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Modeling Credit Risk through Intensity Models

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Abstract

Credit risk arises from the possibility of default of a contingent claim. In this thesis we study the application of intensity models to model credit risk. A general framework for valuation of claims subject to credit risk is established. Additionally, we study credit default swaps, and their implied connection to intensity models. Finally, we study the pricing effect on corporate bonds inducing different correlations between the risk-free rate and the credit spread.

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Chapter 1

Introduction

The general Black-Scholes framework presented in Black and Scholes (1973) and Merton (1973) implicitly assumes no default risk (also called credit risk) from the counterparty involved in a contingent claim. Credit risk arises from the possibility of default in debt contracts or in derivatives transactions. Credit risk matters for investors because of the possibility of getting not only their expected investment returns, but in fact losing their invested capital.

Default risk is associated with the inability of a counterparty to meet its obligations. Throughout history there has been several default events, for example, the announcement on November 26, 2009 of Dubai World, an asset management company that manages a portfolio for the Dubai government, of a possible delay in repayment of its debts.[1] Credit risk has taken a mayor roll in the last decade and specially because of the recent financial crisis that, among other factors, included a credit constriction due to fear of a generalized default in the financial system. A lot of theory has been developed in order to measure credit risk, and there are specialized firms (Standard & Poor's and Moody's among others) that have developed grading scales to analyze the threat of a default event of countries, firms and different securities.

A basic example of a security that involves credit risk are corporate bonds. These bonds are sources of liquidity from which corporations can finance different projects. The repayment of the corporate bond therefore depends on the financial solvency of the firm. In particular, if the firm goes bankrupt, it is most probable that bondholders will not get their expected money completely repayed on schedule. It could also happen that bondholders get nothing of their initial investment back.

A usual assumption is that government bonds issued in the local currency are risk free (for example Swedish Government Bonds denominated in Swedish Crowns). This assumption could be questioned, but in general and for practical purposes, the market assumes that a government can always repay its debt, in the worst case scenario, by "printing" more money. For this type

[1] Dubai World, Wikipedia Article, http://en.wikipedia.org/wiki/Dubai_World

of securities, the Black-Scholes framework and martingale theory applied to arbitrage pricing gives us the *correct* pricing of a bond (see Björk (2004)). In particular, if the contract is a zero-coupon bond, that is a contract that guarantees its holder a payment of one unit of currency at time T, its price at time $t < T$ denoted by $P(t,T)$ would be equal to:

$$P(t,T) = \mathbb{E}^{Q}_{t,r}\left[\mathrm{e}^{-\int_t^T r_s ds} \times 1\right] \tag{1.1}$$

where the martingale measure Q, and the subscripts t, r denote that the expectation should be taken under the risk-neutral measure Q for r_s, for $t \leq s \leq T$ and $r_t = r$.

We can interpret (1.1) as the expected value of one unit of currency discounted to present value. The discounting should be taken not under the objective probability measure P but under the risk-neutral one Q. We emphasize the fact that the discounting factor $\mathrm{e}^{-\int_t^T r_s ds}$ is multiplied by a one, which means that the contract will pay one unit at time T for sure (i.e. with probability one). This is clearly not the case with corporate bonds, where we assume there is a positive probability of a default. The natural question that arises then is how should corporate bonds be priced.

Although the Black-Scholes framework implicitly assume no default risk, Black and Scholes (1973) and Merton (1973) propose a way to price corporate debt. These constitute the so-called firm value models or structural models. They treat the total value of the firm V as a Geometric Brownian Motion (GBM). The value of the firm is the sum of the equity value S and of the corporate debt D. In a very simple model, the equity constitute the remnants of the value of the firm minus the corporate bonds once they mature. In particular, if V at time T (the time of maturity of the bonds) is less than the face value of the debt, the stock will have no value. Thus S can be seen as a call option with underlying process V and strike price the face value of the debt. Within the same approach, the debt is the sum of the facevalue of the bonds discounted and a short position on a put option with underlying process V and strike price the face value of the debt. The authors conclude that the corporate bonds could then be priced easily with a Black-Scholes equation having estimated the proper parameters for the V process. Different and more sophisticated models have been studied within the same framework (structural models), but we are not going to go through them.

An alternative approach to model credit risk is to consider the default event τ and attempt to model it directly. This will lead us to the so-called intensity models. In this thesis, we will explore the modeling of credit risk through intensity models. We will also analyze an instrument called Credit Default Swap, which is a response of the market to credit risk. Finally, we will illustrate the pricing of a zero-coupon bond subject to credit risk with numerical examples inducing parameter dependence.

Chapter 2

Credit Risk Modeling

We will follow Brigo and Mercurio (2006) during this chapter to develop the framework for credit risk modeling.

2.1 Intensity Models

In the simplest intensity model, the default time is modeled as the first jump of a time homogeneous Poisson process. A time homogeneous Poisson process $\{M_t,\ t \geq 0\}$ is a unit-jump increasing, right continuous stochastic process with stationary independent increments. Let τ^1, τ^2, ..., be the first, second, etc., jump times of M. For these processes, there exists a positive constant $\bar{\gamma}$ such that:

$$\mathbb{Q}(M_t = 0) = \mathbb{Q}(\tau^1 > t) = \mathrm{e}^{-\bar{\gamma}t}$$

for all t. More properties of the process include:

$$\lim_{t \to 0} \frac{\mathbb{Q}(M_t \geq 2)}{t} = 0 \quad \text{and} \quad \lim_{t \to 0} \frac{\mathbb{Q}(M_t = 1)}{t} = \bar{\gamma}.$$

The constant $\bar{\gamma}$ further obeys:

$$\bar{\gamma} = \frac{\mathbb{E}(M_t)}{t} = \frac{\mathrm{Var}(M_t)}{t}.$$

In particular, $\bar{\gamma}\tau^1$ is a standard exponential random variable. If we define the default time $\tau := \tau^1$, i.e. as the first jump of a Poisson process, we can calculate its survival probability as:

$$\mathbb{Q}(\tau > t) = \mathrm{e}^{-\bar{\gamma}t}.$$

This result is important because it tells us that survival probabilities have the same structure as *discount factors.* The default intensity $\bar{\gamma}$ plays the same role as interest rates. This property will allow us to view default intensities as *credit spreads.*

Consider now a deterministic time-varying intensity $\gamma(t)$, which is defined as a positive and piecewise right continuous function. Define

$$\Gamma(t) := \int_0^t \gamma(u)du$$

as the cumulated intensity, cumulated hazard rate or Hazard function. If M_t is a standard Poisson process (with intensity one) then a time-inhomogeneous Poisson process N_t with intensity γ is defined as

$$N_t = M_{\Gamma(t)}.$$

The increments of the process N_t are no longer identically distributed due to the time distortion induced by Γ. From the previous equality, we have that N jumps the first time at $\tau \iff M$ jumps the first time at $\Gamma(\tau)$. But since M is a standard Poisson Process with intensity one, then:

$$\Gamma(\tau) =: \xi \sim \text{Exponential Random Variable (1)}$$

i.e. if we transform the first jump time of a Poisson process according to its cumulated intensity, we obtain a standard exponential random variable (independent of all previous processes in the given probability space). Using the transformation, we get:

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\Gamma(\tau) > \Gamma(t)) = \mathbb{Q}(\xi > \Gamma(t)) = \mathrm{e}^{-\Gamma(t)} = \mathrm{e}^{-\int_0^t \gamma(u)du}$$

where again we can see that the Poisson process core structure allow us to view the default survival probability as a discount factor. However, ξ is independent of all default free market quantities and represents an external source of randomness that makes intensity models incomplete.

We can further develop the model to let it capture credit spread volatility. Let intensity be a stochastic process $\mathcal{F}_t$-adapted and right continuous denoted by λ_t. The cumulated intensity or hazard process is the random variable $\Lambda(t) = \int_0^t \lambda_s \, ds$. This process is called a Cox process with stochastic intensity λ_s. The process, conditional on $\mathcal{F}_t$ (or just on $\mathcal{F}_t^\lambda = \sigma(\{\lambda_s : s \leq t\})$, i.e. just on the paths of λ_s), preserves the Poisson process structure and all the facts that we have seen for $\bar{\gamma}$ and $\gamma(t)$ hold for λ_t. We have in particular that $\Lambda(\tau) = \xi$, with ξ being a standard exponential random variable independent of $\mathcal{F}_t$. In a similar way, for the survival probability, we have:

$$\begin{aligned} \mathbb{Q}(\tau > t) &= \mathbb{Q}(\Lambda(\tau) > \Lambda(t)) = \mathbb{Q}(\xi > \Lambda(t)) = \\ &= \mathbb{E}\left[\mathbb{Q}(\xi > \Lambda(t) \mid \mathcal{F}_t^\lambda)\right] = \mathbb{E}\left[\mathrm{e}^{-\int_0^t \lambda_s ds}\right] \end{aligned}$$

which is completely analogous to a zero-coupon bond formula with interest rate process λ_s. We can thus model λ_s as if it was a diffusion process similar to an interest rate. The time varying nature of λ_s can account for the term structure of credit spreads while the stochasticity can be used to introduce credit spread volatility.

