Math 308, Midterm 3 Name: ____________________

May 18, 2016

Instructions.

- Vectors are indicated with arrows, as in

  Let $\vec{v}$ be a vector in $\mathbb{R}^n$.

- You are allowed a two-sided sheet of notes in your own handwriting.
- No calculators.
- There are 5 problems on 7 pages. Make sure your exam is complete.
- **Any cheating observed during, or noticed afterwards when comparing exams, will result in a 0 on this exam. Moreover, such an exam cannot be dropped. This means you lose 28% percent of your total raw grade. It will be difficult to pass the class if this occurs.**

Question	Points	Score
1	10	
2	12	
3	4	
4	15	
5	9	
Total:	50	

1. Check Yes or No for whether the subsets below are subspaces. You also need to EXPLAIN:

- If you check "Yes" explain why by doing one of the following: verifying the three properties, describing it as the span of a set of vectors, the nullspace of a matrix, the range of a linear transformation etc...
- If you check "No" give an explicit example of the property that is not satisfied.

[2 points] (a) The subset of $\mathbb{R}^3$ consisting of the vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a - 3b = 5c$.

√ **Yes** ◯ No Explanation:

Solution: This is the span of $\begin{bmatrix} 1 \\ 0 \\ 1/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3/5 \end{bmatrix}$.

[2 points] (b) The subset of $\mathbb{R}^3$ consisting of the vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x + y + z = 1$.

◯ Yes √ **No** Explanation:

Solution: $\vec{0}$ isn't in there.

[2 points] (c) The set of all points on either the x-axis or the y-axis in $\mathbb{R}^2$.

◯ Yes √ **No** Explanation:

Solution: It fails additivity. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both on the axes but $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ isn't.

[2 points] (d) The set of $\vec{x}$ such that $T(\vec{x}) = \vec{0}$ where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$.

√ **Yes** ◯ No Explanation:

Solution: It is the same as the nullspace of the matrix A^2 which is a subspace.

[2 points] (e) The subset of $\mathbb{R}^2$ consisting of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where $a - b \leq 0$.

◯ Yes √ **No** Explanation:

Solution: It fails scalar multiplication. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in there, but $-1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ isn't.

[12 points] 2. Suppose that A is a 4×3 matrix, and that $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ are in the columnspace of A.

(a) Say if each of the following statements is true, false, or if it could be either true or false from the amount of information given.

1. The vector $A\begin{bmatrix}1\\1\\1\end{bmatrix}$ is in the columnspace of A.
   $\surd$ **True** ◯ False ◯ Not enough information
2. The vector $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ is in the columnspace of A.
   $\surd$ **True** ◯ False ◯ Not enough information
3. The nullspace of A is $\{\vec{0}\}$.
   ◯ True ◯ False $\surd$ **Not enough information**
4. The columnspace of A is $\{\vec{0}\}$.
   ◯ True $\surd$ **False** ◯ Not enough information
5. One of the columns of A is $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$.
   ◯ True ◯ False $\surd$ **Not enough information**

(b) Check off *all* values that the rank of A could be.

☐ 0 ☐ 1 ■ **2** ■ **3** ☐ 4

3. Let $\vec{u} = \begin{bmatrix}1\\3\\2\\-1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}3\\3\\2\\0\end{bmatrix}$.

[2 points] (a) Write down a matrix you would row reduce to find two more vectors, $\vec{x}$ and $\vec{y}$ so that all four span $\mathbb{R}^4$.

Solution:

$$[\vec{u}\ \vec{v}\ \vec{e}_1\ \vec{e}_3\ \vec{e}_3\ \vec{e}_4]$$

[2 points] (b) Do not row reduce! But, describe what you would look for in the echelon form of your matrix from the previous part to determine $\vec{x}$ and $\vec{y}$.

Solution: The first two pivots of the last 4 columns of the row reduced matrix would be the vectors among the $\vec{e}_i$ that we need to add.

4. Suppose you have the following matrices A and A^T that have echelon forms B and C.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 2 & 2 & -3 & 3 \\ -1 & 1 & 0 & 2 & 0 & 3 & -3 & 3 \\ 0 & 1 & -1 & 2 & 2 & 1 & -1 & 4 \\ -1 & 0 & 2 & 1 & -4 & 3 & -2 & -1 \end{pmatrix} \xrightarrow{\text{echelon}} B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 2 \\ 1 & 1 & 2 & 2 & 1 \\ -2 & 2 & 0 & 2 & -4 \\ 1 & 2 & 3 & 1 & 3 \\ 0 & -3 & -3 & -1 & -2 \\ 0 & 3 & 3 & 4 & -1 \end{pmatrix} \xrightarrow{\text{echelon}} C = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the following questions it is okay to use the notation $\vec{a}_i$ for the ith column vector of A, or $\vec{a}_i(r)$ for the ith row vector. Similarly, for B and C. For A^T you can use $\vec{a}_i^T$ and $\vec{a}_i^T(r)$.

[2 points] (a) Find a basis for the row space of A that does NOT use any of the row vectors of A.

Solution: The nonzero rows of B form a basis for the rowspace of A:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 0 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

[2 points] (b) Find a different basis for the row space of A that EXCLUSIVELY uses the row vectors of A.

