictures in order to understand this seemingly very complicated method.

We start from the Black-Scholes partial differential equation (PDE) to show the explicit finite difference approach:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rS = 0$$

At first we need to make approximations of the different derivatives that are part of this PDE in order to discretize the terms. This can be done by Taylor series expansion. For better understanding of the aim, see Figure 5.2.
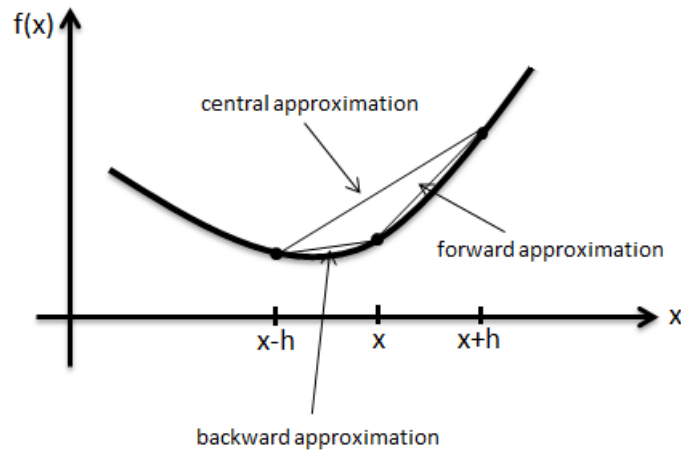

Figure 5.2: **Approximation schemes for the derivative of f**

The forward approximation of $f(x)$ at x is the rearranged equation of the expansion

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}(h)^2 f''(x) + \dots$$

which yields

$$f'(x) = \frac{f(x+h) - f(x)}{h} + o(h)$$

In a similar manner we obtain the first derivative with backward approximation

$$f'(x) = \frac{f(x) - f(x-h)}{h} + o(h)$$

Composing the expansion for forward and backward approximation and rearranging the equation leads to an approximation of the second derivative, which we also need for our procedure

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + o(h)$$

The first derivatives describe the rate of change concerning the chosen variable and the second derivative describes how the rate of change varies.

Due to the fact that the our PDE contains two partial derivatives regarding two different variables, namely the underlying asset S and the time t, we make the approximations as shown above while holding the variables that are not approximated at that moment constant.

Now we construct a grid that shows $M + 1$ equally spaced increments ΔS of the price of the underlying asset in vertical direction, and equally spaced $N + 1$ time intervals Δt in horizontal direction. We have $i = 0, 1, 2, ..., N$ and $j = 0, 1, 2, ..., M$. This shows possible asset prices at different time levels with coordinates (i, j), see Figure 5.3. The option price at the (i, j) point is denoted $f_{i,j}$.
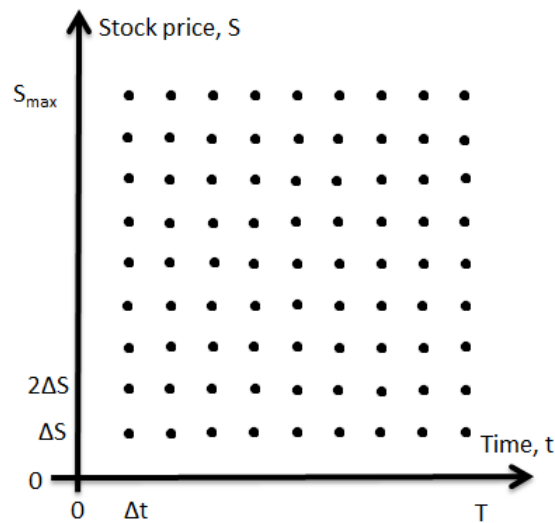

Figure 5.3: **Grid for explicite finite difference approach**

At the upper, lower and right edge of the grid we see the three boundary conditions, which are the highest and lowest expected payoffs, and the expiration time. At $i = 0$ we find the initial asset price which usually is situated near the middle of the vertical axis.

Explicit finite difference method

The derivatives for the Black-Scholes PDE are

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$$

$$\frac{\partial f}{\partial S} = \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{(\Delta S)^2}$$

Thus the difference equation becomes

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + rj\Delta S \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S}$$

$$+ \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{(\Delta S)^2} = rf_{i,j}$$

Multiply by Δt and subtract $f_{i,j}$:

$$f_{i+1,j} + \frac{1}{2}rj\Delta t f_{i+1,j+1} - \frac{1}{2}rj\Delta t f_{i+1,j-1} + \frac{1}{2}\sigma^2 j^2 \Delta t f_{i+1,j+1}$$

$$+ \frac{1}{2}\sigma^2 j^2 f_{i+1,j-1} - \frac{1}{2}\sigma^2 j^2 \Delta t 2 f_{i+1,j} = r\Delta t f_{i,j} + f_{i,j}$$

Collecting the terms for $f_{(i,j)}$, $f_{(i+1,j-1)}$, $f_{(i+1,j)}$ and $f_{(i+1,j+1)}$ yields

$$f_{i,j}(1 + r\Delta t) = f_{i+1,j-1}\left(-\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right) + f_{i+1,j}(1 - \sigma^2 j^2 \Delta t)$$

$$+ f_{i+1,j+1}\left(\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right) \tag{5.9}$$