2.1.1 Dependence between interest rates and the default event

A Poisson process and a Brownian Motion defined on the same probability space are independent (see Bielecki and Rutkowski (2001)). We can thus expect, under deterministic intensities, that r_t and τ will be independent. The dependence can surge from a dependence between the stochastic intensity λ_t and r_t. This will induce a dependence between τ and r_t, coming from λ_t.

Although a first approach in the further development of the model would be to assume independence between the two variables, dependence will become desirable. There is empirical evidence of dependence between the interest rates and credit spreads (see for example Morris et. al. (1998)) and although we are not going to assume nor imply any economic conclusions by taking a specific dependence structure, we would like to be able to measure changes in valuation due to different degrees of dependence. In particular, and for example, when the market perceives a deteriorating condition of solvency within a company (or within a set of companies), there is a *flight to safety,* which means that risk-free bonds are bought driving interest rates down. The first effect would widen the credit spread and the second would lower the interest rate. What is the resulting net effect in the price of a corporate bond?

Consider the price of a zero-coupon corporate bond $\bar{P}(t,T)$. Assume that if the company defaults before T (i.e. $\tau \leq T$), no money is recovered. Following the Black-Scholes framework (taking expectation under the risk-neutral measure Q and $r_t = r$), the price would then be:

$$\begin{aligned}\bar{P}(t,T) =&\mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds} \times 1_{\{\tau>T\}}\right] = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}\ 1_{\{\Lambda(\tau)>\Lambda(T)\}}\right] = \\ =&\mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}\ 1_{\{\xi>\Lambda(T)\}}\right] = \mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{-\int_t^T r_s ds}\ 1_{\{\xi>\Lambda(T)\}} \mid \mathcal{F}_T\right)\right] = \\ =&\mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}\ \mathbb{E}\left(1_{\{\xi>\Lambda(T)\}} \mid \mathcal{F}_T\right)\right] = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}\mathrm{e}^{-\int_t^T \lambda_s ds}\right] = \\ =&\mathbb{E}\left[\mathrm{e}^{-\int_t^T (r_s+\lambda_s)ds}\right]. \end{aligned} \tag{2.1}$$

This formula is general and in the last line, we can see that there will be an impact in the valuation of the bond if r_s and λ_s are dependent. If they are independent however, we can split the last line:

$$\begin{aligned}\bar{P}(t,T) =&\mathbb{E}\left[\mathrm{e}^{-\int_t^T (r_s+\lambda_s)ds}\right] = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}\right]\mathbb{E}\left[\mathrm{e}^{-\int_t^T \lambda_s ds}\right] = \\ =&P(t,T)\ \mathbb{Q}(\tau>T)\end{aligned}$$

i.e. the defaultable zero-coupon bond equals the default-free zero-coupon bond price times the survival probability (under the risk-neutral measure Q).

2.2 General Valuation of Credit Risk

Formula (2.1) works only for bond contracts, and we would like to be able to analyze more general payoffs. We will face the problem from a default free investor entering a contract with a counterparty that has a positive probability of defaulting before maturity.

In general, when one enters a risky contract (risky in the sense of credit risk), one requires a risk premium. In the case of corporate bonds, as we saw above, their yield is higher than risk-free bonds. The positive credit spread implies a lower price for the corporate bonds. This is a typical feature of every contract subject to credit risk: The value of the claim subject to credit risk will always be smaller than a claim with zero default probability.

Let T be the final maturity of the payoff we are going to evaluate. If $\tau > T$ there is no default by the counterparty before the end of the contract and the claim is fulfilled. If $\tau \leq T$ the counterparty cannot fulfill its obligations. We assume now that if the Net Present Value (NPV) of the residual claims is positive for the investor (negative for the counterparty), only a recovery fraction $R \in [0,1]$ of the NPV is recovered. If the NPV is negative for the investor, it is completely paid to the counterparty. We are going to take expectations under the risk-neutral measure Q and the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$ which represents the flow of information on whether the default occurred before time t and on the regular default-free market information up to time t.

Let us call $\Pi^D(t)$ the payoff of a generic defaultable claim at time t and $\mathrm{CF}(u,s)$ the net cashflows of the claim between time u and time $s > u$, discounted back to time u. The payoffs are seen from the investor point of view. Define $\mathrm{NPV}(\tau) = \mathbb{E}_\tau[\mathrm{CF}(\tau,T)]$ and $D(u,s) = \mathrm{e}^{-\int_u^s r_w dw}$, the discount factor for cashflows from time s back to time u. We then have:

$$\begin{aligned}\Pi^D(t) =& 1_{\{\tau > T\}}\mathrm{CF}(t,T) \\ &+ 1_{\{\tau \leq T\}}\left[\mathrm{CF}(t,\tau) + D(t,\tau)\left(R\,(\mathrm{NPV}(\tau))^+ - (-\mathrm{NPV}(\tau))^+\right)\right]\end{aligned} \tag{2.2}$$

i.e. the payoff of a defaultable claim can be divided between the regular default-free payoff if $\tau > T$, plus the payoffs up until time τ if $\tau \leq T$, plus the value of the residual claims at time τ discounted to time t if $\tau \leq T$ (if the residual claims are positive they are multiplied by the recovery rate).

The expected value of (2.2) is the general price of the claim subject to credit risk. If we call $\Pi(t)$ the payoff for an equivalent claim with a default-free counterparty, we have the following proposition:

Proposition 1 (**General credit risk pricing formula).** *At valuation time t and assuming $\{\tau > t\}$, the price of a payoff under credit risk is*

$$\mathbb{E}_t\left[\Pi^D(t)\right] = \mathbb{E}_t[\Pi(t)] - L\,\mathbb{E}_t\left[1_{\{\tau \leq T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+\right] \tag{2.3}$$

where $L = 1 - R$ is the Loss Given Default (rate) and R is assumed to be deterministic and known.

Before proving the proposition, we want to comment on it. It is now clear with the proposition that the value of a generic claim subject to counterparty risk will always be smaller than a similar claim having zero default probability. In particular, the value of the defaultable claim will be the sum of the corresponding default-free claim minus a call option (with strike zero) on the residual NPV value only in scenarios where $\tau \le T$ times L.

A second remark is the equality between (2.1) and (2.3) when it comes to corporate bonds. In particular, for a corporate bond, from the investor point of view, assuming $L = 1$ (no money recovered from the default) and in case $\tau \le T$, $(\mathrm{NPV}(\tau))^+ = (D(\tau,T))^+ = D(\tau,T)$ so (2.3) converts to:

$$\begin{aligned}\mathbb{E}_t\left[D(t,T)\right] - \mathbb{E}_t\left[1_{\{\tau\le T\}}D(t,\tau)D(\tau,T)\right] &= \mathbb{E}_t\left[D(t,T) - 1_{\{\tau\le T\}}D(t,T)\right] = \\ &= \mathbb{E}_t\left[D(t,T)(1 - 1_{\{\tau\le T\}})\right] = \mathbb{E}_t\left[D(t,T)\ 1_{\{\tau>T\}}\right]\end{aligned}$$

which is precisely (2.1). Note that $D(t,\tau)D(\tau,T) = \mathrm{e}^{-\int_t^\tau r_s ds}\mathrm{e}^{-\int_\tau^T r_s ds} = \mathrm{e}^{-\int_t^T r_s ds} = D(t,T)$. We are now going to prove the proposition.

Proof. Since

$$\Pi(t) = \mathrm{CF}(t,T) = 1_{\{\tau>T\}}\mathrm{CF}(t,T) + 1_{\{\tau\le T\}}\mathrm{CF}(t,T)$$

we can rewrite the terms inside the expectations in the right hand side of (2.3) as:

$$\begin{aligned}&1_{\{\tau>T\}}\mathrm{CF}(t,T) + 1_{\{\tau\le T\}}\mathrm{CF}(t,T) + (R-1)\left[1_{\{\tau\le T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+\right] = \\ =&1_{\{\tau>T\}}\mathrm{CF}(t,T) + 1_{\{\tau\le T\}}\mathrm{CF}(t,T) + R\ 1_{\{\tau\le T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+ \\ &-1_{\{\tau\le T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+.\end{aligned} \tag{2.4}$$

The second and fourth terms have expectation equal to:

$$\begin{aligned}&\mathbb{E}_t\left[1_{\{\tau\le T\}}\mathrm{CF}(t,T) - 1_{\{\tau\le T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+\right] = \\ =&\mathbb{E}_t\left[1_{\{\tau\le T\}}\left\{\mathrm{CF}(t,\tau) + D(t,\tau)\mathrm{NPV}(\tau) - D(t,\tau)(\mathrm{NPV}(\tau))^+\right\}\right] = \\ =&\mathbb{E}_t\left[1_{\{\tau\le T\}}\left\{\mathrm{CF}(t,\tau) - D(t,\tau)(-\mathrm{NPV}(\tau))^+\right\}\right]\end{aligned} \tag{2.5}$$

where we use the following property (towering of expectations):

$$\begin{aligned}&\mathbb{E}_t\left[1_{\{\tau\le T\}}\mathrm{CF}(t,T)\right] = \\ =&\mathbb{E}_t\left[1_{\{\tau\le T\}}\left\{\mathrm{CF}(t,\tau) + D(t,\tau)\mathbb{E}_\tau\left[\mathrm{CF}(\tau,T)\right]\right\}\right] \\ =&\mathbb{E}_t\left[1_{\{\tau\le T\}}\left\{\mathrm{CF}(t,\tau) + D(t,\tau)\mathrm{NPV}(\tau)\right\}\right].\end{aligned}$$

Substituting the result of (2.5) into (2.4), and taking the expected value, we get the expected value of (2.2) which is the price of the claim subject to credit risk. □

2.2.1 An Alternative Credit Risk Valuation Formula

A general market practice for valuation of claims with default risk is to discount the cash-flows of the claims incorporating a credit spread, such as in (2.1). We would like to establish a connection between (2.3) and this general market practice.

