Shape-from-operators: recovering shapes from intrinsic differential operators

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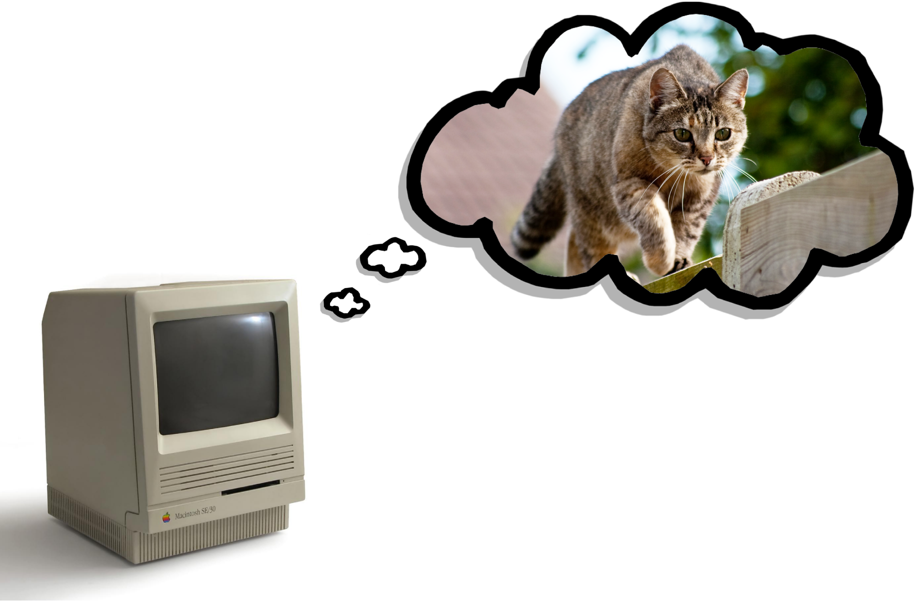
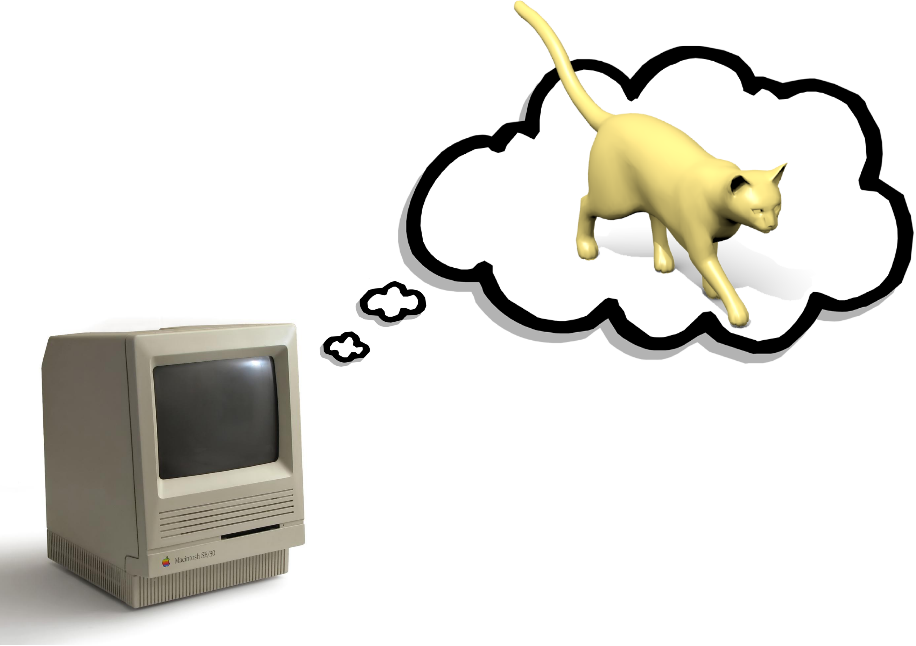
Joint work with: Davide Eynard, Drosos Kourounis and Michael M. Bronstein

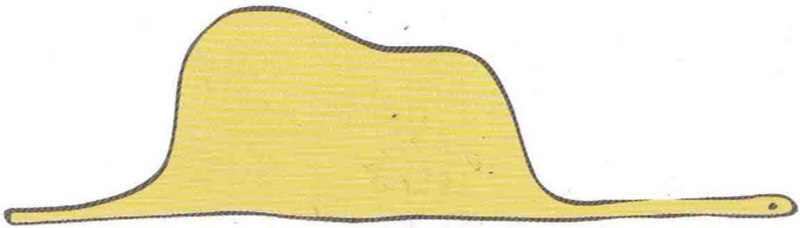

Macintosh SE/30



Slide idea: Crane

"Once upon a time, a child imagined a fierce boa constrictor that swallowed an elephant, giving rise to a most peculiar shape.

[... However] nobody saw in his drawing more than a hat.



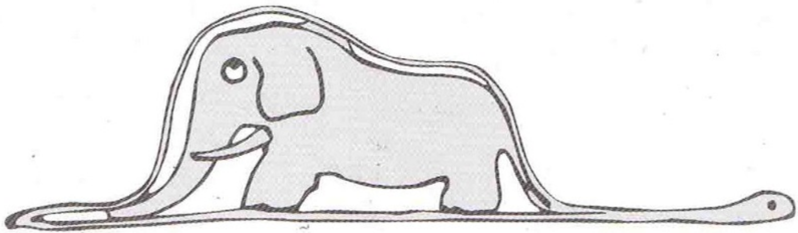
The challenges of non-rigidity

"Once upon a time, a child imagined a fierce boa constrictor that swallowed an elephant, giving rise to a most peculiar shape.

[... However] nobody saw in his drawing more than a hat.

Then the child proceeded to draw an explanatory drawing showing the elephant inside the snake's expansible stomach..."

de Saint-Exupéry, The Little Prince

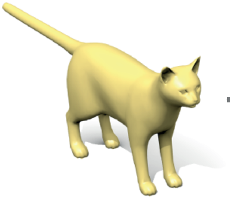



M DK 4281

Can one hear the shape of a drum?