We set

$$a = \frac{1}{1 + r\Delta t}\left(-\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right)$$

$$b = \frac{1}{1 + r\Delta t}(1 - \sigma^2 j^2 \Delta t)$$

$$c = \frac{1}{1 + r\Delta t}\left(\frac{1}{2}rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right)$$

and obtain the final equation

$$f_{i,j} = a_j f_{i+1,j-1} + b_j f_{i+1,j} + c_j f_{i+1,j+1}$$

We have established a method that shows the relation between the option value at time t and the three possible values for time $t+1$. The values for a, b and c less the discount factor $\frac{1}{1+r\Delta t}$ can be interpreted as the probabilities of an increased, decreased or unchanged price of the underlying asset within one time interval Δt. If we sum the probabilities we should obtain the value of 1 which is easily checked and turns out to be true. Moreover $\frac{1}{1+r\Delta t}$ is the well known approximation of $e^{-r\Delta t}$.

The equation for calculation in trinomial tree can thus be expressed as

$$1 + r\Delta t f(t) = e^{r\Delta t} f(t) = [p_u f_u(t+1) + p_m f_m(t+1) + p_d f_d(t+1)] \qquad 0 \le t \le T-1 \quad (5.10)$$

$$f(t) = e^{-r\Delta t} [p_u f_u(t+1) + p_m f_m(t+1) + p_d f_d(t+1)] \qquad 0 \le t \le T-1 \qquad (5.11)$$

From here the computations for trinomial trees are the same as for binomial trees adapted to three probabilities.

Here we showed that backward approximation in the explicit finite difference method leads to (5.11). In the Section 2.7, we showed how to obtain the equivalent formula for binomial trees (2.24) using the backward formula derived through conditional expectation. These are two principles yielding equivalent outcomes which is an interesting and comforting discovery.

5.6   The case of binomial and trinomial equivalence

During the investigation of trinomial trees we came across a really interesting paper written by Rubinstein [18]. This section demonstrates how he proceeded in order to find equivalence between binomial and trinomial trees.

If we study Figure 5.1 again, we can make an interesting observation. Suppose that we double the size of a time step and that we therefore arrive at nodes $(2,4)$, $(2,2)$ and $(2,0)$ after one Δt. We can see that this a trinomial single period tree. This must mean that we can compute the value of an American option at $t(0)$ either by one (5.11) or three equations [where $(1,2)$ is determined through $(2,4)$ and $(2,2)$; $(1,0)$ through $(2,2)$ and $(2,0)$ and finally $(0,0)$ through $(1,2)$ and $(1,0)$], using (2.24) iteratively.

With that in mind, we can introduce the following equations where the left hand sides represent the two time period nodes for the binomial tree, and the right hand sides represent the single time period nodes for the overlapping trinomial tree.

$$f_{uu} = f_U \qquad f_{ud} = f_M \qquad f_{dd} = f_D \qquad (5.12)$$

Now we use the equation that was derived in the explicit finite difference method, accepting that the compound factor in (5.10) which is $(1 + r\Delta t)$ can be substituted by $1 + \ln(r)\Delta t$ (as proposed by Rubinstein 2000 [18]). To make the derivation easier to read, we will denote $\Delta t = h$ and $2\Delta t = 2h$.

$$[1 + (\ln r)2h]f_0 = P_U f_U + (1 - P_U - P_D)f_M + P_D f_D$$

Recalling that we jump over one step for the binomial part of the tree we have

$$r^2 f_0 = p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd}$$

Due to the presumed equivalence between the two ways of expressing the same states, we also have

$$r^2 = (1 + \ln r)2h$$

Since the trinomial interval has twice the size of one binomial it follows that

$$R = r^2 = [1 + 2(\ln R)h]^{1/2} \qquad (5.13)$$

Furthermore it should also be valid to claim that

$$p^2 = P_U, \quad (1 - p) = P_D, \quad 2p(1 - p) = (1 - P_U - P_D)$$

Due to the fact that one period in trinomial spacing $(U, 1, D)$ is double the size of one period in binomial spacing, it is applicable to set

$$U = u^2, \quad D = d^2, \quad 1 = ud \Rightarrow d = \frac{1}{u}$$

From (5.13) we get

$$r = \sqrt{[1 + 2(\ln r)]}$$

From the CRR model [4] we have

$$p = \frac{1}{2} + \frac{1}{2}\frac{\mu\sqrt{h}}{\sigma}$$

From Kamrad and Ritchken [11] we have

$$P_U = \frac{1}{2\lambda^2} + \frac{1}{2\lambda}\frac{\mu\sqrt{2h}}{\sigma} \tag{5.14}$$

and

$$P_D = \frac{1}{2\lambda^2} - \frac{1}{2\lambda}\frac{\mu\sqrt{2h}}{\sigma} \tag{5.15}$$

According to Rubinstein [18] we perform a little algebra and obtain

$$p = \frac{1}{2}\left[\frac{1}{2(\sigma^2 h - \mu^2 h^2)} + \frac{1}{\sqrt{\sigma^2 h - \mu^2 h^2}}\frac{\mu\sqrt{2h}}{\sigma}\right]^{\frac{1}{2}} \tag{5.16}$$