To establish a connection, we are going to alter an assumption we previously did. For the moment being, let us consider a defaultable claim X payable at time T, which value is positive (with probability one) to the investor, i.e. $X \geq 0$ from the investor point of view. In particular, this will imply that $\mathrm{NPV}(\tau) \geq 0$ so that (2.2) becomes:

$$\Pi^D(t) = 1_{\{\tau > T\}} D(t,T)\ X + 1_{\{\tau \leq T\}} D(t,\tau)\ R\ \mathrm{NPV}(\tau).$$

Let us denote the price process of the defaultable claim by V_t, i.e. $V_t = \mathbb{E}_t\left[\Pi^D(t)\right]$. Consider $V_{\tau^-} := \lim_{t\to\tau} V_t$, $(t < \tau)$, i.e. the value of the claim just before default. Assume that the process V_t is almost surely continuous (among other assumptions, see Duffie and Singleton (1999)). This assumption implies $V_{\tau^-} = \mathrm{NPV}(\tau)$ and therefore:

$$\begin{aligned} V_t =& \mathbb{E}_t\left[1_{\{\tau > T\}} D(t,T)\ X + 1_{\{\tau \leq T\}} D(t,\tau)\ R\ \mathrm{NPV}(\tau)\right] \\ =& \mathbb{E}_t\left[D(t,\tau\wedge T)\left\{1_{\{\tau > T\}}\ X + 1_{\{\tau \leq T\}}\ (1-L)\ V_{\tau^-}\right\}\right]. \end{aligned} \tag{2.6}$$

Duffie and Singleton (1999) showed that, under mild conditions, the previous expression is equivalent to:

$$V_t = \mathbb{E}_t\left[\exp\left(-\int_t^T r_s + \lambda_s L\ ds\right) X\right] \tag{2.7}$$

which establishes the connection that we were seeking, in the sense that it discounts the claim X incorporating a credit spread $\lambda_s L$ to the risk-free rate r_s. The connection is such that, under the assumptions made for the claim X, (2.3) and (2.7) are equivalent.

We can in fact relax the assumption of X being positive for the investor, but we would again require a different assumption from the previous treatment. Let us assume that if the counterparty defaults at time $\tau \leq T$, independent of the sign of $\mathrm{NPV}(\tau)$, only the recovered fraction $R \times \mathrm{NPV}(\tau)$ would exchange hands, i.e. the investor would only get $R \times \mathrm{NPV}(\tau)$ if $\mathrm{NPV}(\tau) \geq 0$ and it would only pay $R \times \mathrm{NPV}(\tau)$ if $\mathrm{NPV}(\tau) < 0$. With this assumption, (2.2) again becomes:

$$\Pi^D(t) = 1_{\{\tau > T\}} D(t,T)\ X + 1_{\{\tau \leq T\}} D(t,\tau)\ R\ \mathrm{NPV}(\tau)$$

and following the same argumentation, we again conclude formula (2.7). The difference is that (2.3) and (2.7) are not equivalent anymore, due to the different assumptions.

To conclude this chapter, we would like to provide an informal proof of the equivalence between (2.6) and (2.7), following Duffie and Singleton (1999), and generalizing it to include claims with different time payoffs. Define the process Ψ_t as 0 before the default time τ and 1 afterwards, i.e. $\Psi_t = 1_{\{\tau \leq t\}}$. Let λ_t continue to be the risk-neutral hazard rate for τ. We can write:

$$d\Psi_t = (1 - \Psi_t)\lambda_t \ dt + d\mu_t^1$$

where μ_t^1 is a martingale under Q. The process Ψ_t is the first jump of a Poisson process with hazard rate λ_t. Let also:

$$dV_t = \alpha \ dt + d\mu_t^2$$

where μ_t^2 is a martingale under Q. Consider now the gain process G_t (price plus cumulative dividends) discounted by the risk-free rate. This discounted gain process is defined as:

$$\begin{aligned} G_t =& \exp\left(-\int_0^t r_s ds\right) V_t(1 - \Psi_t) \\ &+ \int_0^t \exp\left(-\int_0^s r_u du\right)(1 - L)V_{s^-} \ d\Psi_s. \end{aligned}$$

The first term is the discounted price of the claim and the second is the discounted payoff of the claim upon default. It must hold then that under Q, G_t is a martingale. Applying Ito's formula, we get:

$$\begin{aligned} dG_t =& -\exp\left(-\int_0^t r_s ds\right) r_t V_t(1 - \Psi_t) \ dt + \exp\left(-\int_0^t r_s ds\right)(1 - \Psi_t) \ dV_t \\ &- \exp\left(-\int_0^t r_s ds\right) V_t \ d\Psi_t + \exp\left(-\int_0^t r_u du\right)(1 - L)V_{t^-} \ d\Psi_t. \end{aligned}$$

Substituting for $d\Psi_t$ and dV_t, as well as V_{t^-} for V_t because of the continuity assumption, and rearranging and grouping terms we get:

$$\begin{aligned} dG_t =& \exp\left(-\int_0^t r_s ds\right)\Big\{ - r_t V_t(1 - \Psi_t) + (1 - \Psi_t)\alpha \\ & -V_t(1 - \Psi_t)\lambda_t + (1 - L)V_t(1 - \Psi_t)\lambda_t \Big\} \ dt + d\mu_t^3 \end{aligned}$$

where μ_t^3 is a martingale under Q. The terms within brackets must be zero (in order for G_t to be a martingale under Q) and can further be expanded:

$$\begin{aligned} 0 =& (1 - \Psi_t)(-r_t V_t + \alpha - V_t\lambda_t + V_t\lambda_t(1 - L)) \\ =& (1 - \Psi_t)(-r_t V_t + \alpha - L \ V_t\lambda_t) \end{aligned}$$

and since $t < \tau$, the right hand parenthesis must equal zero. Hence:

$$\begin{aligned} \alpha =& r_t V_t + L \ V_t \lambda_t \\ =& V_t \ (r_t + \lambda_t L) \end{aligned}$$

i.e. the drift of the price process under Q is $V_t \ (r_t + \lambda_t L)$. We should therefore discount the claims under risk-neutral valuation with the risk-free rate r_s plus the credit spread $\lambda_s L$. We can conclude that a defaultable contract, with promised payoff X_i at time T_i, where $i = \{1, 2, ..., K\}$, has price at time t equal to:

$$V_t = \mathbb{E}_t \left[\sum_{i=1}^{K} \exp \left(\int_t^{T_i} r_s + \lambda_s L \ ds \right) X_i \right]. \tag{2.8}$$

It is a unfortunate that in general (2.3) and (2.8) are not equivalent. It would be convenient to have two different pricing formulas for defaultable claims. In particular, (2.8) is elegant in its simplicity and looks at a first sight easier in calculations than (2.3). Considering pricing with simulation, (2.8) involves only two processes: r_s and λ_s whereas (2.3) involves τ as well . It could then be simpler and faster to make calculations with (2.8).

Additionally, the fact that the discount is done with a spread invites us to treat the sum of the risk-free rate and the spread as a single rate, and use the extensive machinery that has been developed within the defaultable free framework.

Nevertheless, the formulas are equivalent when the contract under consideration is positive in value at all times. Even though this limitation excludes a number of different contracts, a fairly rich family of contracts that we can value using the two formulas remain. An example of these are options on credit default swaps, among others. It would be interesting to compare the two valuation formulas within different claims to explore computational efficiency of either.

Chapter 3

Credit Default Swaps

One response of the market to credit risk was the development of the Credit Default Swap (CDS). A CDS is a contract that ensures protection against default. CDSs became quite liquid in the last few years. It consists of two entities "A" (protection buyer) and "B" (protection seller) that establish the following contract: During a specified time period $[T_a, T_b]$, A will pay B a certain rate s periodically, over some notional (typically 1), with the purpose of protecting itself from a default from a third entity "C". If "C" defaults at time τ, with $T_a < \tau < T_b$, "B" will pay "A" a certain deterministic cash amount L (typically $L =$ notional - recovery $= 1 - R$), and "A" would stop further payments to "B". The payments done by "A" is called the *premium leg* and the payment done by "B", in case of default by "C", is called the *protection leg.*

Although CDSs can be used as speculative instruments, they can also be used for hedging purposes. Suppose that "A" buys a corporate bond issued by "C". This corporate bond would typically have a spread over the risk-free bond. To mitigate the credit risk associated with the bond, "A" could buy the protection leg of a CDS (from "B") with T_b being equal to the time of maturity of the bond. In principle, "A" is *hedged* against any credit event from "C", i.e. if "C" defaults, "A" will get its notional amount back. The combination of both the corporate bond and the protection leg of the CDS would resemble a risk-free bond. By a no-arbitrage argument, s would have to be very similar to the spread of the corporate bond over the risk-free bond. In fact, when the CDS is constituted at time t, s is set so that the expected present value of the two exchange flows is zero (both the protection leg and premium leg). The market quotes both bid and ask prices for s. The bid price is the rate which banks would be willing to pay to buy protection on entity "C" and the ask price is the rate at which they would be willing to sell it.