Kac, 66

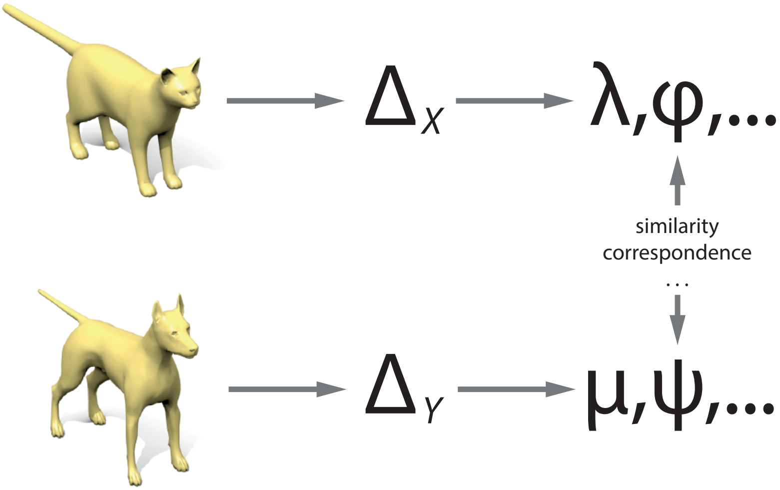
Spectral shape analysis

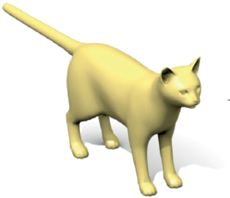


$$\longrightarrow \quad \Delta \quad \longrightarrow \quad \lambda, \varphi, \ldots$$

spectral properties

Spectral shape analysis

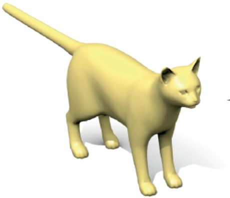




$\longleftarrow \quad \Delta \quad \longleftarrow$ Properties

Shape-from-Operator: find a shape whose Laplacian (or another intrinsic operator) satisfies some properties

Shape-from-Operator inverse problem

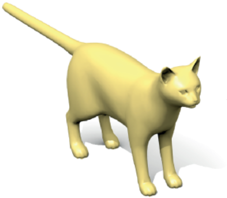


$\longleftarrow \quad \Delta \quad \longleftarrow$ Properties

Shape-from-Operator: find a shape whose Laplacian (or another intrinsic operator) satisfies some properties

- Embedding from angles[1], curvature[2], discrete fundamental forms[3],...
- Edge length from Laplacian[4,5]
- Closest commuting operators[6]
- Laplacian colormaps[7]

[1]Sheffer, de Sturler 2001; [2]Ben-Chen et al. 2008; [3]Wang et al. 2012; [4]Zeng et al. 2012; [5]de Goes et al. 2014; [6]B, Glashoff, Loring 2013; [7]Eynard, Kovnatsky, B 2014

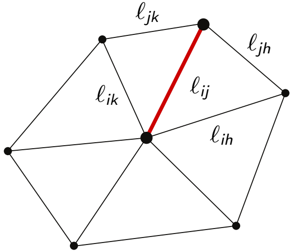


$$\longrightarrow \quad \ell \quad \longrightarrow \quad \Delta$$

- Laplacian is **intrinsic** = expressible in terms of discrete metric (edge lengths)
- Embedding induces a discrete metric (and thus also a Laplacian)
- Not unique: many embeddings give rise to the same metric (isometries)

Laplace-Beltrami discretization (intrinsic)

- Triangular mesh (X, E, F)



Laplace-Beltrami discretization (intrinsic)

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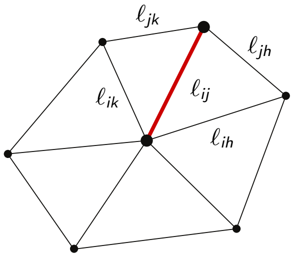
- **Discrete metric** $\boldsymbol{\ell} = (\ell_{ij},\ (i,j) \in E)$



Laplace-Beltrami discretization (intrinsic)

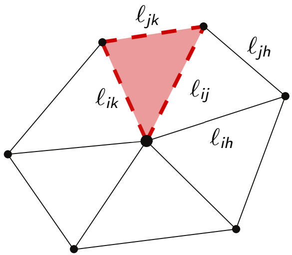
- Triangular mesh (X, E, F)
- **Discrete metric** $\boldsymbol{\ell} = (\ell_{ij},\ (i,j) \in E)$
- **Intrinsic** edge weights

$$w_{ij} = \frac{-\ell_{ij}^2 + \ell_{jk}^2 + \ell_{ik}^2}{8A_{ijk}} + \frac{-\ell_{ij}^2 + \ell_{jh}^2 + \ell_{ih}^2}{8A_{ijh}}$$

where by Heron's formula

$$A_{ijk} = \sqrt{s(s - \ell_{ij})(s - \ell_{jk})(s - \ell_{ik})},$$

and s is the semi-perimeter



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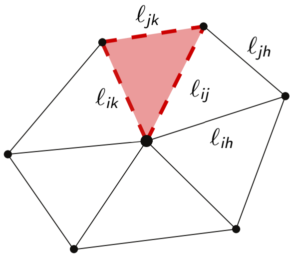


- Laplace-Beltrami operator $|X| \times |X|$ matrix

$$\mathbf{L} = \mathbf{D} - \mathbf{W},$$

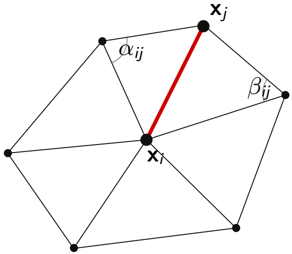
where $\mathbf{D} = \mathrm{diag}(\sum_{i \neq j} w_{ij})$

- Triangular mesh (X, E, F)

- Embedding of the mesh in $\mathbb{R}^3$ specifies the coordinates $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$

