Math 230, Exam 4 Name: ______________________________

November 17, 2018

Instructions.

- You are allowed one side of handwritten notes
- There is a normal table on the last page. You can detach it.
- No calculators.
- You need to use a histogram correction when applying the CLT to discrete quantities.
- There are 6 problems on 6 pages. Make sure your exam is complete.

Run LaTeX again to produce the table

[5 points] 1. Thirteen people are on a bus with 10 stops. Suppose the stop they exit at is independent and random for each person. The bus only stops when someone wants to get off. What is the expected number of stops? Leave your answer in exact form.

Solution: Let $X_i = 1$ if the bus stops at the ith stop, and 0 otherwise. The number of stops is $X = \sum_1^{10} X_i$. So, we have

$$EX = 10EX_1 = 10P(X_1 = 1) = 10(1 - (9/10)^{13}).$$

2. An investment increases by 50% percent with probability p and decreases by 50% percent with probability $1 - p$. You invest \$1. Let M_n be the amount of money you have after n iterations.

[3 points] (a) Define i.i.d. random variables $X_1, X_2, \dots$ with $X_i = \begin{cases} 1.5, & p \\ .5, & 1-p \end{cases}$. What is EX_1?

Solution: $EX_1 = 1.5p + .5(1 - p) = p + .5$

[2 points] (b) Write M_n in terms of the X_i.

Solution: $M_n = X_1 X_2 \cdots X_n$.

[3 points] (c) What is EM_n?

Solution: $EM_n = (p + .5)^n$.

[2 points] (d) What is $E\ln(X_1)$? Don't simplify the ln terms.

Solution: $E\ln(X_1) = \ln(1.5)p + \ln(.5)(1 - p)$.

[2 points] (e) Explain why the law of large numbers implies that $\ln(M_n) \approx nE\ln(X_1)$

Solution: The LLN says that $\ln(M_n)/n$ converges to $E\ln(X_1)$. Rearranging gives the claimed formula.

[2 points] (f) Notice that $M_n = e^{\ln(M_n)}$. Apply this to the righthand side of (e) and your answer to (d) to obtain a formula for M_n with no logarithms. A helpful formula for this is:

$$e^{m(\ln(x)q+\ln(y)w)} = (e^{\ln(x)})^{mq}(e^{\ln(y)})^{mw} = (x^q y^w)^m.$$

Solution:
$M_n/100 \approx e^{n(\ln(1.5)p+\ln(.5)(1-p))} = (1.5^p .5^{1-p})^n = (3^p/2)^n$.

[1 point] (g) If you did everything correctly, your answer could be simplified all the way down to $(3^p/2)^n$. Given that $3^{.6} = 1.93$, do we need $p > .6$ to have a long term positive return? Just write either YES or NO.

Solution: Yes. We need $p > \log_3(2) \approx .63$.

[4 points] 3. (a) The length of human pregnancies is approximately normal with mean 266 days and standard deviation 16 days. What is the probability that a pregnancy lasts less than ('<') 240 days (about 8 months)? Assume days are discrete (i.e. a 244.2345 day pregnancy is not possible). Write your answer in terms of $\Phi(x) = P(Y \leq x)$ with $Y = N(0, 1)$. You do not need to look anything up on the table, or do any arithmetic.

Solution: $X = N(266, 16^2)$. We have

$$P(X \leq 240.5) = P\Big(Y \leq \frac{240.5 - 266}{16}\Big) = P(Y \leq -1.59) = 1 - \Phi(1.59).$$

Okay to leave as $\Phi(-1.59)$.

[4 points] (b) Write an algebraic expression that you could solve for the value c for which there is a 99% probability that a pregnancy lasts longer than c days. You do not need to solve it.

Solution: We want $P(X \geq c) = .99$. This means that $P\Big(Y \geq \frac{c - .5 - 266}{16}\Big) = .99$. Looking at the table we need

$$\frac{c - .5 - 266}{16} = -2.33.$$

[4 points] 4. Suppose that there are on average 100 bear attacks per decade. Use normal approximation to the Poisson to estimate the probability that X, the number of bear attacks next decade, has $90 \leq X \leq 120$. Actually write down the table values for Φ. Recall if $Z = \text{Poi}(\lambda)$ then $EZ = \text{var}(Z) = \lambda$.

Solution: We can assume that $X = N(100, 10^2)$. We then want

$$P(89.5 \leq X \leq 119.5) = P(\frac{-10.5}{10} Y \leq \frac{19.5}{10})$$

$$= P(-1.05 \leq Y \leq 1.95) = \Phi(1.95) - \Phi(-1.05) = \Phi(1.95) + \Phi(1.05) - 1.$$

[8 points] 5. A marathon runner has observed that her mile splits for the first 16 miles of a marathon are i.i.d. with mean 7 minutes and standard deviation .5 minutes. Use normal approximation to estimate the probability she runs the first 16 miles in less than (‘$<$’) 108 minutes. Actually lookup Φ in the table. Note that $7 * 16 = 112$ and $.5^2 = .25$.

Solution: Let T_i be the duration of her ith mile. The time to run 20 miles is $T = \sum_1^{20} T_i$. Notice that $\text{var}(T) = 16 * .25 = 4$ and so $\sigma(T) = 2$. We can write

$$P(T \leq 120) \approx P(X \leq \frac{108 - 112}{.2}) = P(X \leq -2) = 1 - \Phi(2).$$

[6 points] 6. (a) Mattual, a mutual fund, claims they can with probability more than 1/2 beat the average return of the stock market each month. You decide to start monitoring their gains for the next 25 months. How many of these months would Mattual have to outperform the market for you to be 95% certain of their claim?

It is sufficient to call this number x and write an algebraic equation you would solve.

Solution: Let $X = \sum_1^n 25X_i$ be the number of months they outperform the market. We need x so that $P(X \geq x) = .05$. Rescaling this is the same as

$$P(Y \geq \frac{x - .5 - 25/2}{5/2})$$

with $Y = N(0, 1)$. We know that $P(Y \geq 1.65) = .05$. Thus, we would solve

$$\frac{x - .5 - 25/2}{5/2} = 1.65.$$

So $x = 18$.

[4 points] (b) Suppose Mattual beats the market 20 of those 25 months. Give a 95% percent confidence interval for their probability p of outperforming the market.

Solution: $\mathbf{E}\hat{p} = p$ and $\operatorname{var}\hat{p} = p(1-p)/n$. We have observed that $\hat{p} = 4/5$, and want to be inside 2 standard deviations of this. Notice that $\sqrt{(4/5)(1/5)/25} = 2/25$

$$[\frac{4}{5} - 2(2/25), \frac{4}{5} + 2(2/25)] = [.64, .96].$$

.png .pdf .jpg .jpeg .bmp .tiff .tif .gif .eps .ps .eps.gz .ps.gz .eps.Z