We define for (5.14)

$$a \equiv \frac{1}{2\lambda^2}, \quad b \equiv \frac{1}{2\lambda}\frac{\mu\sqrt{2h}}{\sigma} \Rightarrow b^2 = \frac{1}{2\lambda^2}\frac{\mu^2 h}{\sigma^2}$$

and can write that

$$P_U = a + b \Rightarrow p^2 = a + b \tag{5.17}$$

If the two models are equivalent we should obtain from (5.15)

$$P_D = (1-p)^2 = a - b$$

This will not be true, unless if we choose that

$$u = e^{\sqrt{\sigma^2 h - \mu^2 h}} \tag{5.18}$$

According to Kamrad and Ritchken [11] we have

$$U = e^{\lambda\sigma\sqrt{2h}}$$

From this follows that

$$u^2 = e^{\lambda \sigma \sqrt{2h}}$$

$$(2\ln u)^2 = \lambda^2 \sigma^2 (2h)$$

$$\lambda^2 = \frac{\sigma^2 2h}{(2\ln u)^2}$$

$$\frac{1}{2\lambda^2} = \frac{1/4(\sigma^2 h)}{(\ln u)^2}$$

Now we substitute what we proposed in (5.18) into the last equation which yields

$$\frac{1}{2\lambda^2} = \frac{(1/4)\sigma^2}{\sigma^2 - \mu^2 h} \tag{5.19}$$

Now we set

$$a - b^2 = \frac{1}{2\lambda^2} - \frac{1}{2\lambda^2} \frac{\mu^2 h}{\sigma}^2$$

$$= \frac{1}{2\lambda^2} \frac{\sigma^2 - \mu^2 h}{\sigma^2}$$

and substitute $\dfrac{1}{2\lambda^2}$ with the right hand side of (5.19). That gives

$$\frac{(1/4)\sigma^2}{\sigma^2 - \mu^2 h} \frac{1}{2\lambda^2} \frac{\sigma^2 - \mu^2 h}{\sigma^2} = \frac{1}{4}$$

Therefore

$$a - b^2 = \frac{1}{4} \Rightarrow a = b^2 + \frac{1}{4}$$

From (5.17) we have

$$p = \sqrt{a + b}$$

We plug in our expression for a and obtain

$$p = \sqrt{b^2 + b + \frac{1}{4}} = b + \frac{1}{2}$$

From this follows

$$(1 - p)^2 = [1 - (b + \frac{1}{2})]^2 = 1 - 2(b + \frac{1}{2}) + (b + \frac{1}{2})^2$$

$$= 1 - 2b - 1 + b^2 + b + \frac{1}{4} = b^2 + \frac{1}{4} - b = a - b$$

66

This means that $(1 - p)^2 = a - b$ under condition $u = e^{\sqrt{\sigma^2 h - \mu^2 h^2}}$.

Therefore we can draw the conclusion that binomial models, with every other period skipped, and trinomial models, under lognormal transformation, coincide if we choose

$$u = e^{\sqrt{\sigma^2 h - \mu^2 h}}$$

and

$$p = \frac{1}{2} \left[\frac{1}{2(\sigma^2 h - \mu^2 h^2)} + \frac{1}{\sqrt{\sigma^2 h - \mu^2 h^2}} \frac{\mu \sqrt{2h}}{\sigma} \right]^{\frac{1}{2}}$$

as the parameters. This leads to a more efficient algorithm because fewer calculations are necessary in order to price an American option as explained in the beginning of this section.

5.7 More on the connection between binomial and trinomial trees

This way of overlapping in the previous section implies also that we can use the Cox, Ross, Rubinstein parameters $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$ and p^* see (2.7) if volatility is constant, i.e., the spacing between the nodes is of the same distance. In Derman et al [5] we found support for that.

Taking into consideration that we jump over one time interval in the binomial part of the tree and summing the above parameters, we obtain values for

$$p_1 = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2$$

$$p_3 = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2$$

$$p_2 = 1 - p_1 - p_3$$

This result can give a base for the state space of implied trinomial trees but it is not the only one. Another method was for instance developed by Jarrow and Rudd [10].

'Implied' tree means that the volatility differs from time step to time step thus giving unequal spacing between nodes. These kind of trees can match prices of standard options better than constant volatility trees because the state space is chosen in advance which helps out with fact that we discovered in Boyle's approach where we had two constraints and five unknowns.

With implied trees three of the unknowns are eliminated and we can solve for transition probabilities.

Chapter 6

Conclusion

When we studied the course *Introduction to financial matehematics* during our first year and met the theory of Option Pricing and Binomial Models for the very first time, we had no idea that we just saw the tip of an iceberg. It seemed as if binomial trees were easy models but as it turns out, they are not! We started Chapter 2, by making a quick review of the binomial model, its properties and the basics of option pricing. We then moved on and showed how to determine probabilities for up and downward movements because we cannot simply assume that the chance of those movements is 50% each. No arbitrage arguments and martingales led us to risk neutral probability measures and we showed how they are derived in the Cox-Ross-Rubinstein model. We then continued to show where the factors for up and downward movements in the Cox, Ross and Rubinstein model actually come from. For proper understanding of the Option Pricing formula it is furthermore essential to have knowledge about random processes and their partial sums and how they function under conditional expectation. Thus we proved in Sections 2.7 and 2.8, that the Option Pricing formula is a consequence of these conditions and approximates a stochastic process in continuous time. Examples show the implementation of this process on a European call and an American Put option.