- Embedding $\mathbf{X}$ **induces** the metric

$$\ell = (\|\mathbf{x}_i - \mathbf{x}_j\|, (i,j) \in E)$$



Pinkall and Polthier, 1993

Laplace-Beltrami discretization (extrinsic)

- Triangular mesh (X, E, F)

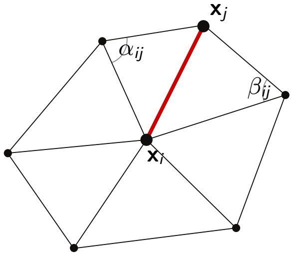
- Embedding of the mesh in $\mathbb{R}^3$ specifies the coordinates $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$

- Embedding $\mathbf{X}$ **induces** the metric

$$\ell = (\|\mathbf{x}_i - \mathbf{x}_j\|, (i,j) \in E)$$

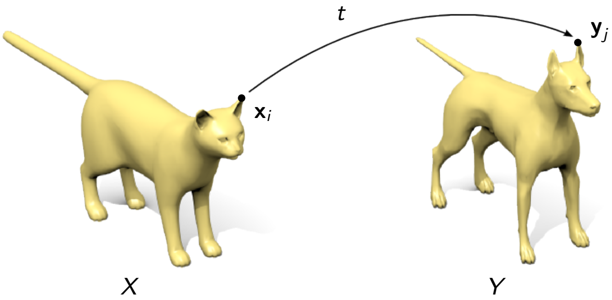
- **Cotangent weights**

$$w_{ij} = \begin{cases} \dfrac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2}, & (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$



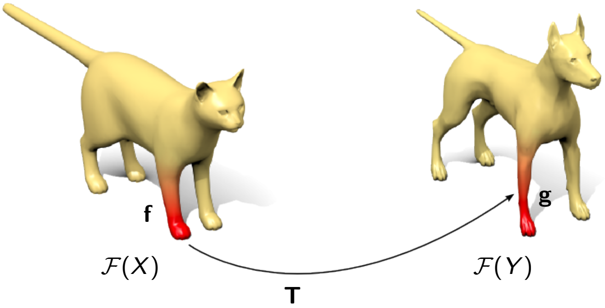
Pinkall and Polthier, 1993

Functional maps



Point-wise maps $t\colon X \to Y$

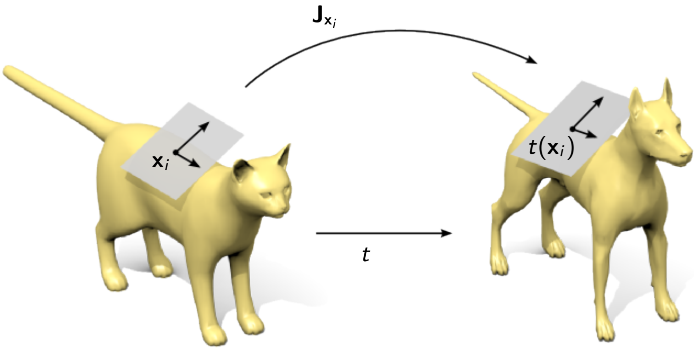
Ovsjanikov et al., 2012



Functional maps $\mathbf{T}: \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$

Ovsjanikov et al., 2012

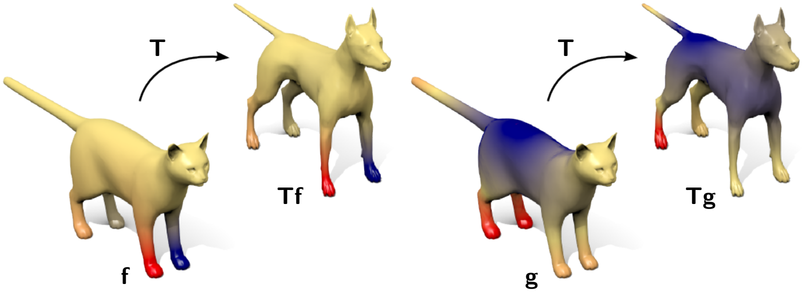
Shape difference operators



Distortions induced by a map = change of inner products of **vectors**

Rustamov et al., 2013

Shape difference operators



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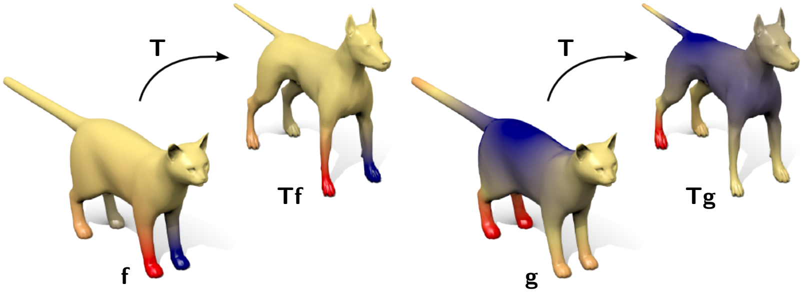

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{F}(X)} \neq \langle \mathbf{Tf}, \mathbf{Tg} \rangle_{\mathcal{F}(Y)}$$

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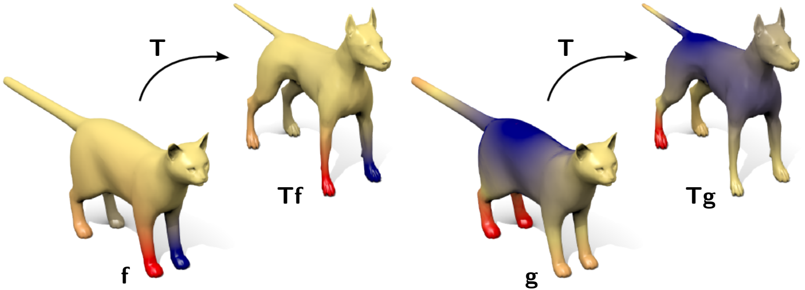
Shape difference operators