Solution: The nonzero pivots of C tell us that the first, second and fourth rows of A form a basis for the rowspace of A:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 2 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 2 \\ 1 \\ -2 \\ 4 \end{bmatrix}.$$

[2 points] (c) Find a basis for the column space of A that does NOT use any column vectors of A.

Solution: The nonzero rows of C form a basis for the column space of A:

$$\begin{bmatrix}1\\0\\1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix}.$$

[2 points] (d) Find a different basis for the column space of A that EXCLUSIVELY uses the column vectors of A.

Solution: The pivots of B tell us that the first, second and third column vectors of A form a basis for the column space of A:

$$\vec{a}_1, \vec{a}_2, \vec{a}_3.$$

Here are the matrices again:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 2 & 2 & -3 & 3 \\ -1 & 1 & 0 & 2 & 0 & 3 & -3 & 3 \\ 0 & 1 & -1 & 2 & 2 & 1 & -1 & 4 \\ -1 & 0 & 2 & 1 & -4 & 3 & -2 & -1 \end{pmatrix} \xrightarrow{\text{echelon}} B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 2 \\ 1 & 1 & 2 & 2 & 1 \\ -2 & 2 & 0 & 2 & -4 \\ 1 & 2 & 3 & 1 & 3 \\ 0 & -3 & -3 & -1 & -2 \\ 0 & 3 & 3 & 4 & -1 \end{pmatrix} \xrightarrow{\text{echelon}} C = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[2 points] (e) If the 5×9 augmented matrix $[A \mid \vec{e}_1]$ reduced to a consistent matrix then $\vec{e}_1$ is in the column space of A. Write down the SMALLEST dimension augmented matrix possible to determine if $\vec{e}_1$ is in the column space of A.

Solution: It is a 5×4 matrix:

$$[\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \mid \vec{e}_1].$$

[2 points] (f) Without doing any further calculations, find the nullity of A^T.

Solution: A^T has the same rank as A, which is 3. Notice that A^T is a 8×5 matrix. The rank theorem says that $3 + \text{nullity}(A^T) = 5$, thus the nullity is 2.

[3 points] (g) Find a BASIS for the nullspace of A^T

Solution: Looking at C we have the solution set $A^T\vec{x} = \vec{0}$ is of the form

$$\begin{bmatrix} -x_3 - 2x_5 \\ -x_3 - x_5 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix}.$$

Letting $x_3 = s$ and $x_5 = t$ this has the form

$$s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

So, $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for the nullspace of A^T.

5. Let S be the subspace of $\mathbb{R}^3$ given by vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $a - 2b + 5c = 0$.

[2 points] (a) Find a basis for S.

Solution: This is a plane, or a subspace of dimension 2, so it suffices to find two linearly independent vectors. We know that $a = 2b - 5c$ so we can write the vectors in S as

$$\begin{bmatrix} 2b - 5c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}.$$

So, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ form a basis.

[2 points] (b) Suppose $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$. Explain why a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^n$ is completely determined by the values $T(\vec{u}), T(\vec{v})$, and $T(\vec{w})$.

Solution: We can write any element $\vec{x} \in \mathbb{R}^3$ as a linear combination of the basis elements. By linearity and scaling of T we can write $T(\vec{x})$ as a linear combination of $T(\vec{u}), T(\vec{v})$, and $T(\vec{w})$.

[1 point] (c) Find a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that the kernel of T is precisely S. By the previous part a satisfactory answer need only specify what it does to a basis for $\mathbb{R}^3$.
Hint: *You may use the fact that $\vec{w} = \langle 1, -2, 5 \rangle$ does not belong to S.*

Solution: A quick solution would notice that points in S satisfy $a - 2b + 5c = 0$ so the map

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - 2b + 5c \\ 0 \end{bmatrix}.$$

sends just S to $\vec{0}$.
Another way to go is to let $\vec{u}$ and $\vec{v}$ be the basis vectors from part (a). Since $\vec{w}$ is not on the plane S the set $\{\vec{u}, \vec{v}, \vec{w}\}$ forms a basis for $\mathbb{R}^3$. We can define $T(\vec{u}) = T(\vec{v}) = \vec{0}$ and $T(\vec{w}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (any nonzero vector would be fine here.) Since any $\vec{x} \in \mathbb{R}^3$ can be written as $c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$ this determines the map T.

[2 points] (d) Describe a linear transformation $F\colon \mathbb{R}^5 \to \mathbb{R}^3$ such that the range of F is S.

Solution: Define $F(\vec{x}) = A\vec{x}$ with

$$A = \begin{bmatrix} 2 & -5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Alternatively, we could specify $F(\vec{e}_1) = \vec{u}$,$F(\vec{e}_2) = \vec{v}$ and $F(\vec{e}_i) = \vec{0}$ for $i = 3, 4, 5$.

[2 points] (e) What is the dimension of the kernel of such a linear transformation F as in part (d)?
Hint: *it is not necessary to use your answer from part (d)*

Solution: The dimension of the kernel of F is the same as the nullspace of A. Since it's range is S, it has dimension 2 and therefore nullity $5 - 2 = 3$.