San Francisco State University
ECON 715

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Final Exam - Solution

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Time: 120 minutes

Name: ______________________________

Instructions:
1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (25 points). Suppose that the market value of an asset, at time t, is given by $V(t)$ - a twice differentiable function of time. Suppose that the interest rate per period (year) is r.

   (a) The owner of the asset wishes to maximize the present value of the asset, by selling it at the right time. Write the optimization problem of the owner, and derive the first order necessary condition.

   Optimization problem is:

$$\max_t PV(t) = V(t) e^{-rt}$$

   First order necessary condition for maximum is:

$$\begin{aligned} \frac{d}{dt} PV(t) &= V'(t) e^{-rt} - rV(t) e^{-rt} = 0 \\ &\Rightarrow V'(t) - rV(t) = 0 \end{aligned}$$

   (b) Provide economic intuition of the first order necessary condition from the previous section.

   The first order condition can be written as

$$\frac{V'(t)}{V(t)} = r$$

   This means that the owner should keep the asset until the growth rate of its value equalizes to the interest rate.

(c) Suppose the value of the asset evolves according to $V(t) = Ke^{f(t)}$, where $f(t) = 0.12\sqrt{t}$, where t is time in years and the interest rate is $r = 2\%$. Find the optimal holding time, t^*, of the asset.

$$\begin{aligned}
\frac{V'(t)}{V(t)} &= r \\
\frac{Ke^{f(t)} \cdot f'(t)}{Ke^{f(t)}} &= r \\
f'(t) &= 0.12 \cdot 0.5t^{-0.5} = r \\
t^* &= \left(\frac{0.06}{r}\right)^2 = \left(\frac{0.06}{0.02}\right)^2 = 3^2 = 9 \text{ years}
\end{aligned}$$

(d) Suppose that $f(t)$ is unknown, but it is given that f is increasing and concave ($f'(t) > 0\ \forall t$, and $f''(t) < 0\ \forall t$). Prove that the optimal holding time, t^*, is decreasing in interest rate r.

Proof 1. Taking differential of both sides of the optimality condition $f'(t) = r$:

$$\begin{aligned} f''(t)\, dt &= dr \\ &\Rightarrow \frac{dt}{dr} = \frac{1}{f''(t)} < 0 \end{aligned}$$

The last inequality follows from the given that $f''(t) < 0$.

Proof 2. Write the F.O.N.C. as implicit function:

$$F(t,r) = f'(t) - r = 0$$

Then, by the implicit function theorem:

$$\frac{dt}{dr} = -\frac{F_r}{F_t} = -\frac{-1}{f''(t)} = \frac{1}{f''(t)} < 0$$

(e) Provide economic intuition for the result in the previous section.

The interest rate is the opportunity cost of holding the asset for additional time. The owner can sell the asset, and invest the money at interest rate r. Thus, higher interest rate, means that the opportunity cost of holding the asset is higher, and the seller would like to sell it earlier.

2. (25 points). Consider a monopoly that sells a single product to n segmented markets. The revenue function in market i, as a function of quantity of product sold in that market, is $R_i(Q_i)$, with $R_i'(Q_i) > 0$ and $R_i''(Q_i) < 0$. Assume that the total cost function, $C(Q) = C\left(\sum_{i=1}^{n} Q_i\right)$, is increasing and strictly convex.

   (a) Write the optimization problem of the monopoly, and derive the first order necessary condition.

   Profit maximization problem:

$$\max_{Q_1,...,Q_n} \pi(Q_1, ..., Q_n) = \sum_{i=1}^{n} R_i(Q_i) - C(Q)$$

   First order necessary condition:

$$\frac{\partial}{\partial Q_i}\pi(Q_1, ..., Q_n) = R_i'(Q_i) - C'(Q) = 0 \quad \forall i = 1, ..., n$$

   (b) Provide economic interpretation of the first order necessary condition from the previous section.

   The above condition states that the Marginal Revenue (MR) in all markets must be equal to the common marginal cost.

(c) Show that the monopoly's selling price in market i is equal to the markup rate μ_i above the marginal cost, i.e.

$$p_i = [1 + \mu_i]\, MC$$

where the markup rate is $\mu_i = 1/(|\eta_i| - 1)$, and η_i is the price elasticity of demand in market i.