Consider the following example to clarify CDSs: Suppose that company "A" enters into a 5-year CDS contract with "B", where "A" agrees to pay 90 basis points (0.0090) annually for protection against default by company "C", for a notional coverage of \$100 million. If "C" does not default during the 5 years, "A" will receive no payoff and pays to "B" \$900,000 every year during the duration of the contract. Suppose on the contrary that there is a credit event by "C" a quarter of time into the fourth year. "A" will pay the accrued payment from the beginning of the year until the quarter of the year (i.e. $0.25 * 900,000$) and receive \$100 million $\times$ L. In particular if the five year risk-free bond yields 5%, the yield of the corporate bond issued by "C" should be approximatedly 5.90%. If it is more than this, an investor can do an arbitrage by borrowing at the risk-free rate and by buying the corporate bond and buying CDS protection. If it is less, the arbitrage strategy would consist in shorting (selling) the corporate bond, buying risk-free bonds and selling CDS protection.

We are going to analyze now the pricing of a CDS, following Brigo and Mercurio (2006). First we want to remark that we will not consider default events from the counterparties involved in the CDS ("A" or "B"). We assume this entities will surely make the payments they are committed to and therefore we only examine the default event coming from a third counterparty "C". It is important to keep this in mind because, for example, the counterparty who sells the protection might be involved with "C", thus making it more likely that if "C" defaults, "B" will default on its promised payments. This situation aroused in the recent financial crisis when the investment bank Lehman Brothers went bankrupt in September 2008. There were a numerous amount of CDSs concerning Lehman Brothers, and the collapse of the bank threatened at some point the financial solvency of the banks that sold or bought these CDSs.

3.1 Different CDSs Formulations and Pricing Formulas

Consider a CDS valid during the time interval $[T_a, T_b]$. The premium leg pays the rate s at times $T_{a+1}, T_{a+2}, ..., T_b$ or until default time τ of the reference entity "C", if $\tau \in [T_a, T_b]$. The protection leg pays the *deterministic* amount L at the default time τ, again if $\tau \in [T_a, T_b]$. This particular CDS is called a "running CDS" (RCDS). Formally, we may write the discounted cashflows of the RCDS, denoted by Π_{RCDS}, seen from "B" at time t as:

$$\begin{aligned}\Pi_{RCDS}(t) = &\sum_{i=a+1}^{b} D(t, T_i)\ \alpha_i\ s\ 1_{\{\tau \geq T_i\}} \\ &+ D(t, \tau)(\tau - T_{\beta(\tau)-1})\ s\ 1_{\{T_a < \tau < T_b\}} - D(t, \tau)\ L\ 1_{\{T_a < \tau \leq T_b\}} \end{aligned} \tag{3.1}$$

where $\alpha_i = T_i - T_{i-1}$ and $T_\beta(t)$ is the first date among the T_i's that follows t, i.e. $T_{\beta(\tau)-1}$ is the last T_i before the default event. The first term of the previous expression is the present value of the premium leg, the second term is the accrued present value of the premium leg (in case of default) and the third term is the present value of the protection leg.

A variation of the RCDS is if we don't consider the exact default time τ for exchanging cashflow but instead the first time T_i following it, i.e. if we consider $T_{\beta(\tau)}$. This is called a "postponed payoff RCDS" (PRCDS). In a similar manner:

$$\Pi_{PRCDS}(t) = \sum_{i=a+1}^{b} D(t,T_i)\ \alpha_i\ s\ 1_{\{\tau>T_{i-1}\}} - \sum_{i=a+1}^{b} D(t,T_i)\ L\ 1_{\{T_{i-1}<\tau\leq T_i\}}.$$

In general, we can compute the CDS price according to risk-neutral valuation. If we denote $CDS(t,s,L)$ the price at time t, seen from "B", for the standard RCDS described above, we then have:

$$CDS(t,s,L) = \mathbb{E}_t\left[\Pi_{RCDS}(t) \mid \mathcal{G}_t\right]. \tag{3.2}$$

Notice that the above expectation is conditional on the filtration $\mathcal{G}_t$. In some cases it is better to continue computing prices as expectations conditional on the usual default-free filtration $\mathcal{F}_t$. The change of filtration is possible and in particular we can write (3.2) as:

$$CDS(t,s,L) = \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t \mid \mathcal{F}_t)}\mathbb{E}_t\left[\Pi_{RCDS}(t) \mid \mathcal{F}_t\right]. \tag{3.3}$$

For the proof, see Brigo and Mercurio (2006). We can therefore write the price of the RCDS, using (3.1) as:

$$\begin{aligned} CDS(t,s,L) =&\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t \mid \mathcal{F}_t)}\left\{\sum_{i=a+1}^{b}\alpha_i\ s\ \mathbb{E}_t\left[D(t,T_i)1_{\{\tau\geq T_i\}} \mid \mathcal{F}_t\right]\right.\\ &+s\ \mathbb{E}_t\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})\ 1_{\{T_a<\tau<T_b\}} \mid \mathcal{F}_t\right]\\ &\left.-L\ \mathbb{E}_t\left[D(t,\tau)\ 1_{\{T_a<\tau\leq T_b\}} \mid \mathcal{F}_t\right]\right\}. \end{aligned} \tag{3.4}$$

If we consider again the price of a zero-coupon defaultable corporate bond, ignoring any recovery in case of default, we have:

$$\begin{aligned} \bar{P}(t,T) = 1_{\{\tau>t\}}\ \bar{P}(t,T) =&\mathbb{E}_t\left[D(t,T)\ 1_{\{\tau>T\}} \mid \mathcal{G}_t\right]\\ =&\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t \mid \mathcal{F}_t)}\mathbb{E}_t\left[D(t,T)\ 1_{\{\tau>T\}} \mid \mathcal{F}_t\right] \end{aligned}$$

where we have changed the filtration again. Rearranging terms, we conclude:

$$\mathbb{E}_t\left[D(t,T)\ 1_{\{\tau>T\}} \mid \mathcal{F}_t\right] = \bar{P}(t,T)\ \mathbb{Q}(\tau>t \mid \mathcal{F}_t).$$

Substituting this last equation into (3.4) we obtain:

$$CDS(t,s,L) =\frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t \mid \mathcal{F}_t)}\left\{\sum_{i=a+1}^{b}\alpha_i\ s\ \bar{P}(t,T_i)\ \mathbb{Q}(\tau>t \mid \mathcal{F}_t)+\right.$$

$$+s\ \mathbb{E}_t\left[D(t,\tau)(\tau - T_{\beta(\tau)-1})\ 1_{\{T_a<\tau<T_b\}} \mid \mathcal{F}_t\right] +$$
$$\left.-L\ \mathbb{E}_t\left[D(t,\tau)\ 1_{\{T_a<\tau\leq T_b\}} \mid \mathcal{F}_t\right]\right\}. \tag{3.5}$$

A CDS agreed at time t is fixed with the rate $s(t)$ such that the contract has value zero at t, i.e. $CDS(t, s(t), L) = 0$. Solving this equation for $s(t)$, using (3.5) gives us the expression for the CDS forward rate $s(t)$:

$$s(t) = \frac{L\ \mathbb{E}_t\left[D(t,\tau)\ 1_{\{T_a<\tau\leq T_b\}} \mid \mathcal{F}_t\right]}{\sum_{i=a+1}^{b} \alpha_i\ \bar{P}(t,T_i)\ \mathbb{Q}(\tau > t \mid \mathcal{F}_t) + \text{accrual}_t}$$

where $\text{accrual}_t = \mathbb{E}_t\left[D(t,\tau)(\tau - T_{\beta(\tau)-1})\ 1_{\{T_a<\tau<T_b\}} \mid \mathcal{F}_t\right]$.

We can further develop (3.5) if we assume independence between interest rates and the default time τ, i.e. $D(u,t)$ independent of τ for all possible $0 < u < t$.

