Low-rank optimization on the set of low-rank non-symmetric matrices

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1 Introduction

We address unconstrained convex optimization problems with low-rank solutions. We propose an algorithm that alternates between fixed-rank optimization and rank-one updates. The fixed-rank optimization is characterized by different factorization models. The search space is nonlinear but is equipped with a particular Riemannian structure that leads to efficient computations. For each of the factorization we present a second-order trust-region algorithm with a guaranteed quadratic rate of convergence. Overall, the proposed optimization scheme converges super-linearly to the global solution while still maintaining complexity that is linear in the number of rows of the matrix. The performance of the proposed algorithm is illustrated on the low-rank matrix completion problem.

2 Problem formulation

We focus on the following convex unconstrained problem

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times m}} \quad f(\mathbf{X}) \qquad (1)$$

where f is a smooth convex function with the additional information that the solution $\mathbf{X}^*$ of the problem (1) is low-rank. Programs of this type have attracted much attention in the recent years as efficient convex relaxations of intractable rank minimization problems [Faz02, KO09, CCS10] with *trace* (nuclear) norm or *tikhonov* regularization. While most fixed-rank optimization algorithms assume a priori the search space and solve for a local minimum, we perform a systematic and efficient search for the global minimum of (1). The usefulness of this scheme has been presented in [MMBS11] in the context of trace norm.

Alternating between fixed-rank optimization and rank-one updates ensures that all intermediate iterates are low-rank while converging super-linearly to global minimum. Local minima are then escaped by incrementing the rank until the global minimum in reached. The rank-one update is always selected to ensure a decrease of the cost. This scheme of low-rank optimization differs from standard approaches which require singular value thresholding operations at each iteration [CCS10, MGC11].

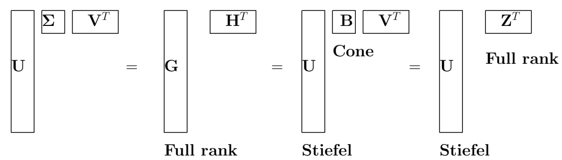

Figure 1: Different factorization models of a p-rank non-symmetric matrix.

3 Factorization models

We discuss the geometry associated with each of the factorization model and give the necessary insights to devise second-order schemes, shown in Figure 1. The search space of p-rank manifold is the quotient manifold defined by

$$\begin{array}{lcl} \mathbb{R}^{n\times p}\times\mathbb{R}^{m\times p}/\mathrm{GL}(r) & \leftarrow & \mathbf{GH}^T \\ \mathrm{St}(p,n)\times S_{++}(p)\times \mathrm{St}(p,m)/O_p & \leftarrow & \mathbf{UBV}^T \\ \mathrm{St}(p,n)\times\mathbb{R}^{m\times p}/O_p & \leftarrow & \mathbf{UZ}^T \end{array}$$

where $\mathrm{GL}(r)$ is the set of full rank $r\times r$ matrices and O_p is the set of $p\times p$ orthogonal matrices.

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