Math 308: Midterm 4 Review

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June 2, 2016

Contents

1 Determinants

1.1 Computing the determinant

1. What is $\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ -2 & 1 & 0 \end{vmatrix}$?

Solution: -15

1.2 Properties of the determinant

1.2.1 Invertibility

2. Which of the following matrices are invertible?

   (a) $A = \begin{bmatrix} 8 & 3 \\ 11 & 4 \end{bmatrix}$

   Solution: $\det(A) = 32 - 33 = -1 \neq 0$, so the matrix is invertible.

   (b) $B = \begin{bmatrix} 5 & 0 & 1 & 7 \\ 1 & -1 & 4 & 3 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & -4 \end{bmatrix}$.

   Solution: Expanding along the second column, we have

   $$\det(B) = (-1)\begin{vmatrix} 5 & 1 & 7 \\ 1 & 0 & 2 \\ 2 & 0 & -4 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = -8.$$

   So B is invertible.

1.2.2 Geometric

3. Kimberly is using a 3d electron microscope to look at the cell structure of a macrophage. She wants the macrophage to appear as large as possible, but also wants to distort certain directions to accentuate the mitochondria structure. There are two linear transformations that accomplish this:

$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \vec{x} \qquad S(\vec{x}) = B\vec{x} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix} \vec{x}.$$

Which linear transformation will enlarge the macrophage more and why?

Solution: $|\det A| = 3 < |\det B| = 4$. So, S is the one. The absolute value of the determinant of the linear transformation matrix is the resulting volume of the unit box. Since S of the unit box is larger than T of it, it follows from linearity and scaling of linear transformations that every subset of $\mathbb{R}^3$ will have large volume as the image of S.

1.2.3 Cramer's Rule

Recall **Cramer's Rule** says that $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$ Where $\operatorname{adj}(A)$ is the adjoint of A given by

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Here C_{ij} is the i,jth cofactor of A (obtained by deleting the ith row and jth column from A then taking the determinant of the resulting submatrix.

4. Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & -2 & 4 \\ 0 & -2 & 1 & 0 \end{bmatrix}$. You are told that $\det(A) = -15$. Find the bottom right entry of A^{-1}.

Solution: We need to compute $C_{nn} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & -2 \end{vmatrix}$. It's easier to compute the determinant of the transpose:

$$\begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 3 \\ 2 & 1 & -2 \end{vmatrix} = -4 - 3 = -7.$$

The solution is then $7/15$.

2 Eigenvalues and vectors

2.1 Finding eigenvalues

5. Find the eigenvalues of the following matrices.

   (a) $A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$.

   Solution:

   $$\det(A - \lambda I) = (2-\lambda)(2-\lambda) + 3 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5).$$

   So A has eigenvalues 3 and 5.

   (b) $B = \begin{bmatrix} 1 & 4 & 2 & -7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

   Solution: Since $B - \lambda I$ is a diagonal matrix its determinant is the product of the diagonal entries.

   $$\det(B - \lambda I) = (1-\lambda)(3-\lambda)(-3-\lambda)(5-\lambda)$$

   So the eigenvalues are $\lambda = 1, 3, -3, 5$.

6. Recall that the determinant of a matrix and it's transpose are always the same. If A is a square matrix, argue that A and A^T must also have the same characteristic polynomial, and therefore the same eigenvalues.

   Solution: There are a couple ways to see this. The most intuitive is observing that subtracting a matrix by λI only affects the diagonal entries, which stay the same when we take a transpose. Therefore $A^T - \lambda I = (A - \lambda I)^T$. These two matrices will have the same determinant, and the rest follows.

2.2 Finding eigenspaces

7. Find the eigenspaces of the following matrices.

   (a) $A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$, using the eigenvalues you found before.

(b) $F = \begin{bmatrix} 3 & 4 & -1 \\ -1 & -2 & 1 \\ 3 & 9 & 0 \end{bmatrix}$, given its eigenvalues are 2 and -3.

2.3 Understanding peculiarities of eigenvalues

8. Find the eigenvalues of $\begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix}$.

Solution: The characteristic polynomial is $(1-\lambda)(-2-\lambda) - 10 = (\lambda + 4)(\lambda - 3)$. So we have -4 and 3.

Now, find which lines in $\mathbb{R}^2$ get fixed by the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

Explain your answer. (Hint: find the eigenvectors.)

Solution: First let's find the eigenspace for -4. This amounts to finding the nullspace of

$$\begin{bmatrix} 5 & 5 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

So this has eigenspace: Span $\{[1,1]\}$. Let's choose $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to be our eigenvector. (Note we could have chosen anything in the span.)

Similarly, we need to find the nullspace of:

$$\begin{bmatrix} -2 & 5 \\ 2 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 5 \\ 0 & 0 \end{bmatrix}$$

This is Span $\left\{ \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} \right\}$. Let $\vec{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

The two lines fixed by our linear transformation are $t\vec{u}_1$ and $s\vec{u}_2$. This is because they are eigenvalues. When we take $T(t\vec{u}_1) = At\vec{u}_1 = -4t\vec{u}_1$, this stays on the same line. Similarly for $\vec{u}_2$.