In the Chapter 3, we showed that the binomial model converges to the Geometric Brownian Motion with both normal and lognormal distribution. Black and Scholes, and Merton had already assumed that the dynamic of risky security returns follows a Geometric Brownian Motion and here it was shown that this also applies for the Cox-Ross-Rubinstein model.

Except the famous CRR model, there are many other binomial models and we spent the whole Chapter 4, on presenting some of them. We found out that we obtain different values for the parameters which are addressed in the respective models. This occurs because some models use normal distribution and their corresponding expected value and variance, others use lognormal distribution values. Another reason was that in some models there were three unknowns to be searched for (u, d, p) but only two constraint functions. Therefore values for p or u could be chosen so that the problem with too many unknowns could be eliminated. In the Chapter 4, we also demonstrate how Jarrow and Rudd found that the limit for $p^* = \frac{1}{2}$ and that Leisen and Reimer could estimate the binomial model so that it converges faster to the

Black-Scholes model than for example the Cox, Ross and Rubinstein model.

In Chapter 5, we extended the binomial model by adding the possibility of movements into a third direction. Thus we established the trinomial model, its probability distribution and properties. We continued with the presentation of Boyle's method [2] to derive transition probabilities. As for the binomial model we showed furthermore that we can obtain different results depending on the choice of approximation technique for the discretization of the continuous distribution, and how the random variable, that describes the return of the underlying security, is chosen to be distributed with its corresponding mean and variance. We also made an attempt to derive the risk neutral probability and discovered that there is no unique probability distribution that generates a contingent claim in the replicating portfolio. We also demonstrated how the Black-Scholes partial differential equation can be discretized and further processed so that an option value equation is obtained. This trinomial equation is the equivalent to the binomial option pricing formula which we obtained through conditional expectation in the Chapter 2. This similarity through different methods stroke us as interesting and we looked deeper into this seemingly present connection. The result that we showed in part 5.6 states that the binomial option pricing model and the explicit finite difference method coincide under a certain parametrization. We then extended our thoughts about the connection of binomial and trinomial models because we discovered that there are models assuming constant volatility and models assuming varying volatility. Therefore we give a brief introduction to implied trinomial trees which closes the work on this paper.

However, we are not satisfied. Many of our topics have the potential to be investigated further and more models should be examined. For example we would like to know if there are more theories about the speed of convergence of the binomial model to the Black-Scholes formula. We were happy to find the Leisen - Reimer model that converges faster as the CRR model, but are there models that are even faster and does the trinomial model converge as well? It would also have been interesting to do some numerical computations and to see if binomial models, trinomial models and the Black-Scholes formula will produce the same results. Also, how big is the approximation error and how many time steps need to be implemented into the respective models in order to obtain similar results?

Besides we have only studied the basic ideas of option pricing. As it was mentioned before, there exists lots of different kinds of options in the market. We can use our current knowledge and the basic algorithm of option pricing to learn more about different kinds of options in future studies. Firstly, in our approach we have not covered jump diffusion, which is an important dilemma in option pricing. Merton [15] has already done that, so working in option pricing with jump diffusion and expanding it in different models would be really vital and interesting. Then, we can study Asian options, non-standard American options, Swap options, Barrier options and Currency options [7]. Additionally, if we look at markets where there exist some complicated options, we have to estimate their price by a numerical approach, namely Monte Carlo and Quasi-Monte Carlo Simulation [6]. So, one further step in future studies could be this matter. It is an important subject because if we want to do estimations we will have some errors and consequently we will have to use our knowledge from Statistical Inferences [21],[6] to find confidence intervals, conduct some different statistic tests to find the

error of estimation, analyze the type of error and so on. Moreover, we need to apply our mathematical knowledge to computer programming. We have already learnt some, and one can learn some more by continuing his/her study in the "Financial Engineering Program" to learn computer programs as for example "Numerical Methods with Matlab", "Analytical Finance with Matlab", "Time Series Analysis" with R Program, "Java for Analytical Finance" and so forth.

Moreover we discovered the distinction between constant volatility tree models and implied volatility models in a late phase of our work. This concept is valid for binomial trees as well as for trinomial trees and is a subject that really needs further exploring. But as we have seen during our work, there are many published papers to go through and when we started to study one of them it referred to another paper which again referred to another paper. It seems like a never-ending stream of information and theories, unfortunately not possible to go through within the time frame for this work to be done.

As we already have said, we have found the tip of an iceberg; but the good thing is, that we know that there is enough material to choose from when we have to select topics for upcoming projects!

6.1 Summary of reflection of objectives in the thesis

The examination goals have been met as listed below.