Consider the value of the premium leg (PR) of the CDS at time 0:

$$\begin{aligned}
PR =& \mathbb{E}\left[D(0,\tau)(\tau - T_{\beta(\tau)-1})\ s\ 1_{\{T_a<\tau<T_b\}}\right] + \\
&+ \sum_{i=a+1}^{b} \mathbb{E}\left[D(0,T_i)\ \alpha_i\ s\ 1_{\{\tau\geq T_i\}}\right] \\
=& \mathbb{E}\left[\int_{t=0}^{\infty} D(0,t)(t - T_{\beta(t)-1})\ s\ 1_{\{T_a<t<T_b\}} 1_{\{\tau\in[t,t+dt]\}}\right] + \\
&+ \sum_{i=a+1}^{b} \mathbb{E}\left[D(0,T_i)\right]\ \alpha_i\ s\ \mathbb{E}\left[1_{\{\tau\geq T_i\}}\right] \\
=& \int_{t=T_a}^{T_b} \mathbb{E}\left[D(0,t)(t - T_{\beta(t)-1})\ s\ 1_{\{\tau\in[t,t+dt)\}}\right] + \\
&+ \sum_{i=a+1}^{b} P(0,T_i)\ \alpha_i\ s\ \mathbb{Q}(\tau \geq T_i) \\
=& \int_{t=T_a}^{T_b} \mathbb{E}\left[D(0,t)\right](t - T_{\beta(t)-1})\ s\ \mathbb{E}\left[1_{\{\tau\in[t,t+dt)\}}\right] + \\
&+ \sum_{i=a+1}^{b} P(0,T_i)\ \alpha_i\ s\ \mathbb{Q}(\tau \geq T_i) \\
=& s \int_{t=T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\ \mathbb{Q}(\tau \in [t, t+dt)) + \\
&+ s \sum_{i=a+1}^{b} P(0,T_i)\ \alpha_i\ \mathbb{Q}(\tau \geq T_i) \\
=& s\left\{ - \int_{t=T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\ d_t\mathbb{Q}(\tau \geq t) + \right. \\
&\left. + \sum_{i=a+1}^{b} P(0,T_i)\ \alpha_i\ \mathbb{Q}(\tau \geq T_i) \right\}.
\end{aligned} \tag{3.6}$$

Similarly, for the protection leg (PL):

$$
\begin{aligned}
PL =& \mathbb{E}\left[D(0,\tau)\ L\ 1_{\{T_a<\tau\le T_b\}}\right] \\
=& L\ \mathbb{E}\left[\int_{t=0}^{\infty} D(0,t) 1_{\{T_a<t\le T_b\}} 1_{\{\tau\in[t,t+dt)\}}\right] \\
=& L \int_{t=T_a}^{T_b} \mathbb{E}\left[D(0,t) 1_{\{\tau\in[t,t+dt)\}}\right] \\
=& L \int_{t=T_a}^{T_b} \mathbb{E}\left[D(0,t)\right] \mathbb{E}\left[1_{\{\tau\in[t,t+dt)\}}\right] \\
=& L \int_{t=T_a}^{T_b} P(0,t)\ \mathbb{Q}(\tau\in[t,t+dt)) \\
=& -L \int_{t=T_a}^{T_b} P(0,t)\ d_t\mathbb{Q}(\tau\ge t).
\end{aligned} \tag{3.7}
$$

In principle, both (3.6) and (3.7) are model independent given the initial zero-coupon risk-free bond price curve observed at time 0 in the market, and the survival probabilities of τ at time 0. A possible way to strip survival probabilities, is to equate both values for different maturities, starting from $T_b = 1y$, finding the market implied survival probabilities $\{\mathbb{Q}(\tau \ge t), t \le 1y)\}$, and "bootstrapping" up to $T_b = 10y$. In practice one can consider deterministic intensities (in particular piecewise linear or constant between different maturities) of the underlying hazard rate function to facilitate the stripping of survival probabilities. We will make a concrete example of this later on. One could also strip survival probabilities from the zero-coupon corporate bond price curve, assuming that there exists enough bonds issued by entity "C" to make this procedure possible.

Before we give a concrete example of how to strip survival probabilities with the CDSs market data, we want to establish a simple relationship between the CDS forward rate s and the intensity process γ.

3.2 Simple Relationship Between s and γ.

A simpler formula to calibrate a deterministic constant intensity $\gamma(t) = \gamma$ to a single CDS that is of common use in the market is:

$$\gamma = \frac{s}{L}.$$

The formula tells us that given a constant hazard rate (and subsequent independence between τ and r_t), s can be interpreted as a credit spread or as a default probability. In particular, s is equal to the loss incurred by the default times the probability of instant default, i.e. $s = L \times \gamma$.

To derive the formula, we need to change the CDS cashflows. We will assume independence between τ and r_t and we will also assume that the premium leg pays the rate s continuously until τ or T_b, i.e. the premium leg pays $s\ dt$ for the interval $[t, t + dt]$ given that default has not happened. Discounting each flow by $D(0, t)$ and adding all the flows, the premium leg (PR) is worth:

$$\begin{aligned} PR =& \mathbb{E}\left[\int_0^{T_b} D(0,t)1_{\{\tau>t\}}\ s\ dt\right] \\ =& s\int_0^{T_b} \mathbb{E}\left[D(0,t)\right]\mathbb{E}\left[1_{\{\tau>t\}}\right] dt \\ =& s\int_0^{T_b} P(0,t)\ \mathbb{Q}(\tau > t)\ dt. \end{aligned} \tag{3.8}$$

Assume also that default comes from a constant intensity model like we have considered, i.e. $\mathbb{Q}(\tau > t) = \mathrm{e}^{-\gamma t}$. Then:

$$d\ \mathbb{Q}(\tau > t) = -\gamma \mathrm{e}^{-\gamma t} dt = -\gamma\ \mathbb{Q}(\tau > t)\ dt$$

and substituting this last expression in (3.7), we obtain:

$$PL = -L\int_0^{T_b} P(0,t)\ d_t\mathbb{Q}(\tau \geq t) = L\gamma\int_0^{T_b} P(0,t)\ \mathbb{Q}(\tau > t)\ dt. \tag{3.9}$$

Equating (3.8) and (3.9) gives us the desired result.

The relationship is an approximation due to the assumption of continuous payments in the premium leg, but nevertheless gives a quick calibration of default intensities and s.

3.3 Implied Hazard Rates within CDS quotes

It is possible to strip survival probabilities from market instruments, such as CDSs and corporate bonds. We will review with some detail the procedure concerning CDSs. Consider again a simple intensity model and specifically assume the default intensity γ to be deterministic and *piecewise* constant, i.e.:

$$\gamma(t) = \gamma_i \text{ for } t \in [T_{i-1}, T_i)$$

where the T_i's span the different relevant maturities. With this particular form of the default intensity, the cumulated intensity becomes:

$$\Gamma(t) = \int_0^t \gamma(u)du = \sum_{i=1}^{\beta(t)-1} (T_i - T_{i-1})\gamma_i + (t - T_{\beta(t)-1})\gamma_{\beta(t)}$$

where $\beta(t)$ is the index of the first T_i following t. For simplicity of notation, denote:

$$\Gamma_j = \int_0^{T_j} \gamma(u)du = \sum_{i=1}^{j}(T_i - T_{i-1})\gamma_i$$

The survival probabilities become:

$$\mathbb{Q}(\tau > t) = \exp\left(-\int_0^t \gamma(u)du\right) = \exp\left(-\Gamma_{\beta(t)-1} - (t - T_{\beta(t)-1})\gamma_{\beta(t)}\right)$$

And its derivative:

$$d\mathbb{Q}(\tau > t) = -\gamma_{\beta(t)} \exp\left(-\Gamma_{\beta(t)-1} - (t - T_{\beta(t)-1})\gamma_{\beta(t)}\right)$$

Substituting these expressions into (3.6) and (3.7) gives us, after some calculations, the price of a CDS valid during the time interval $[T_a, T_b]$ at time 0:

$$\begin{aligned}
&CDS_{a,b}(0, s, L; \Gamma(\cdot)) = \\
&= s \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1}))P(0,u)(u - T_{i-1})\ du + \\
&+ s \sum_{i=a+1}^{b} P(0, T_i)\ \alpha_i\ \exp(-\Gamma(T_i)) - \\
&- L \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1}))P(0,u)\ du
\end{aligned}$$

The substitution is valid because if we assume $\gamma(t)$ as we did (deterministic), we have independence between the interest rates and the default time τ, thus the use of (3.6) and (3.7) is justified.

In the market, $T_a = 0$ and s is usually quoted for $T_b = 1y, 2y, 3y, ..., 10y$ with the T_i's resetting quarterly. We would then solve:

$$CDS_{0,1y}(0, s^{1y}, L; \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma^1) = 0;$$
$$CDS_{0,2y}(0, s^{2y}, L; \gamma^1; \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \gamma^2) = 0; ...$$

and so on, each time finding the new intensity parameters. Note that s^{1y} is the market quote for the 1 year CDS, and so on.

3.3.1 A numerical example

We are now going to give a numerical example of the previous procedure. We decided to strip survival probabilities of the CDSs from the company Parmalat[1] that went into financial solvency problems in the final quarter of 2003, going bankrupt in December 2003. We took the CDSs quotes from Brigo and Mercurio (2006) and stripped the survival probabilities for four different dates: September 10, November 28, December 8, and December 10 of 2003. Although

[1] Parmalat, Wikipedia Article, http://en.wikipedia.org/wiki/Parmalat

the quotes are the same and our results are very similar to Brigo's and Mercurio's, we didn't expect them to be exactly equal due to the possible differences in the zero-coupon discount curve used. The Matlab source codes for the pricing and stripping of the CDSs can be find in appendix A.1 and A.2.

The following pages contain the results for the four different dates. We include for each date a table containing 5 different maturities CDSs, corresponding to 1, 3, 5, 7 and 10 years, with their respective market quotes, stripped piecewise constant intensity, and stripped survival probability. For each date we also include the recovery rate R used for the calibration, as well as a plot of the piecewise constant intensity and the survival probability.

3.3.2 Comments on the numerical example

The intensity in general is highest in the first period (the only exception being September 10). The market is perceiving the first period as the most risky. In fact, given the lower intensities in the following periods, the market perceives that if the firm survives the first year, its situation will improve considerably.