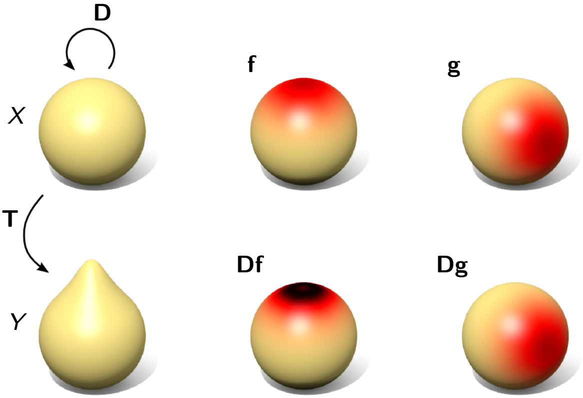
$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{F}(X)} \neq \langle \mathbf{Tf}, \mathbf{Tg} \rangle_{\mathcal{F}(Y)}$$

Riesz theorem: there exists a unique self-adjoint linear operator $\mathbf{D} : \mathcal{F}(X) \to \mathcal{F}(X)$ such that

$$\langle \mathbf{Tf}, \mathbf{Tg} \rangle_{\mathcal{F}(Y)} = \langle \mathbf{f}, \mathbf{Dg} \rangle_{\mathcal{F}(X)}$$

Rustamov et al., 2013

Shape difference operators



- Captures the **difference** in the geometry of the two shapes
- **Depends** on choice of inner product

Rustamov et al., 2013

Shape difference discretization

- **area**-based,

$$\langle f, g \rangle_{L^2(X)} = \int_X f(x) g(x) d\mu(x)$$

$$\mathbf{D} = \mathbf{V}_{X,Y} = \mathbf{A}_X^{-1} \mathbf{T}^\top \mathbf{A_Y} \mathbf{T}$$

Rustamov et al., 2013

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- **conformal**-based,

$$\langle f, g \rangle_{H^1(X)} = \int_X \nabla f(x) \nabla g(x) d\mu(x)$$

$$\mathbf{D} = \mathbf{R}_{X,Y} = \mathbf{W}_X^\dagger \mathbf{T}^\top \mathbf{W}_Y \mathbf{T}$$

Rustamov et al., 2013

Shape difference discretization

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- if $\mathbf{V} = \mathbf{I}$, the map preserves the areas

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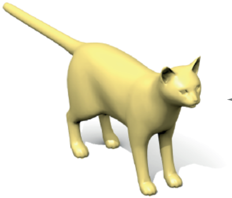
- if $\mathbf{R} = \mathbf{I}$, the map preserves the angles

- if $\mathbf{V} = \mathbf{R} = \mathbf{I}$, the map is an isometry

Rustamov et al., 2013

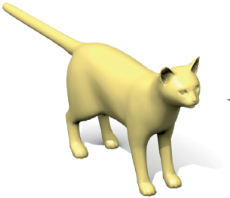


$$\mathbf{O}_0$$

Generic **Shape-from-Operator (SfO)** problem: given some intrinsic operator $\mathbf{O}_0$, find an embedding $\mathbf{X}$ by minimizing some cost function

$$\min_{\mathbf{X}} \mathcal{E}\left(\mathbf{O}(\ell(\mathbf{X})), \mathbf{O}_0\right)$$

Boscaini, Eynard, Kouronis, Bronstein 2014



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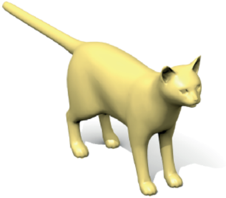
$$\min_{\mathbf{X}} \mathcal{E}\big(\mathbf{O}(\ell(\mathbf{X})), \mathbf{O}_0\big)$$

Note: $\mathbf{O}$ depends on $\mathbf{X}$ indirectly through the discrete metric $\ell(\mathbf{X})$, very hard for optimization!

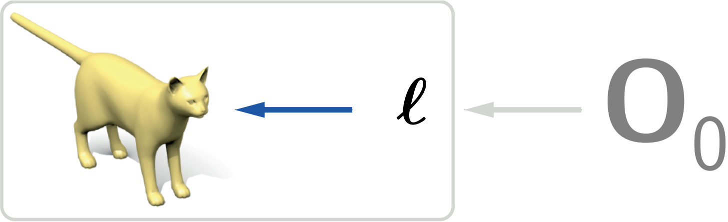
Boscaini, Eynard, Kouronis, Bronstein 2014



$$\longleftarrow \quad \boldsymbol{\ell} \quad \longleftarrow \quad \mathbf{O}_0$$

- **Metric-from-Operator (MfO):** $\min_{\boldsymbol{\ell}} \mathcal{E}\big(\mathbf{O}(\ell), \mathbf{O}_0\big)$ s.t. triangle inequality
- **Shape-from-Metric (SfM):** $\min_{\mathbf{X}} \sum_{ij \in E}^{n} (\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij})^2,$

Boscaini, Eynard, Kouronis, Bronstein 2014



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Boscaini, Eynard, Kouronis, Bronstein 2014

Shape-from-metric

Special setting of **MDS**: given a metric ℓ, find its Euclidean realization by minimizing the stress

$$\min_{\mathbf{X}} \sum_{i,j=1}^{n} v_{ij}(\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij})^2,$$

where

$$v_{ij} = \begin{cases} 1 & \text{if } ij \in E, \\ 0 & \text{otherwise} \end{cases}$$

Leeuw et al., 1977

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SMACOF algorithm: fixed point iteration of the form

$$\mathbf{X} \leftarrow \mathbf{Z}^{\dagger}\mathbf{B}(\mathbf{X})\mathbf{X}$$

where

$$\mathbf{Z} = \begin{cases} -v_{ij} & \text{if } i \neq j, \\ \sum_{i \neq j} v_{ij} & \text{if } i = j \end{cases} \qquad \mathbf{B}(\mathbf{X}) = \begin{cases} -\dfrac{v_{ij}\ell_{ij}}{\|\mathbf{x}_i - \mathbf{x}_j\|} & \text{if } i \neq j \text{ and } \mathbf{x}_i \neq \mathbf{x}_j, \\ 0 & \text{if } i \neq j \text{ and } \mathbf{x}_i = \mathbf{x}_j, \\ \sum_{i \neq j} b_{ij} & \text{if } i = j \end{cases}$$