The revenue in any market is $R(Q) = P(Q)\,Q$ (here we omit the market subscript i for ease of notation). Thus, the Marginal Revenue can be expressed as a function of price elasticity of demand:

$$MR = \frac{d}{dQ}R(Q) = \frac{dP}{dQ} \cdot Q + P = P\left[1 + \frac{dP}{dQ}\frac{Q}{P}\right] = P\left[1 + \frac{1}{\eta}\right] = P\left[1 - \frac{1}{|\eta|}\right]$$

Using the $MR = MC$ profit maximization condition:

$$\begin{aligned} P\left[1 - \frac{1}{|\eta|}\right] &= MC \\ P &= \left[\frac{|\eta|}{|\eta| - 1}\right] MC = \left[1 + \frac{|\eta|}{|\eta| - 1} - 1\right] MC \end{aligned}$$

Thus, the selling price in market i is

$$P_i = \left[1 + \frac{1}{|\eta_i| - 1}\right] MC = [1 + \mu_i]\, MC$$

(d) Prove that if the demand becomes perfectly elastic, the monopoly price becomes the same as in perfect competition, $P = MC$ (markup rate is zero). That is, prove that

$$\lim_{|\eta_i| \to \infty} P_i = MC$$

$$\lim_{|\eta_i| \to \infty} P_i = \lim_{|\eta_i| \to \infty} \left[1 + \frac{1}{|\eta_i| - 1}\right] MC = \left[1 + \frac{1}{\infty - 1}\right] MC = [1 + 0]\, MC = MC$$

(e) Prove that the critical value of the profit function is a unique global maximum. Clearly state the theorems used in your proof.

Proof. It is given that $R_i(Q_i)$ is strictly concave $\forall i$, and therefore $\sum_{i=1}^{n} R_i(Q_i)$ is strictly concave (sum of strictly concave functions is str. concave). The cost function is given to be strictly convex, so $-C(Q)$ is strictly concave (f is concave if and only is $-f$ is convex, strict or not). Thus, the profit function is a sum of strictly concave functions and therefore strictly concave. Consequently, the critical point of a strictly concave function is automatically a unique global maximum.

3. (20 points). Suppose that consumer A derives utility from quantities of two goods, x, y, and his utility function is $u(x, y)$, which is strictly quasiconcave. The prices of the goods are p_x, p_y and consumer's income is I.

   (a) Write the consumer's utility maximization problem, assuming that he must spend all his income on the two goods.

   Consumer's problem:

$$\max_{x,y} u(x, y)$$
$$s.t.$$
$$p_x x + p_y y = I$$

   (b) Write the Lagrange function associated with the consumer's problem, and derive the first order necessary conditions for constrained optimum.

   The Lagrange function:

$$\mathcal{L} = u(x, y) - \lambda [p_x x + p_y y - I]$$

   The first ordered necessary conditions are:

$$\begin{aligned}
\mathcal{L}_\lambda &= -[p_x x + p_y y - I] = 0 \\
\mathcal{L}_x &= u_x(x, y) - \lambda p_x = 0 \\
\mathcal{L}_y &= u_y(x, y) - \lambda p_y = 0
\end{aligned}$$

(c) Suppose that consumer B has utility function $v(u(x,y))$, where v is monotone increasing and differentiable function, and $u(x,y)$ is the utility of consumer A. Prove that consumer A and consumer B will always choose the same consumption bundle, if they have the same budget constraint.

The condition for optimal consumption bundle for consumer A, from $\mathcal{L}_x = \mathcal{L}_y = 0$ is

$$\frac{u_x(x,y)}{u_y(x,y)} = \frac{p_x}{p_y}$$

The Lagrange function for consumer B is:

$$\mathcal{L} = v(u(x,y)) - \lambda[p_x x + p_y y - I]$$

The first ordered necessary conditions are:

$$\begin{aligned} \mathcal{L}_\lambda &= -[p_x x + p_y y - I] = 0 \\ \mathcal{L}_x &= v'(u(x,y))\,u_x(x,y) - \lambda p_x = 0 \\ \mathcal{L}_y &= v'(u(x,y))\,u_y(x,y) - \lambda p_y = 0 \end{aligned}$$

These imply the same optimality condition of consumption bundle

$$\frac{v'(u(x,y))\,u_x(x,y)}{v'(u(x,y))\,u_y(x,y)} = \frac{u_x(x,y)}{u_y(x,y)} = \frac{p_x}{p_y}$$

Thus, given that consumer A and B have the same budget, they will always chose the same bundle, if the optimal bundle is unique.

(d) Prove that first order necessary conditions in part b characterize a unique constrained global maximum.