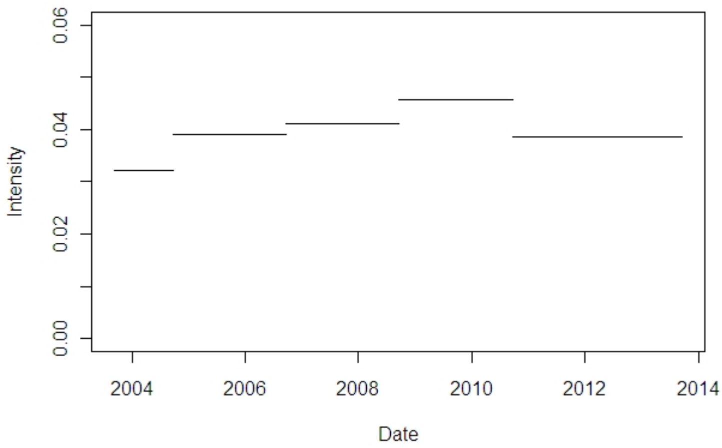
The intensity is a probability per unit of time so it has to be non-negative. For the last date (December 10), the intensity is negative for the period comprising the second and third year. This negativity gives rise to an increasing survival probability with respect to time. This is clearly pathological. Brigo and Mercurio attribute it to the CDS name being too *distressed.* The interpretation that we give to this explanation is the following: the quotes for the CDS are very high if they are compared to names that are not facing financial solvency problems. The quotes indicate a near-term high probability of default by the company. The piecewise constant intensity model sets a constant intensity for the whole first year, and the relatively big difference between the first year maturity quote and the third year maturity quote suggests that the market perceives that if the company survives the first year its situation will improve considerably. In fact, the situation could be such that the market is perceiving that if the company survives the following *months* its situation will improve. Thus a constant intensity for the whole first year and for the second and third year might a bad approach to model the situation, since during the first months of the CDS the constant intensity might be very high decreasing rapidly before the end of the first year. An ideal solution would be to have quotes that cover maturities in between periods, for example, a 6 month maturity quote, as well as a 2 year maturity quote.

In any way, the CDS quotes imply very high probabilities of default, which are a warning sign by themselves. It is also possible that the quotes are not completely reliable given the probable lack of liquidity these instruments experience when the name is under considerable financial solvency problems.

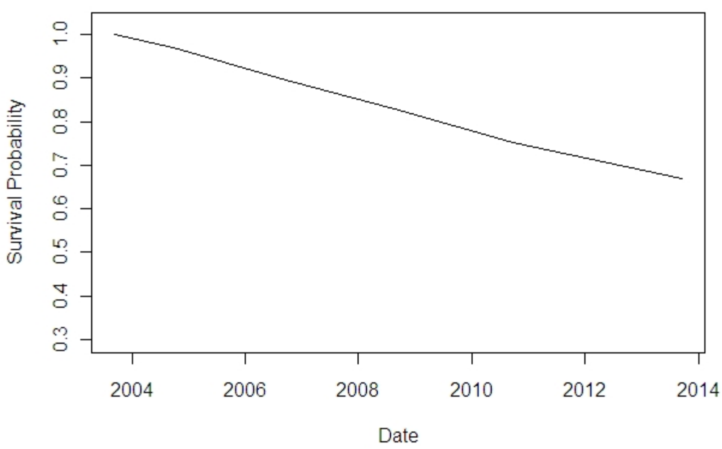
A final comment is that the example uses a simple piecewise constant intensity model. Another way to model the intensity, for example, is to assume that it is linear in between maturity dates. Nevertheless the piecewise constant model appears to be more solid and gives a fair explanation of how the market perceives the situation of the name under consideration.

Maturity (Years)	Maturity (Dates)	s^{MKT}	Intensity γ	Survival Probability
1	20-Sep-04	0.01925	0.0323	0.9683
3	20-Sep-06	0.0215	0.0390	0.8956
5	20-Sep-08	0.0225	0.0410	0.8251
7	20-Sep-10	0.0235	0.0456	0.7532
10	20-Sep-13	0.0235	0.0386	0.6707

(a) Results for September 10, 2003



(b) Piecewise Constant Intensity

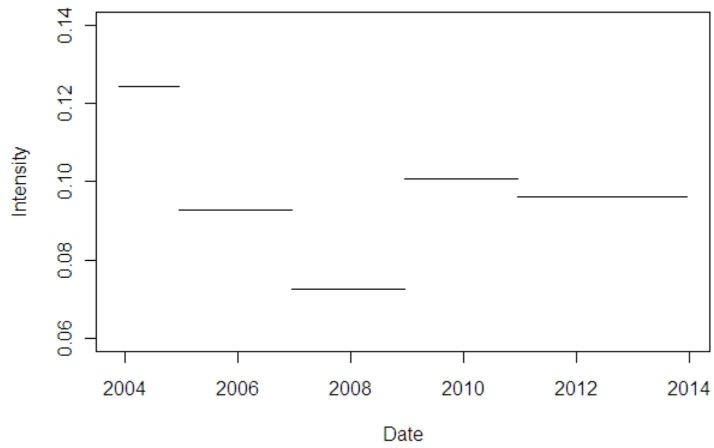


(c) Survival Probability

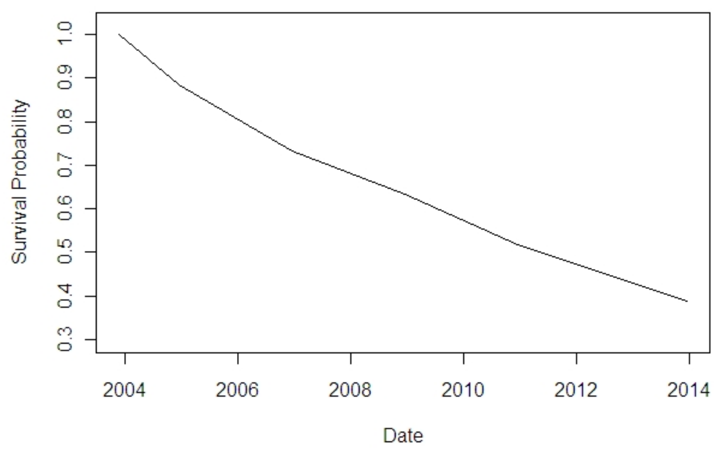
Figure 3.1: Calibration done for September 10, 2003. R=40%.

Maturity (Years)	Maturity (Dates)	s^{MKT}	Intensity γ	Survival Probability
1	20-Dec-04	0.0725	0.1242	0.8832
3	20-Dec-06	0.0630	0.0928	0.7336
5	20-Dec-08	0.0570	0.0725	0.6345
7	20-Dec-10	0.0570	0.1008	0.5187
10	20-Dec-13	0.0570	0.0962	0.3887

(a) Results for November 28, 2003



(b) Piecewise Constant Intensity

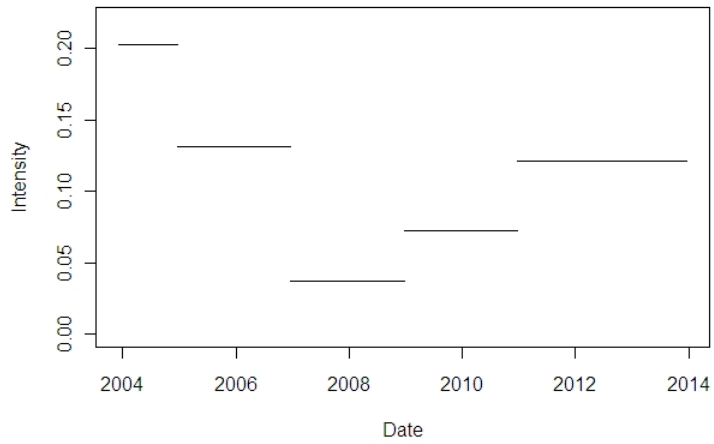


(c) Survival Probability

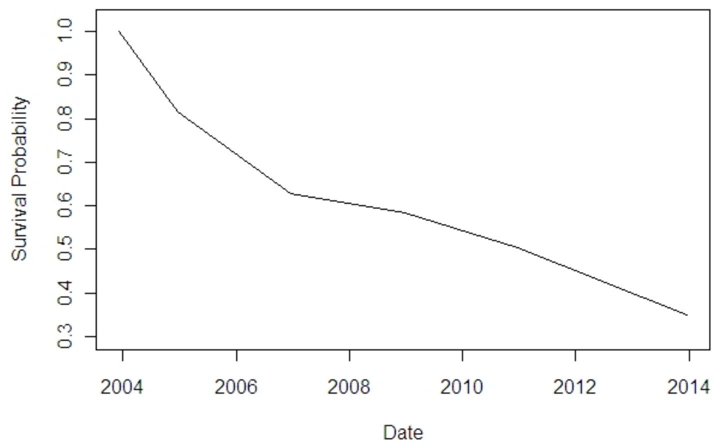
Figure 3.2: Calibration done for November 28, 2003. R=40%.