Leeuw et al., 1977

From operators to shapes

$$\boldsymbol{\ell} \longleftarrow \mathbf{O}_0$$

- **Metric-from-Operator (MfO):** $\min_{\boldsymbol{\ell}} \mathcal{E}\big(\mathbf{O}(\boldsymbol{\ell}), \mathbf{O}_0\big)$ s.t. triangle inequality

- **Shape-from-Metric (SfM):** $\min_{\mathbf{X}} \sum_{ij \in E}^{n} (\|\mathbf{x}_i - \mathbf{x}_j\| - \ell_{ij})^2,$

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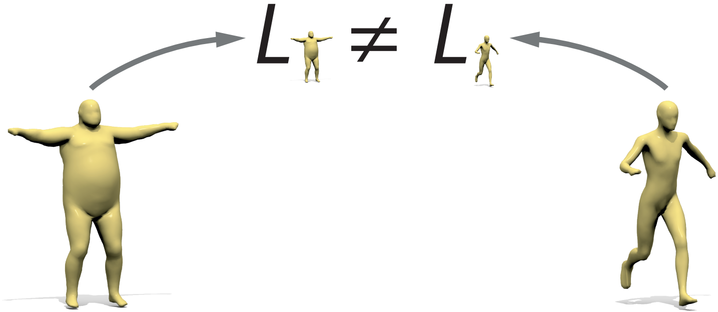


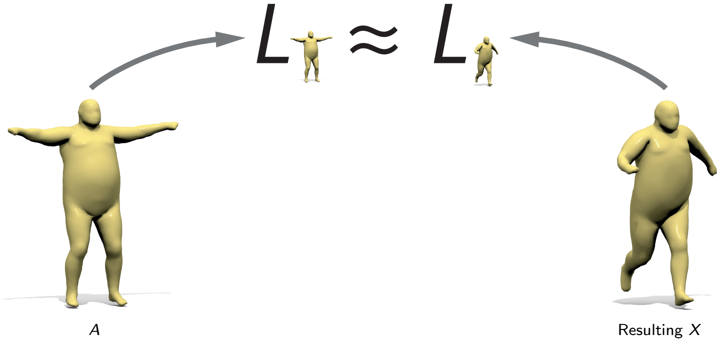
A Initial X

Given a reference Laplacian operator $\mathbf{L}_A$, and a corresponding initial shape X, deform X by minimizing

$$\min_{\mathbf{X}} \|\mathbf{L}(\ell(\mathbf{X})) - \mathbf{L}_A\|$$

Boscaini, Eynard, Kouronis, Bronstein 2014

Shape-from-Laplacian



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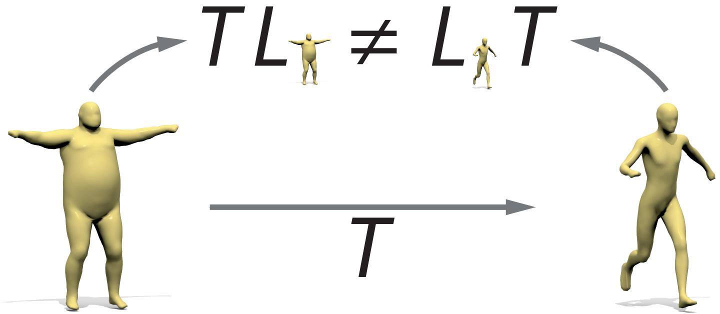
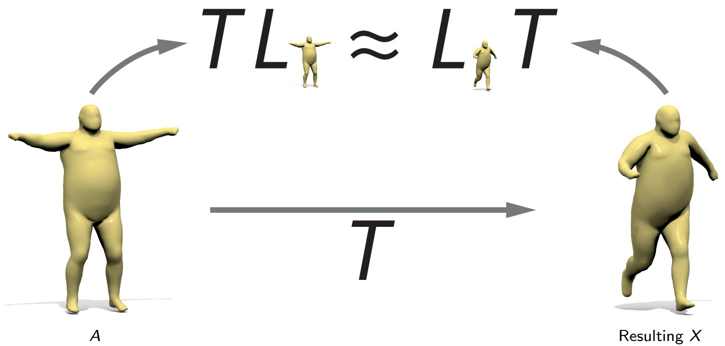


A Initial *X*

Given a reference Laplacian operator $\mathbf{L}_A$, and a initial shape X related by functional correspondence T, deform X by minimizing

$$\min_{\mathbf{X}} \|\mathbf{T}\mathbf{L}(\ell(\mathbf{X})) - \mathbf{L}_A\mathbf{T}\|$$

Boscaini, Eynard, Kouronis, Bronstein 2014



Given a reference Laplacian operator $\mathbf{L}_A$, and a initial shape X related by functional correspondence T, deform X by minimizing

$$\min_{\mathbf{X}} \|\mathbf{TL}(\ell(\mathbf{X})) - \mathbf{L}_A\mathbf{T}\|$$

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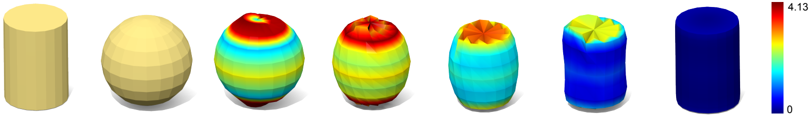
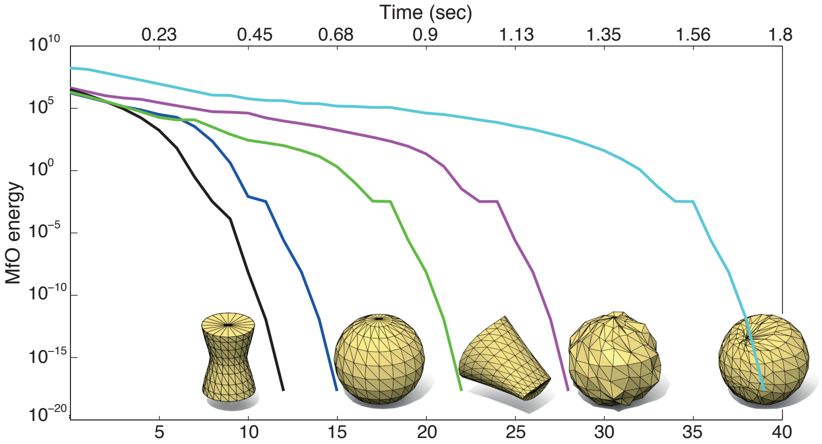
Shape-from-Laplacian convergence



Reference Y Initialization X Iteration 1 4 6 7 9

Convergence of our method in the shape-from-Laplacian optimization problem.
Colors show vertex-wise MfO energy contribution

Shape-from-Laplacian convergence



Convergence of our method in the shape-from-Laplacian optimization problem using different initializations.