6.1.1 Objective 1 - Knowledge and understanding

In the introduction and Chapter 2 we have shown that we are familiar with options and the importance of their proper pricing. Furthermore, in Chapters 4 and 5, we have explained and compared several lattice models and conditions that influence the parameters which are necessary in order to use option pricing formulas in continuous or discrete time respectively. This shows a deep understanding of the topic. We also presented derivations of the factors which are only possible if an extensive knowledge of mathematics is present, e.g. calculus, probability theory, algebra and stochastic processes. Our knowledge of these areas can be recognized in all the calculations, but as an example we would like to mention the calculation of u and d factors in Section 2.6 where we use expected value and variance from probability theory, Maclaurin expansion from calculus and a system of equations from linear algebra.

As we also mentioned in the introduction, there are different kinds of options. In fact, some of them are very complicated. Apparently, the concept can be developed and extended in many ways and options are constantly subject for new and further research. Our own research papers span over the time space between 1973 and 2001 which is 29 years, but there are of course more papers and literature published up to today. Moreover, the evolution of computers allows

71

for development of software that is able to estimate option prices and this is a field that is still developing.

6.1.2 Objective 2 - Ability to search, collect, evaluate and interpret

The topic for this thesis is very theoretical and it was crucial to scan many papers in order to find the results that we were looking for. It was even more important to realize what we were reading for understanding *why* we found so many different results. We had to use text books as for example [7] and [21], and also lecture notes [14], [17], with the purpose of really getting a grip on the derivations. An example that really emphasizes this is the demonstration of the case of binomial and trinomial equivalence in Section 5.6. The paper by Rubinstein referred to the Explicite Finite Difference approach by Brennan and Schwartz. We found it important to have an understanding of this paper in order to properly understand Rubinstein, so we studied Brennan and Schwartz as well (which is represented in Section 5.5). This paper in turn was hard to understand so we had to search in one of our textbooks [7] as well. This helped very much but was not sufficient why we consulted a book from our course in *Calculus* as well. Similar approaches applied to contents in other sections and in that way we collected sources and aggregated the contents of the thesis.

6.1.3 Objective 3 - Identify, formulate and solve problems

In the first year of our program, we came (among others) across option pricing formulas, binomial models and the Black-Scholes formulae in *Introduction to financial mathematics*. We wondered how up and down factors were set, probabilities were determined and on what ground the size of time intervals were chosen, but we did not have sufficient knowledge to understand this. Therefore, the problem formulation of this thesis is a follow up on questions that we already wanted to get answers to in the past. As we found out and explained carefully, there are different ways for solving these problems as we have shown throughout this thesis.

6.1.4 Objective 4 - Communication of our project to different groups

We believe that our thesis can be read and understood by a range of people with varying previous knowledge about option pricing and the mathematics behind it. The reader that is interested in an overview of options and how they are priced can be satisfied with Sections 2.1, 2.3 - 2.4 and 2.7 - 2.9, whereas the more interested financial economist can include the rest of Chapter 2 and Chapters 4 and 5, or at least the results provided at their end. The mathematician might be more interested in calculations, but should be able to understand the financial background as well. A financial analyst or engineer will grasp the whole thesis.

6.1.5 Objective 5 - Ability to put our work into a societal context and its value within it

Option pricing has concerned people for a long time and it still does. The purchase of an option is an expense and the demand for correct pricing is required in public use, which is a societal aspect. There are different approaches and models and we have shown how some of them are connected and what the reason for the complexity and the different results is. We surveyed many papers, extracted relevant information, deepened our own understanding and where finally able to present them in a simpler version. We have not invented new theories, but we have collected an extensive amount of them into one place. We believe that this is valuable for operators in the financial market since it increases the understanding for the difficulties in correct pricing strategies and could help to decide which strategy to choose.

Bibliography

[1] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):pp. 637–654, 1973.

[2] Phelim P. Boyle. A lattice framework for option pricing with two state variables. *The Journal of Financial and Quantitative Analysis*, 23(1):pp. 1–12, 1988.

[3] Michael J. Brennan and Eduardo S. Schwartz. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. *The Journal of Financial and Quantitative Analysis*, 13(3):pp. 461–474, 1978.

[4] John C Cox, Stephen A Ross, and Mark Rubinstein. Option pricing: A simplified approach. *Journal of financial Economics*, 7(3):229–263, 1979.

[5] Emanuel Derman, Iraj Kani, and Neil Chriss. Implied trinomial tress of the volatility smile. *The Journal of Derivatives*, 3(4):7–22, 1996.

[6] Paul Glasserman. *Monte Carlo methods in financial engineering*. Springer, New York, 2004.

[7] John Hull. *Options, futures, and other derivatives*. Pearson Prentice Hall, Upper Saddle River, NJ, 2009.

[8] George M. Jabbour, Marat V. Kramin, and Stephen D. Young. Two-state option pricing: Binomial models revisited. *Journal of Futures Markets*, 21(11):987–1001, 2001.

[9] Robert Jarrow. *Derivative securities*. South-Western College Pub, Cincinnati, Ohio, 2000.

[10] Robert A Jarrow and Andrew Rudd. *Option pricing*. Dow Jones-Irwin Homewood, IL, 1983.

[11] Bardia Kamrad and Peter Ritchken. Multinomial approximating models for options with k state variables. *Management science*, 37(12):1640–1652, 1991.