2.4 Applications

2.4.1 Population modeling

9. The lemur population in a dangerous part of Madagascar has the following features:

   - There are three stages, $p_{1,2} = 1/3$ is the probability of going from stage 1 to stage 2, and $p_{2,3} = 1/4$ is the probability of surviving from stage 2 to stage 3.
   - Stage 2 lemurs produce (on average) 2 offspring. Stage 3 lemurs produce on average 1 offspring.

   (a) Write a matrix, A, that models the lemur population.

   Solution: $A = \begin{pmatrix} 0 & 2 & 1 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$

   (b) Find the characteristic polynomial of this matrix.

   Solution: $\det(A - \lambda I) = 3\lambda^3 - 2\lambda - \frac{1}{4}$.

   (c) The roots of the answer from the previous part are $-.74, -.13, .88$. Explain why the eigenvectors of A form a basis for $\mathbb{R}^3$.

   Solution: Every eigenvalue has an eigenvector. So we have three eigenvectors. These are linearly independent since if they were dependent they would have the same eigenvalue. The Big Theorem guarantees that 3 linearly independent vectors in $\mathbb{R}^3$ form a basis.

   (d) Letting $\lambda_1 = -.73, \lambda_2 = -.13$ and $\lambda_3 = .88$ these have eigenvectors:

   $$\vec{u}_1 = \begin{bmatrix} 1 \\ -.45 \\ .15 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ -2.6 \\ 5.07 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} .67 \\ .25 \\ .08 \end{bmatrix}$$

   Write a formula for A^{100}.

   Solution: Let $P = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]$ and D be the diagonal matrix of eigenvalues. Then $A^{100} = PD^{100}P^{-1}$.

(e) Do you predict the lemur population will survive or go extinct? Explain.

Solution:
They will go extinct. The largest eigenvalue is .88, which as we take powers of will go to 0.

2.4.2 Centrality measures

10. If A is the adjacency matrix representing wikipedia's link structure, and $\vec{c}$ is the importance rating of each page, explain why expecting $\vec{c}$ t satisfy the equation:

$$\vec{c} = \frac{1}{\lambda} A\vec{c},$$

is reasonable.

Solution: This equation says that the score c_i of the ith website is $\frac{1}{\lambda}$ times the sum of the scores of its neighbors.

2.5 Change of basis

11. Let $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \right\}$ and let $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$. Find $\vec{x}_{\mathcal{B}_1}$ if $\vec{x}_{\mathcal{B}_2} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}_2}$. You do not need to find any inverses, or multiply any matrix. It is sufficient to introduce labels for matrices and write an equation just involving those labels.

Solution: Call the basis vectors of $\mathcal{B}_1$, $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and those of $\mathcal{B}_2$, $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Write $B_1 = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$ and $B_2 = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$. We have

$$\vec{x}_{\mathcal{B}_1} = B_1^{-1} B_2 \vec{x}_{\mathcal{B}_2}.$$

2.6 Diagonalization

2.6.1 Powers of matrices

12. Suppose that $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are eigenvalues of A, with eigenvalues λ_1, λ_2 and λ_3. Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. Also, suppose that $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_{\mathcal{B}}$. Write a formula for $A^k\vec{x}$.

Solution: $A^k\vec{x} = c_1\lambda_1^k\vec{u}_1 + c_2\lambda_2^k\vec{u}_2 + c_3\lambda_3^k\vec{u}_3$.

2.6.2 Exponential of a matrix

13. Suppose that $A = \begin{bmatrix} \ln 2 & 0 & 1 \\ 0 & \ln 3 & 0 \\ 0 & 0 & \ln 4 \end{bmatrix}$. What is e^A?

Solution: The eigenvalues are $\ln 2, \ln 3$ and $\ln 4$. We need to find the eigenvectors. The first one is the nullspace of

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & \ln 3 - \ln 2 & 0 \\ 0 & 0 & \ln 4 - \ln 2 \end{bmatrix},$$

this is $t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The second one is the nullspace of

$$\begin{bmatrix} \ln 2 - \ln 3 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \ln 4 - \ln 3 \end{bmatrix},$$

which is $t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The last one is the nullspace of

$$\begin{bmatrix} \ln 2 - \ln 4 & 0 & 1 \\ 0 & \ln 3 - \ln 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is $t\begin{bmatrix} -1/(\ln 2 - \ln 4) \\ 0 \\ 1 \end{bmatrix}$. So, our eigenvectors in a matrix are

$$P = \begin{bmatrix} 1 & 0 & -1/(\ln 2 - \ln 4) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This means that

$$e^A = P \begin{bmatrix} e^{\ln 2} & 0 & 1 \\ 0 & e^{\ln 3} & 0 \\ 0 & 0 & e^{\ln 4} \end{bmatrix} P^{-1}.$$

14. Find a 2×2 matrix, A, such that

$$e^A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Solution: We want a matrix with eigenvalue 1 and an eigenbasis of

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The characteristic polynomial has to be $(1 - \lambda)(2 - \lambda)$. So let's try the simplest matrix with this property

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Does this have the correct eigenvectors? Well, $A\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. So these are eigenvectors. Thus, our matrix A works.