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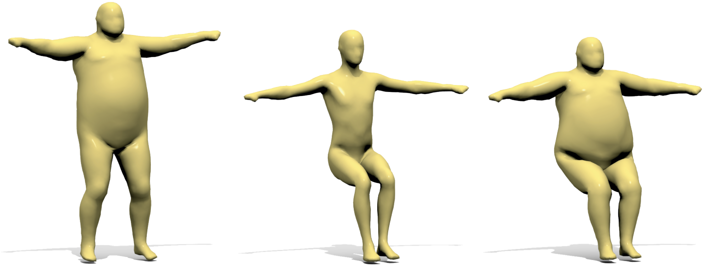
Style transfer by shape-from-Laplacian

Reference Y Initialization X Result

"Modify X such that $\mathbf{L}_X$ becomes as similar as possible to reference Laplacian $\mathbf{L}_Y$"

Boscaini, Eynard, Kouronis, Bronstein 2014

Style transfer by shape-from-Laplacian



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Boscaini, Eynard, Kouronis, Bronstein 2014

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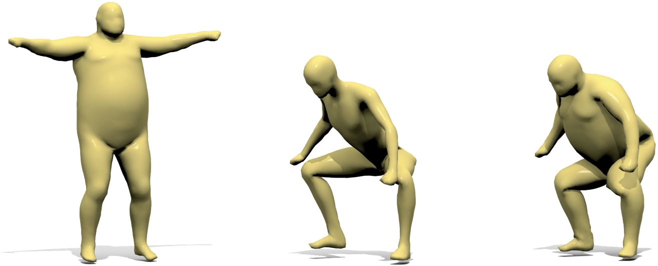

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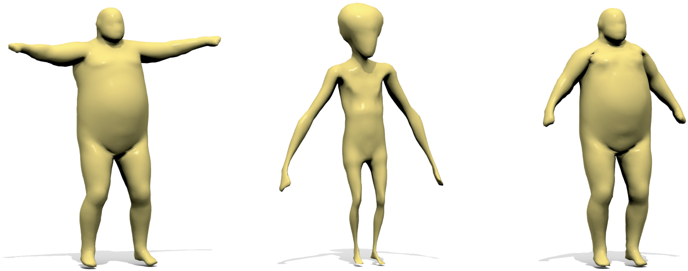

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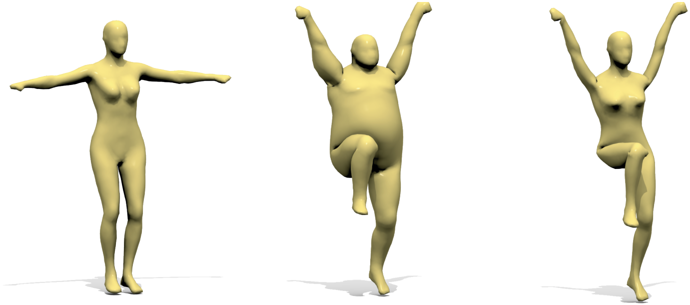


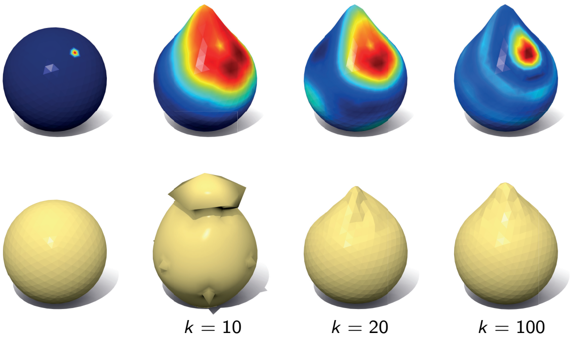
Reference Y | Initialization X | Result

"Modify X such that $\mathbf{L}_X$ becomes as similar as possible to reference Laplacian $\mathbf{L}_Y$"

Boscaini, Eynard, Kouronis, Bronstein 2014

Sensitivity to map quality

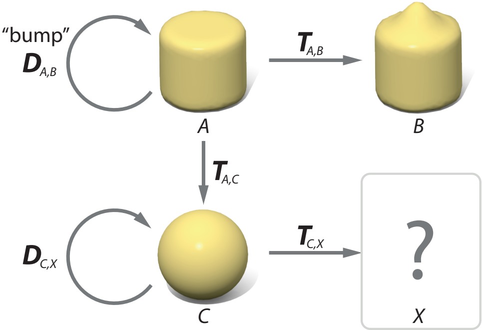
Functional map approximated as a matrix $\mathbf{T} \approx \mathbf{\Psi}\mathbf{C}^\top\mathbf{\Phi}^\top$ of rank k using the first functions in Fourier expansion (larger k = better map)



Shape-from-Laplacian result for different quality of the map $\mathbf{T}$
(initial shape: sphere, reference Laplacian: bumped sphere)