The first order conditions determine the unique global maximum because the constrain set is convex and objective function is strictly quasiconcave (given). We need to prove that the budget set, B, is convex, where B is defined as:

$$B = \{(x, y) \,| p_x x + p_y y = I\}$$

Let (x_1, y_1) and (x_2, y_2) be two bundles in the budget set and let $\alpha \in (0, 1)$. We need to verify that $\alpha (x_1, y_1) + (1 - \alpha) (x_2, y_2) \in B$. Since the two bundles are in the budget set, we have

$$\begin{aligned} p_x x_1 + p_y y_1 &= I \\ p_x x_2 + p_y y_2 &= I \end{aligned}$$

Multiply the first by α and the second by $(1 - \alpha)$ and add up:

$$\begin{aligned} p_x \alpha x_1 + p_y \alpha y_1 &= \alpha I \\ p_x (1 - \alpha) x_2 + p_y (1 - \alpha) y_2 &= (1 - \alpha) I \\ p_x [\alpha x_1 + (1 - \alpha) x_2] + p_y [\alpha y_1 + (1 - \alpha) y_2] &= I \end{aligned}$$

4. (5 points). Prove that any Cobb-Douglas function $u(x_1, ...x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$, with $\alpha_i > 0$ $\forall i = 1, ..., n$, is strictly quasiconcave. Clearly state the theorems used in your proof.

   The given function can be written as

$$\exp\left(\sum_{i=1}^{n} \alpha_i \ln x_i\right)$$

   The $\ln(\cdot)$ function is strictly concave ($\frac{d}{dx}\ln(x) = x^{-1}$, $\frac{d^2}{dx^2}\ln(x) = -x^{-2} < 0$), the weighted sum of strictly concave functions $\sum_{i=1}^{n} \alpha_i \ln x_i$ is strictly concave. The exponential function $\exp(\cdot)$ is monotone increasing, and since monotone increasing transformation of strictly concave function is strictly quasiconcave, $u(x_1, ...x_n)$ is strictly quasiconcave.

5. (15 points). Suppose Oscar has wealth w, his preferences over risky alternatives are described by Expected Utility Theory, and his utility over certain outcomes is represented by $u(\cdot)$, which is strictly increasing and strictly concave. He can divide his wealth between investment in risky asset, with random return r, and a risk-free asset with guaranteed return r_f. Let the amount invested in risky asset be $x \in [0, w]$.

   (a) Write Oscar's optimal investment problem.

   The amount x is invested in risky asset with return r and the rest $w - x$ is invested in risk-free asset with return r_f. Thus, the future wealth is $x(1+r) + (w-x)(1+r_f) = w(1+r_f) + x(r-r_f)$. The optimal investment problem is therefore:

$$\max_{0 \leq x \leq w} E\left[u\left(w\left(1+r_f\right)+x\left(r-r_f\right)\right)\right]$$

   (b) Write the first order necessary condition for optimal x.

   The first order condition for interior optimum, x^*, is

$$E\left[u'\left(w\left(1+r_f\right)+x^*\left(r-r_f\right)\right)\left(r-r_f\right)\right]=0$$

(c) Prove that Oscar will invest a positive amount in risky asset ($x^* > 0$) if and only if the expected return on the risky asset is greater than the risk-free return: $E(r) > r_f$.

The condition of positive investment in risky asset, $x^* > 0$, is (slope of utility at $x = 0$ is positive):

$$\begin{aligned} E\left[u'\left(w\left(1+r_f\right)\right)\left(r-r_f\right)\right] &> 0 \\ u'\left(w\left(1+r_f\right)\right) E\left(r-r_f\right) &> 0 \end{aligned}$$

Which holds if and only if

$$E(r) > r_f$$

6. (10 points). Consider the Matlab script below:

```
1    clear
2    syms  x y p_x p_y I k a
3    u(x,y) = a*log(x) + (1-a)*log(y);
4    L(x,y,k) = u(x,y) - k*(p_x*x + p_y*y - I);
5    FOC = gradient(L,[x,y,k])==0;
6    [x,y,k] = solve(FOC,[x,y,k]);
7    demand = subs([x,y],[a,p_x,p_y,I],[0.5,2,3,100]);
```

(a) What is the purpose of the entire program?

The program defines a lagrange function corresponding to a consumer utility maximization problem, and solves for the demand.

(b) What is the purpose of line 5?

Defines the First Order Necessary Conditions for the optimization problem.

(c) what is the purpose of line 6?

Solves (symbolically) for the demand `(x,y)` and the lagrange multiplier (here `k`).

(d) What is the purpose of line 7? In particular, what is the purpose of the numerical values in line 7?

The command in line 7 substitutes the numerical values of the parameters into the symbolic solution, to get the numerical demand and lagrange multiplier. Here a is the weight on x in the utility function, so the utility is $u(x,y) = 0.5 \ln x + 0.5 \ln y$. The values 2,3 are the prices p_x, p_y and consumer's income is $I = 100$.