[12] Masaaki Kijima. *Stochastic processes with applications to finance*. Chapman and Hall CRC, Boca Raton, Fla, 2003.

[13] Dietmar PJ Leisen and Matthias Reimer. Binomial models for option valuation-examining and improving convergence. *Applied Mathematical Finance*, 3(4):319–346, 1996.

[14] Anatoliy Malyarenko. *Lecture notes, Stochastic Processes*. 2012.

[15] Robert C. Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):pp. 141–183, 1973.

[16] Jr. Rendleman, Richard J. and Brit J. Bartter. Two-state option pricing. *The Journal of Finance*, 34(5):pp. 1093–1110, 1979.

[17] Jan Röman. *Lecture notes, Analytical Finance I*. http://janroman.dhis.org/, 2012.

[18] Mark Rubinstein. On the relation between binomial and trinomial option pricing models. *The Journal of Derivatives*, 8(2):47–50, 2000.

[19] Yisong Tian. A modified lattice approach to option pricing. *Journal of Futures Markets*, 13(5):563–577, 1993.

[20] Lenos Trigeorgis. A log-transformed binomial numerical analysis method for valuing complex multi-option investments. *The Journal of Financial and Quantitative Analysis*, 26(3):pp. 309–326, 1991.

[21] Dennis Wackerly. *Mathematical statistics with applications, 7th edition*. Thomson Brooks/Cole, Belmont, CA, 2008.

Appendix A

Appendix

In this paper we have used Black-Scholes model 1973 and their result several times. So, it might be a good idea to see how they obtain their formula. To begin with, we explain the derivation of Black-Scholes partial differential equation (PDE)and then we will derive Black-Scholes formula for European Call Option [1],[17]. We got a considerable help from lecture notes of the course *"Analytical Finance I"* [17] which we have studied in our program to explain these derivations properly.

A.1 Black-Scholes PDE

Again, we consider a market which consist of two type of financial instruments, bond and stocks. So, we will have [1],[17]:

$$\begin{cases} dB(t) = rB(t)dt \\ B(0) = 1 \end{cases}$$

Which means the return on bound is simply $B(t) = e^{rt}$. And we assume that the stock price will follow the following stochastic process [1],[17]:

$$\begin{cases} dS(t) = \alpha S(t)dt + \sigma S(t)dW \\ S(0) = S_0 \end{cases}$$

Where, in deterministic part α is drift coefficient and in stochastic part which follows Geometric Brownian Motion (Wiener Process), σ is diffusion coefficient. Now, we can construct our portfolio $h = (h_B, h_S)$, where h_B represents the number of bond and h_S represents the number of stocks. So, the value process will be defined as [1],[17]:

$$V(t) = h_B(t)B(t) + h_S(t)S(t)$$

and to have our portfolio self-financed we must have [1],[17]:

$$dV(t) = h_B(t)dB(t) + h_S(t)dS(t)$$

Moreover, we have invested our money in two categories stock and bond. Let's say we have decided to invest x percent of our money in bond and y percent of our money in stock. x and y can get negative value, since its possible to short one of securities to long the other, but under condition $x + y = 1$. To hold this condition, we define a relative portfolio $u = (x, y)$ in such a way that $x(t) + y(t) = 1$ holds. So, we will obtain following relative portfolio [1],[17]:

$$\begin{cases} x(t) = \frac{h_B(t)B(t)}{V} \Rightarrow h_B = \frac{xV}{B} \\[2mm] y(t) = \frac{h_S(t)S(t)}{V} \Rightarrow h_S = \frac{yV}{S} \end{cases}$$

rewriting the self-financed process equation in terms of the relative portfolio and substituting the value of dB and dS we will obtain [1],[17]:

$$dV = V(rx + \alpha y)dt + V(\sigma y)dW \tag{A.1}$$

Moreover, if we suppose $V(t) = V(t, S(t))$ we can apply Itô formula and we will obtain [1],[17]:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$$
$$= V_t dt + V_S(\alpha S dt + \sigma S dW) + \frac{1}{2}V_{SS}\sigma^2 S^2(dW)^2$$

substituting $(dt)^2 = 0$, $(dW)^2 = dt$, collecting the terms which are multiplying by dt and multiplying right hand side by V/V we will obtain [1],[17]:

$$dV = \frac{V\left(V_t + V_S\alpha S + \frac{1}{2}V_{SS}\sigma^2 S^2\right)dt}{V} + \frac{V(V_s\sigma S)dW}{V} \tag{A.2}$$

comparing (A.1) and (A.2), we will obtain:

$$y = \frac{V_S S}{V}$$
$$rx + \alpha y = \frac{V_t + V_S\alpha S + \frac{1}{2}V_{SS}\sigma^2 S^2}{V}$$

substituting $y = \dfrac{V_S S}{V}$ in (A.2), we will obtain:

$$dV = V\left(\frac{V_t + \frac{1}{2}V_{SS}\sigma^2 S^2}{V} + \alpha y\right)dt + Vy\sigma dW$$