Ovsjanikov et al. 2012; Boscaini, Eynard, Kouronis, Bronstein 2014

Shape-from-difference operator

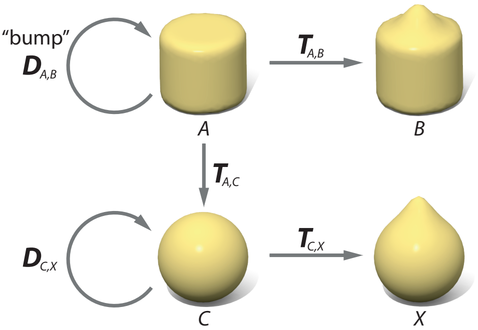


Deform initial shape X to make it different from C same way as B is different from A

$$\min_{\mathbf{X}} \quad \|\mathbf{D}_{C,X}(\ell(\mathbf{X}))\mathbf{T}_{A,C} - \mathbf{T}_{A,C}\mathbf{D}_{A,B}\|$$

Boscaini, Eynard, Kouronis, Bronstein 2014

Shape-from-difference operator



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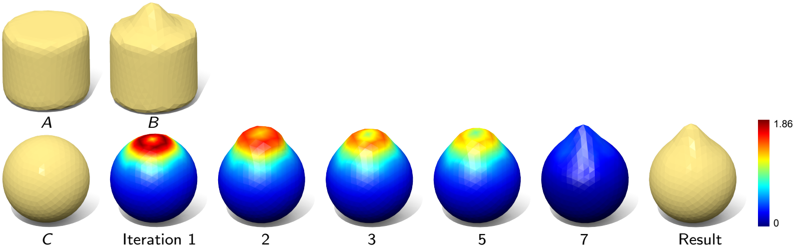
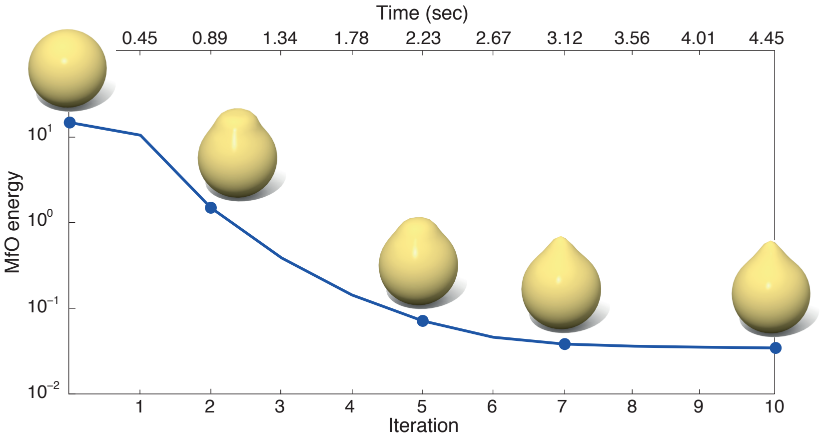
Boscaini, Eynard, Kouronis, Bronstein 2014

Shape-from-difference convergence



Convergence of our method in the shape-from-difference optimization problem.
Colors show vertex-wise MfO energy contribution

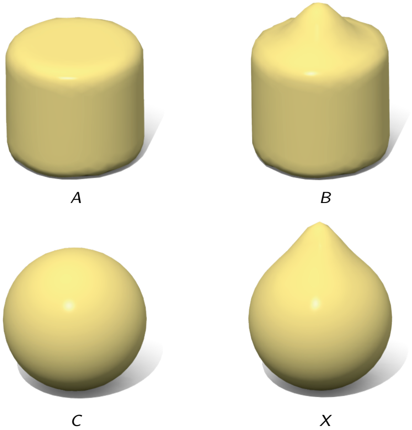
Shape-from-difference convergence



Convergence of our method in the shape-from-difference optimization problem.

Boscaini, Eynard, Kouronis, Bronstein 2014

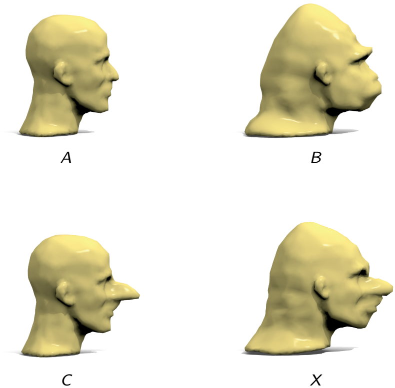
Analogy synthesis by shape-from-difference



"Find X such that the difference operator between C, X is as similar as possible to the given difference operator between A, B"

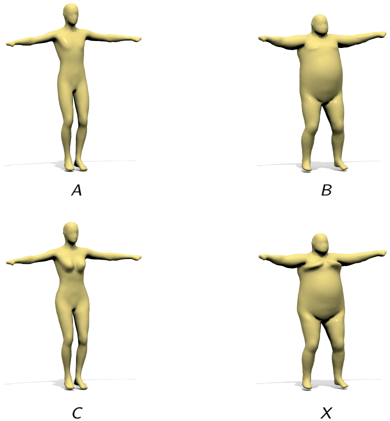
Boscaini, Eynard, Kouronis, Bronstein 2014

Analogy synthesis by shape-from-difference



"Find X such that the difference operator between C, X is as similar as possible to the given difference operator between A, B"

Analogy synthesis by shape-from-difference



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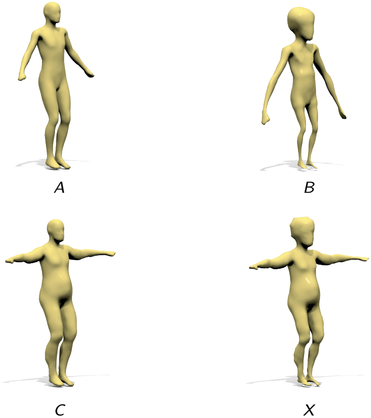
Boscaini, Eynard, Kouronis, Bronstein 2014

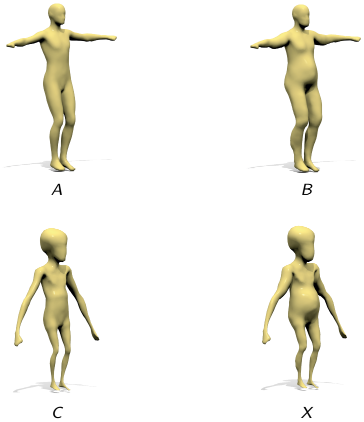
Analogy synthesis by shape-from-difference