Maturity (Years)	Maturity (Dates)	s^{MKT}	Intensity γ	Survival Probability
1	20-Dec-04	0.1450	0.2026	0.8166
3	20-Dec-06	0.1200	0.1317	0.6275
5	20-Dec-08	0.0940	0.0370	0.5827
7	20-Dec-10	0.0850	0.0727	0.5039
10	20-Dec-13	0.0850	0.1210	0.3505

(a) Results for December 8, 2003



(b) Piecewise Constant Intensity

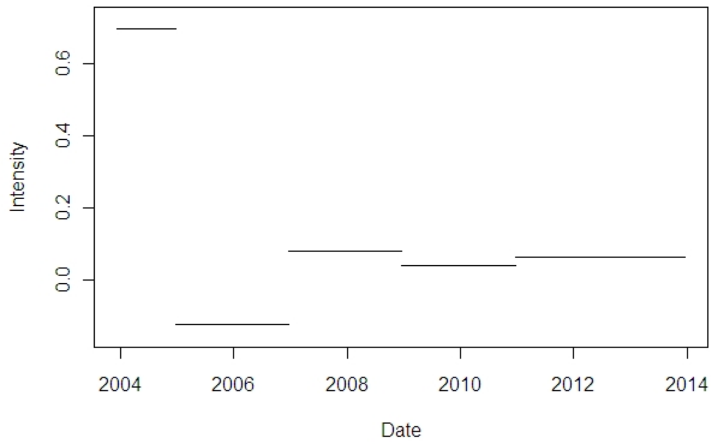


(c) Survival Probability

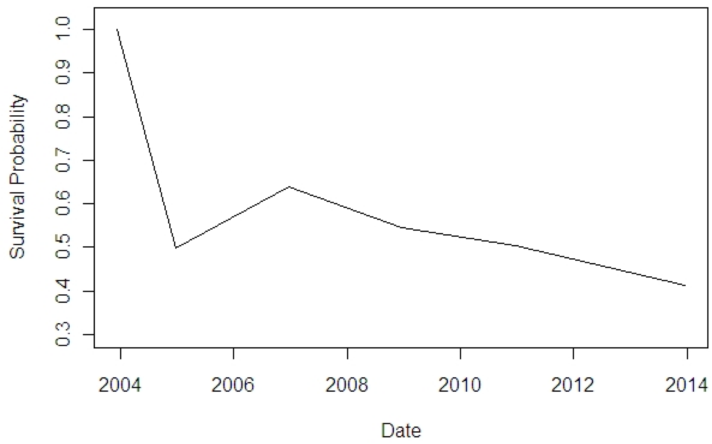
Figure 3.3: Calibration done for December 8, 2003. R=25%.

Maturity (Years)	Maturity (Dates)	s^{MKT}	Intensity γ	Survival Probability
1	20-Dec-04	0.5050	0.6956	0.4988
3	20-Dec-06	0.2100	-0.1231	0.6380
5	20-Dec-08	0.1500	0.0784	0.5454
7	20-Dec-10	0.1250	0.0403	0.5031
10	20-Dec-13	0.1100	0.0648	0.4143

(a) Results for December 10, 2003



(b) Piecewise Constant Intensity



(c) Survival Probability

Figure 3.4: Calibration done for December 10, 2003. R=15%.

Chapter 4

Pricing Under Dependence of r_t and λ_s

4.1 Corporate Bonds

In the second chapter we studied the pricing formula of a claim subject to credit risk. We want to focus now on how are defaultable claims prices affected by a dependence between the default time τ and r_t, specifically the price of a corporate bond. This dependence can surge from a dependence between the stochastic intensity λ_s and r_t.

From what we have seen, it is natural to model λ_t as a diffusion process similar to an interest rate. Consider the following model for r_t and λ_t:

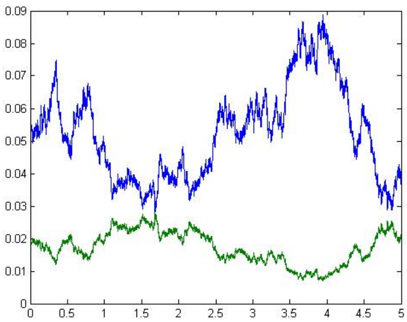
$$\begin{aligned} dr_t &= dr = \kappa(\theta - r)\ dt + \sigma_1\sqrt{r}\ dW \\ r_0 &= c_1 \\ d\lambda_t &= d\lambda = \gamma(\varphi - \lambda)\ dt + \sigma_2\sqrt{\lambda}\ dV \\ \lambda_0 &= c_2 \end{aligned} \tag{4.1}$$

where $\kappa, \theta, \sigma_1, c_1$ as well as $\gamma, \varphi, \sigma_2, c_2$ are positive constants and W and V are standard Brownian motions.

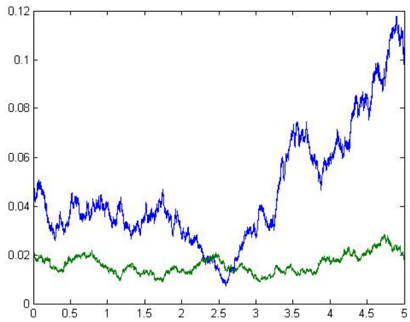
Individually, (4.1) is known as the CIR model for interest rates, first proposed by Cox-Ingersoll-Ross (Cox et. al. (1985)). With the additional condition of $2\kappa\theta > \sigma_1^2$, r_t follows a non-central chi square distribution. Its value is positive (with probability one) and tends to θ with *velocity* κ. The decision of modeling r_t with the CIR model is arbitrary, but the model has properties that can be desirable from an economical perspective. In particular, the properties are the positivity of the process, the mean-reverting property to the value θ and in less extent, the volatility directly proportional to the level of the process.

The properties that apply to r_t also apply to λ_t (with the condition of $2\gamma\varphi > \sigma_2^2$) and the dependence between the two processes is established between the correlation of the processes W and V. The correlation $\rho = dWdV$ can take any value in the interval $[-1, 1]$.

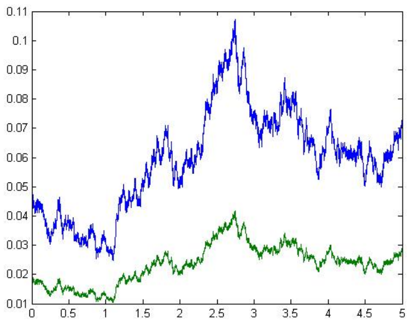
We will use the following values for the different parameters throughout the following examples: $\kappa = 0.3,\ \theta = 0.05,\ \sigma_1 = 0.10,\ c_1 = 0.05$ and $\gamma = 0.3,\ \varphi = 0.02,\ \sigma_2 = 0.06,\ c_2 = 0.02$. The choice of the values is arbitrary, making sure the conditions for the positiveness of the processes hold. Figure 4.1 shows three realizations of both processes, with $\rho = -1,\ \rho = 0$, and $\rho = 1$, with a time horizon of 5 (as in 5 years).



(a) $\rho = -1$



(b) $\rho = 0$



(c) $\rho = 1$

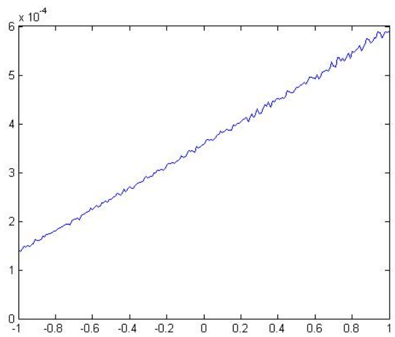
Figure 4.1: Three realizations of r_t and λ_t.

We are now going to explore the pricing of a corporate bond inducing different correlation values for the processes W and V. Assume $L = 1$ and using formula (2.1) for $t = 0$ and $T = 5$, with the parameters specified before, a discretization for both r_t and λ_t was done for the interval $[0, 5]$ with $\Delta t = \frac{1}{5*100}$. The number of different realizations (simulations) done for both processes was 35,000. Because of computational limitations, and after analyzing different results, we chose to have a big number of simulations (35,000) and relatively big time steps (Δt), instead of less number of simulations and smaller time steps, because with smaller time steps and less simulated paths the results were much more volatile but within the same range.

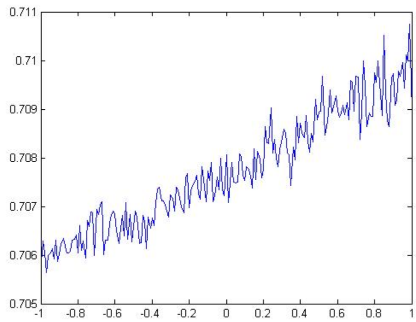
The Matlab source code can be find in appendix A.3. The following results were obtained for different values of ρ:

ρ	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
$\mathrm{var}(\int_0^5 r_s + \lambda_s\ ds)$	0.0001	0.0002	0.0002	0.0003	0.0004	0.0004	0.0005	0.0005	0.0006
$\bar{P}(0,5)$	0.7057	0.7067	0.7066	0.7072	0.7081	0.7081	0.7089	0.7095	0.7089

The following figures are the results in graphics. In the two figures, the x-axis is the value of ρ:



(a) $\mathrm{var}(\int_0^5 r_s + \lambda_s\ ds)$



(b) $\bar{P}(0,5)$

Negative correlation between r_t and λ_t diminishes the volatility of the sum (and hence of the average process), and the volatility increases as the correlation increases. The price of the corporate bond increases as the correlation increases. This increase in price is probably due to the increase of volatility of the sum of the processes. Ekström and Tysk (2007) prove that under certain properties of the parameters of a diffusion process, the price of a bond is increasing in the volatility of the underlying process. It is not straightforward to prove that the sum of r_t and λ_t fulfill those properties because the process is not a typical diffusion.