77

if we just multiply $\dfrac{V_t + \frac{1}{2}V_{SS}\sigma^2 S^2}{V}$ by r/r, we will get:

$$dV = V\left(r\frac{V_t + \frac{1}{2}V_{SS}\sigma^2 S^2}{rV} + \alpha y\right)dt + Vy\sigma dW$$

comparing the last equation with (A.1) will give us:

$$x = \frac{V_t + \frac{1}{2}V_{SS}\sigma^2 S^2}{rV}$$

and since $x + y = 1$:

$$\frac{V_t + \frac{1}{2}V_{SS}\sigma^2 S^2}{rV} + \frac{SV_S}{V} = 1$$

multiplying both hand side with rV will give us Black-Scholes PDE:

$$V_t + rSV_S + \frac{1}{2}V_{SS}\sigma^2 S^2 = rV$$

It is common to write Black-Scholes PDE as [1],[17]:

$$\Theta + rS\Delta + \frac{1}{2}\Gamma\sigma^2 S^2 = rV \qquad (A.3)$$

Where [1],[17]:

$\Theta = \dfrac{\partial V}{\partial t}$, represents the change in value with respect to the time

$rS\Delta = rS\dfrac{\partial V}{\partial S}$, represents the change in value with respect to stock price

$\dfrac{1}{2}\Gamma\sigma^2 S^2 = \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial V}{\partial S}$, represents the change in value with respect to volatility of stock price

rV, represents the expected return, which means in risk-neutral world the expected return must be equal to risk-free interest rate.

A.2   Black-Scholes formula for European Call option

In this section we will try to go through Black and Scholes approach to find the fair price of European call option. We have already derived the Black-Scholes PDE and we know that the payoff to the European call option is $\max(S_T - K, 0)$. So we can write [1],[17]:

$$\begin{cases} V_t + rSV_S + \frac{1}{2}V_{SS}\sigma^2 S^2 = rV \\[2mm] V_T = \max(S_T - K, 0) \end{cases}$$

To solve this PDE, we can use a trick. Using this trick we suppose the $V(t, S(t))$ is solution to Black-Scholes PDE. where the stock price following the stochastic process [1],[17]:

$$\begin{cases} dS(t) = \alpha S(t)dt + \sigma S(t)dW \\ S(0) = S_0 \end{cases}$$

Applying Itô formula to $V(t, S(t))$, we will obtain [1],[17]:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$$

$$= V_t dt + V_S(\alpha S dt + \sigma S dW) + \frac{1}{2}V_{SS}\sigma^2 S^2(dW)^2$$

$$= \left(V_t + rSV_s + \frac{1}{2}V_{SS}\sigma^2 S^2\right)dt + \sigma S V_s dW$$

where Itô formula gives $\alpha = r$, $(dt)^2 = 0$ and $(dW)^2 = dt$. Substituting $V_t + rSV_s + \frac{1}{2}V_{SS}\sigma^2 S^2 = rV$, we will obtain:

$$dV = rVdt + \sigma S V_s dW$$

taking integral:

$$\int_t^T dV = \int_t^T (rV(u)du + \sigma S(u)V_S(u)dW(u))$$

$$V_T - V_t = r\int_t^T V(u)du + \sigma \int_t^T S(u)V_S(u)dW(u)$$

taking expectation will yield:

$$E[V_T - V_t] = E\left[r\int_t^T V(u)du + \sigma \int_t^T S(u)V_S(u)dW(u)\right]$$

since the expectation of a sum is the sum of expectation, the expectation of a constant is just that constant, i.e., $E[V_t] = V_t$ and the expectation of stochastic part is zero, we will obtain:

$$E[V_T] = V_t + r\int_t^T E[V(u)]du + 0$$

denoting $E[V_T] = m$ and taking derivative of both hand side with respect to time, we can use the fundamental theorem of calculus and we will obtain:

$$m = V_t + r\int_t^T mdu$$

$$\Rightarrow \frac{dm}{dt} = 0 + rm$$

$$\Rightarrow \frac{dm}{m} = rdt$$

solving this differential equation we will obtain:

$$\int_t^T \frac{dm}{m} = r \int_t^T dt$$

$$\Rightarrow \ln(m_T - m_t) = r(T-t)$$

$$\Rightarrow m_T = m_t e^{r(T-t)}$$

$$\Rightarrow E[V_T] = V_t e^{r(T-t)}$$

$$\Rightarrow V_t = e^{-r(T-t)} E[V_T]$$

since $V_T = \max\{S_T - K, 0\}$ and random variable S_T follows the stochastic process under equivalent martingale probability measure, we can write [1],[17]:

$$c(t, S(t)) = V(t, S(t)) = e^{-r(T-t)} E^Q[\max\{S_T - K, 0\}|\mathscr{F}_t]$$

which implies that the fair price, $c(t, S(t))$ of an European Call Option under martingale probability measure , is discounted expected payoff. Now, we have the formula to calculate the price of European call option. The next step is to find the equation for S_T. Let denote $X(t) = \ln S(t)$, where we know our process has the following form.