"Find X such that the difference operator between C, X is as similar as possible to the given difference operator between A, B"

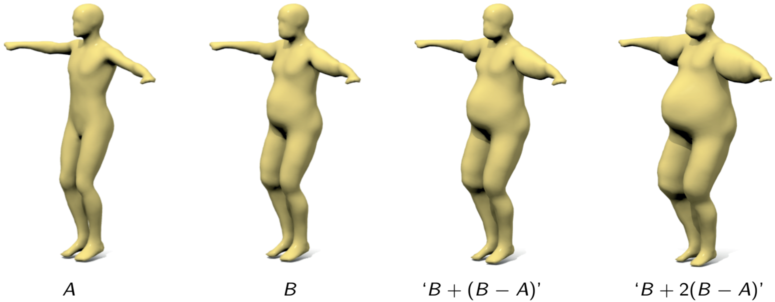
Boscaini, Eynard, Kouronis, Bronstein 2014

Analogy synthesis by shape-from-difference



"Find X such that the difference operator between C, X is as similar as possible to the given difference operator between A, B"

Shape exaggeration



Shape exaggeration obtained by applying the difference operator between A, B to B several times.

Boscaini, Eynard, Kouronis, Bronstein 2014

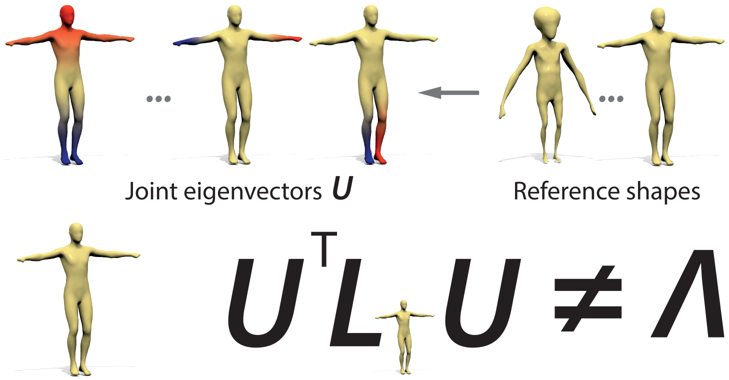
Shape-from-eigenvectors

Kovnatsky et al., 2013

Shape-from-eigenvectors

What is the shape whose Laplacian is diagonalized by the joint eigenvectors?

Kovnatsky et al., 2013

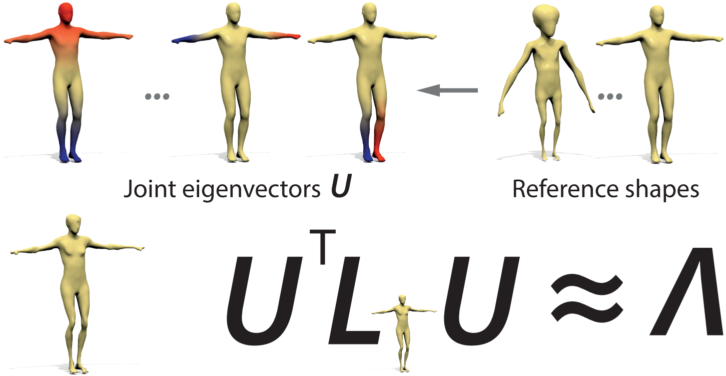


Given an orthonormal basis $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_r)$ on $\mathcal{F}(X)$ and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$, deform shape X such that $\mathbf{U}$ is an (approximate) eigenbasis of its Laplacian

$$\min_{\mathbf{X}} \|\mathbf{W}(\ell(\mathbf{X}))\mathbf{U} - \mathbf{A}(\ell(\mathbf{X}))\mathbf{U}\mathbf{\Lambda}\| + \mu\|\boldsymbol{\ell} - \boldsymbol{\ell}_0\|$$

Boscaini, Eynard, Kouronis, Bronstein 2014

Shape-from-eigenvectors

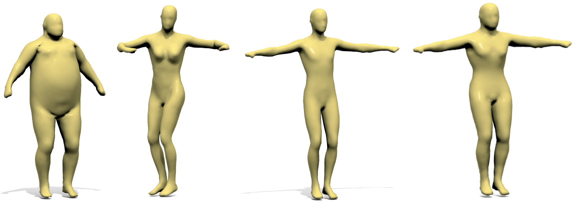


Given an orthonormal basis $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_r)$ on $\mathcal{F}(X)$ and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$, deform shape X such that $\mathbf{U}$ is an (approximate) eigenbasis of its Laplacian

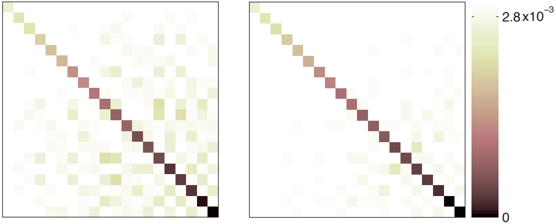
$$\min_{\mathbf{X}} \|\mathbf{W}(\boldsymbol{\ell}(\mathbf{X}))\mathbf{U} - \mathbf{A}(\boldsymbol{\ell}(\mathbf{X}))\mathbf{U}\boldsymbol{\Lambda}\| + \mu\|\boldsymbol{\ell} - \boldsymbol{\ell}_0\|$$

Boscaini, Eynard, Kouronis, Bronstein 2014

‘Intrinsic average shape’ by shape-from-eigenvectors



Reference shapes   Initialization   Result



$|\mathbf{U}^\top \mathbf{L}_{\mathrm{init}} \mathbf{U}|$ $|\mathbf{U}^\top \mathbf{L}_{\mathrm{res}} \mathbf{U}|$

“Modify initial shape such that its Laplacian is diagonalized by a given basis $\mathbf{U}$”

Boscaini, Eynard, Kouronis, Bronstein 2014

Conclusions

- New generic framework for shape-from-operator inverse problems
- Variety of applications in shape editing
- Other intrinsic operators
- Other shape representations
- Different solutions to shape-from-metric problem