Finally, we decided also to do a simulation of τ, and use it to price the bond again, using $\bar{P}(0,5) = \mathbb{E}\left[\mathrm{e}^{-\int_0^5 r_s ds} \times 1_{\{\tau>5\}}\right]$. This simulation is more demanding in number of computations, and with the same number of simulations and size of time steps, the results were far more volatile. The following figure illustrates the price of the bond using this method:

The advantage of the first method is thus evident, proving the superiority of (2.8) over (2.3) in computational terms. A final comment is that pricing corporate bonds assume a term structure of the interest rates. In general the market uses CDSs rates to establish an approximation of $\mathrm{e}^{-\int_t^T \lambda_s\ ds}$ that is assumed to be *correct*. Another assumption is that r_t and λ_t are independent. The possibility of introducing different models for the process λ_t or for the sum of the processes gives rise to endless pricing techniques, that can be compared with the actual market prices, and the market does not always has to be right.

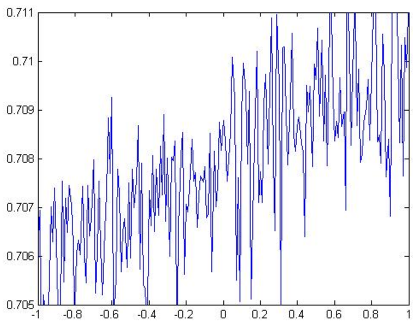


Figure 4.2: $\bar{P}(0,5)$ simulating τ

Chapter 5

Conclusion

Credit risk is the risk associated to claims that have a positive probability of default. Credit risk is important because investors are not guaranteed to get the expected return on their investments, but instead they can lose their invested capital if default occurs.

In this thesis we studied the modeling of credit risk through intensity models. The general approach of these models is to model the default event τ as the first jump of a Poisson process. The survival probability of τ takes the form of a discount factor, and the default intensity plays the same role as an interest rate process. This default intensity can be modeled as a constant, as a deterministic time-varying function, or as a stochastic process.

We studied a general valuation formula of claims subject to credit risk considering a stochastic default intensity. The formula is general in the sense that it can be applied to any claim subject to credit risk, and is independent of the structure of the default intensity. In particular, the formula does not assume independence between the default intensity and the risk-free process. The case where the contingent claim is a corporate bond was studied with more detail.

The Credit Default Swap (CDS) is an instrument that provides an option to investors to mitigate credit risk. The pricing of a model independent CDS was studied, as well as the particular case where the default intensity is independent of the risk-free process. A usual market practice of "stripping" survival probabilities from CDS quotes was also studied.

Finally, we studied the pricing effects in corporate bonds of inducing different correlations between the default intensity and the risk-free process. In particular, a negative correlation between the processes lowers the volatility of the sum of the processes. This lowers the price of the corporate bond, and the price increases with the correlation. Additionally, and for this particular claim, it was seen that it is more efficient to discount the cashflow with the sum of the risk-free process and the credit spread, than to simulate the default event by itself. Further proposed research include if this superiority is also evident in other contingent claims subject to default risk.

Appendix A

Matlab Source Codes

A.1 CDS pricing

```
%CDS pricing function
%a=Beginning time of the CDS (usually 0)
%b=Maturity in years of the CDS
%s=Market quote
%L=Loss given default
%gamma=Intensity vector
%PEURC=Default-free zero-coupon bond prices

function z=CDS(a,b,s,L,gamma,PEURC);

j=(b-a)*4;
Gamma_j=zeros(1,j+1);
for i=2:(j+1)
       Gamma_j(i)=Gamma_j(i-1)+gamma(i);
end

fir=0;
sec=0;
thir=0;
for i=1:j
    fir=fir+gamma(i)*quad(@(u)FIRST(u,Gamma_j,gamma,PEURC,i-1),i-1,i);
    sec=sec+exp(-Gamma_j(i))*PEURC(i*90);
    thir=thir+gamma(i)*quad(@(u)THIRD(u,Gamma_j,gamma,PEURC,i-1),i-1,i);
```

```
end
fir=s*fir;
sec=s*sec;
thir=L*thir;

z=fir+sec-thir;
return;

%First argument auxiliary function for the CDS pricing
function y=FIRST(u,Gamma_j,gamma,PEURC,mmin)
days=floor(u*90);
days(days == 0) = 1;
y=exp(-Gamma_j(mmin+1)-gamma(mmin+2).*(u-mmin)).*PEURC(days)'.*(u-mmin);
return;

%Third argument auxiliary function for the CDS pricing
function y=THIRD(u,Gamma_j,gamma,PEURC,mmin)
days=floor(u*90);
days(days == 0) = 1;
y=exp(-Gamma_j(mmin+1)-gamma(mmin+2).*(u-mmin)).*PEURC(days)';
return;
```

A.2 Code for stripping out constant intensities

```
%Function to strip out constant intensities
%L=Loss given default
%ba=Maturity terms
%sa=Market quotes
%gammai=Intensity parameters
function z=calib(PEURC);
L=1-0.15;
ba=[1 3 5 7 10];
%sa=[.01925 .0215 .0225 .0235 .0235];
%sa=[.0725 .0630 .0570 .0570 .0570];
%sa=[.1450 .1200 .0940 .0850 .0850];
sa=[.5050 .2100 .1500 .1250 .1100];
```

```
gamma1=.05;
gamma2=.05;
gamma3=.05;
gamma4=.05;
gamma5=.05;

gamma1=fzero(@(gamma1)CDS(0,ba(1),sa(1)/4,L,[[gamma1/4*ones(1,5)]
[gamma2/4*ones(1,8)] [gamma3/4*ones(1,8)] [gamma4/4*ones(1,8)]
[gamma5/4*ones(1,12)]],PEURC),gamma1);
gamma2=fzero(@(gamma2)CDS(0,ba(2),sa(2)/4,L,[[gamma1/4*ones(1,5)]
[gamma2/4*ones(1,8)] [gamma3/4*ones(1,8)] [gamma4/4*ones(1,8)]
[gamma5/4*ones(1,12)]],PEURC),gamma2);
gamma3=fzero(@(gamma3)CDS(0,ba(3),sa(3)/4,L,[[gamma1/4*ones(1,5)]
[gamma2/4*ones(1,8)] [gamma3/4*ones(1,8)] [gamma4/4*ones(1,8)]
[gamma5/4*ones(1,12)]],PEURC),gamma3);
gamma4=fzero(@(gamma4)CDS(0,ba(4),sa(4)/4,L,[[gamma1/4*ones(1,5)]
[gamma2/4*ones(1,8)] [gamma3/4*ones(1,8)] [gamma4/4*ones(1,8)]
[gamma5/4*ones(1,12)]],PEURC),gamma4);
gamma5=fzero(@(gamma5)CDS(0,ba(5),sa(5)/4,L,[[gamma1/4*ones(1,5)]
[gamma2/4*ones(1,8)] [gamma3/4*ones(1,8)] [gamma4/4*ones(1,8)]
[gamma5/4*ones(1,12)]],PEURC),gamma5);

z=[gamma1;gamma2;gamma3;gamma4;gamma5];
return;
```

A.3 Code for pricing a defaultable corporate bond

```
% Corporate bond pricing simulation with r and lambda
function [Y]=simr;

T=5; %years
j=100; %time steps by year
n=35000; %number of simulations
deltat=1/j;

%r process parameters
```

```
k=0.3;
theta=0.05;
r0=0.05;
sigma1=0.1;

%lambda process parameters
g=0.3;
gamma=0.02;
lambda0=0.02;
sigma2=0.06;

J=T*j;
zz=0.01;
RES=zeros(3,2/zz+1);
for m=1:(2/zz+1)
    rho=(m-1)/(1/zz)-1 %different values of the correlation
    if rho==1
        C2=[1 1; 0 0];
    elseif rho==-1
        C2=[1 -1; 0 0];
    else
        C2=chol([1 rho;rho 1]);
    end

    R=zeros(n,J+1);
    L=zeros(n,J+1);
    INT=zeros(n,J);
    R(:,1)=r0;
    L(:,1)=lambda0;

    for i=2:(J+1)
        ale=randn(n,2)*C2;

        deltar=k*(theta*ones(n,1)-R(:,i-1))*deltat +
         sigma1 * sqrt(deltat) * sqrt(R(:,i-1)).*ale(:,1);
        R(:,i)=max(0.0001,R(:,i-1)+deltar);
```

```
        deltal=g*(gamma*ones(n,1)-L(:,i-1))*deltat +
         sigma2 * sqrt(deltat) * sqrt(L(:,i-1)).*ale(:,2);
        L(:,i)=max(0.0001,L(:,i-1)+deltal);

        if i==2
            INT(:,i-1)=(R(:,i-1)+L(:,i-1))*deltat;
        else
           INT(:,i-1)=(R(:,i-1)+L(:,i-1))*deltat+INT(:,i-2);
        end
    end

    RES(1,m)=rho;
    RES(2,m)=var(INT(:,J)/T);
    RES(3,m)=mean(exp(-INT(:,J)));
end
Y=RES;
return;
```

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