$$\begin{cases} dS(t) = rS(t)dt + \sigma S(t)dW \\ S(0) = S_0 \end{cases}$$

applying Itô formula, we will obtain [1],[17]:

$$dX = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S} dS + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} (dS)^2$$

$$= 0 + \frac{1}{S} (rSdt + \sigma SdW) + \frac{1}{2} \frac{-1}{S^2} \sigma^2 S^2 (dW)^2$$

$$= \left(r - \frac{1}{2}\sigma^2 \right) dt + \sigma dW$$

taking the integral we will obtain:

$$X_T - X_t = \left(r - \frac{1}{2}\sigma^2 \right)(T-t) + \sigma(W_T - W_t)$$

substituting $X(t) = \ln S(t)$ and taking exponent of both hand side we will obtain:

$$S_T = S_t e^{\left(r - \frac{1}{2}\sigma^2 \right)(T-t) + \sigma(W_T - W_t)}$$

since we want the price process to follow a Geometric Brownian Motion, the Wiener process part can be re-written as $W_T - W_t = \sqrt{T-t}z$ where, $z \sim N\left[\left(r - \frac{1}{2}\sigma^2 \right)(T-t), \sigma^2(T-t) \right]$ [1],[17].

Remark A.2.1. *The current explanation can make it clear why Black and Scholes, and Merton considered that dynamic of risky asset or stock is following the Geometric Brownian Motion. As we saw, with this assumption we can find a random variable U such that our random variable will have normal distribution, i.e., $U \sim N\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right]$. Having a normal random variable U we can transform our random variable to standard normal random variable Z by letting $Z = \frac{U-\mu}{\sigma}$ [21] where μ stands for mean and σ is standard deviation and then, we can use normal cure areas to find the probability.*

In next step, we will introduce the following notations:

$$\tilde{r} = r - \frac{1}{2}\sigma^2$$
$$\tau = T - t$$
$$\sqrt{\tau}z = W_T - W_t$$
$$y = \tilde{r}\tau + \sigma\sqrt{\tau}z \Rightarrow z = \frac{y - \tilde{r}\tau}{\sigma\sqrt{\tau}}$$
$$S_T = S_t e^y$$

Now, if we calculate $S_T - K$ we will obtain, $y_0 = \ln\left(\frac{K}{S_t}\right)$. So, we introduce new point

$z_0 = \dfrac{\ln\left(\frac{K}{S_t}\right) - \tilde{r}\tau}{\sigma\sqrt{\tau}}$. Substituting, all notation above we can calculate the price of European Call Option.

$$
\begin{aligned}
c(t, S(t)) = V(t, S(t)) &= e^{-r(T-t)} E^Q[\max\{S_T - K, 0\}|\mathscr{F}_t] \\
&= e^{-r\tau} \int_{-\infty}^{\infty} \max(S_T - k, 0)\phi(z)dz \\
&= e^{-r\tau} \int_{z_0}^{\infty} (S_T - k)\phi(z)dz \\
&= e^{-r\tau} \int_{z_0}^{\infty} (S_t e^y - k)\phi(z)dz
\end{aligned}
$$

we know that for standard normal distribution function, the cumulative distribution function

is $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. substituting we will get:

$$
\begin{aligned}
c(t, S(t)) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left(\int_{z_0}^{\infty} (S_t e^y - k) e^{-z^2/2} dz \right) \\
&= \frac{S_t e^{-r\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{y - z^2/2} dz - \frac{k e^{-r\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-z^2/2} dz \\
&= \frac{S_t e^{-r\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{y - z^2/2} dz - k e^{-r\tau} N(-z_0) \\
&= \frac{S_t e^{-r\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{\tilde{r}\tau + \sigma\sqrt{\tau}z - z^2/2} dz - k e^{-r\tau} N(-z_0) \\
&= \frac{S_t e^{-r\tau} e^{\tilde{r}\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-(z - \sigma\sqrt{\tau})^2/2} e^{\sigma^2\tau} dz - k e^{-r\tau} N(-z_0) \\
&= \frac{S_t e^0}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-(z - \sigma\sqrt{\tau})^2/2} dz - k e^{-r\tau} N(-z_0) \\
&= S_t N\left(-z_0 + \sigma\sqrt{\tau}\right) - k e^{-r(T-t)} N\left(-z_0\right)
\end{aligned}
$$

To summarize we can calculate the price of a European Call option using the following formula [1],[17]:

$$
c(t, S(t)) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)
$$

$$
d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{(T-t)}}
$$

$$
d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)}
$$

With the similar approach we can find the price of European Put Option, i.e., $p(t, S(t))$, where the payoff is $\max(K - S_T, 0)$ [1],[7],[17].

$$
p(t, S(t)) = K e^{-r(T-t)} N(-d_2) S_t N(-d_1)
$$

Remark A.2.2. *As we explained in previous remark, Black-Scholes formula has a big advantage. So, for pricing European call and put options we need to calculate the value for d_1 and d_2 and then we can find the value for $N(d_1)$ and $N(d_1)$ from standard normal random variable table. Finally, substituting inputs we can calculate the fair price of European Call or put options.*

Remark A.2.3. *For pricing American put option, because of the possibility of early exercises, we do not have a widely used formula. One needs to use some numerical method like Monte-Carlo simulation [6] to simulate the process and estimate the price or in simple cases, it is possible to use the backward equation to calculate the price of American put option via lattice approach.*