Example 7.16.8 *Obtain the state transition matrix for the system expressed as :*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

PU : May-06

$$\left[\text{Ans.} : \begin{bmatrix} e^{-t} & t e^{-t} & \frac{1}{2} t^2 e^{-t} \\ 0 & e^{-t} & t e^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix}\right]$$

Example 7.16.9 *Obtain the state transition matrix for system :*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

PU : Dec.-98,10, May-12

$$\left[\text{Ans.} : \begin{bmatrix} e^{-3t} & 0.5\, e^{-t} - 0.5\, e^{-3t} \\ 0 & e^{-t} \end{bmatrix}\right]$$

Example 7.16.10 *For a system* $\dot{X} = AX$, $X(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ *when* $X(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ *and*

$X(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ *for* $X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *Determine the system matrix A.*

PU : May-11

$$\left[\text{Ans.} : \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]$$

Example 7.16.11 *Find out the time response for unit step input of a system given by*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 5 \end{bmatrix} U(t) \quad \text{and} \quad Y(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(t)$$

and $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

PU : May-07, Dec.-12

$$[\text{Ans.} : Y_1(t) = 3(e^{-t} - e^{-2t}),\ Y_2(t) = -5 + 3\,(2e^{-2t} - e^{-t})]$$

Example 7.16.12 *Obtain the complete time response of system given by,*

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} X(t) \quad \text{where } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Y(t) = [1 \ \ -1]\, X(t)$$

PU : Dec.-07

$$\left[\text{Ans.} : Y(t) = \frac{3}{\sqrt{2}} \sin \sqrt{2}\, t\right]$$

□□□

8 Feedback Control System Characteristics

Syllabus

Error signal analysis, Sensitivity of control systems to parameter variations, Disturbance signals in a feedback control system, Control of the transient response, Steady-state error, The cost of feedback.

Contents

8.1 Introduction

It is seen that feedback is most important property of control systems. Essentially the feedback is used to reduce the difference between reference input r(t) and actual output c(t), which is called tracking error e(t). The feedback not only reduces the tracking error but also reduces, the sensitivity of the system to parameter variations and unwanted internal and external disturbances.

A system without feedback is called open loop system. The Fig. 8.1.1 shows open loop system with $T_d(s)$ as disturbances. In such system the output is highly sensitive to the disturbances and changes in parameters of G(s). But use of negative feedback, the sensitivity of system to such parameter variations and disturbances can be greatly reduced. This chapter analyses the various effects of feedback on the characteristics of control systems.

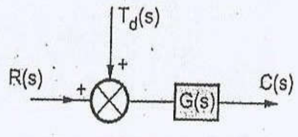


Fig. 8.1.1 Open loop system

8.2 Error Signal Analysis

Let us analyse the effect of disturbances and noise on error. Consider a feedback control system with three inputs $R(s), T_d(s)$ and $N(s)$ as shown in the Fig. 8.2.1.

$$T_d(s) = \text{Disturbance input}$$

$$N(s) = \text{Noise input}$$

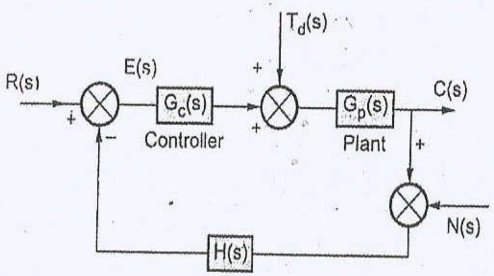


Fig. 8.2.1

Using block diagram reduction rules for multiple input system the expression for C (s) is,

$$C(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)}R(s)+\frac{G_p(s)}{1+G_c(s)G_p(s)H(s)}T_d(s)-\frac{G_c(s)G_p(s)H(s)}{1+G_c(s)G_p(s)H(s)}N(s)$$

Assume $H(s) = 1$ and $E(s) = R(s)-C(s)$

Using the expression of C(s) in E(s),

$$E(s) = \frac{1}{1+G_c(s)G_p(s)}R(s)-\frac{G_p(s)}{1+G_c(s)G_p(s)}T_d(s)+\frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)}N(s)$$

The error $E(s) = R(s)-C(s)$ is called tracking error.

The open loop transfer function L(s) is defined as the product of $G_c(s)\ G_p(s)$.

$$\therefore \quad L(s) = G_c(s)G_p(s) = \text{Open loop transfer function}$$

While F(s) is defined as $1+G_c(s)G_p(s)$ i.e. $1+L(s)$

$$\therefore \quad F(s) = 1+L(s)$$

While the sensitivity function is defined as reciprocal of F(s).

$$\therefore \quad S(s) = \frac{1}{F(s)} = \frac{1}{1+L(s)}$$

The complementary sensitivity function is defined as,

$$S_c(s) = L(s)S(s) = \frac{L(s)}{1+L(s)}$$

Using in the expression for E(s),

$$\boxed{E(s) = S(s)R(s)-S(s)G_p(s)T_d(s)+S_c(s)N(s)}$$

Thus to minimize the error both S(s) and $S_c(s)$ must be small and both are the functions of controller $G_c(s)$. Hence proper design of controller can minimize the tracking error.

It can seen that, $\boxed{S(s)+S_c(s)=1}$

Thus if S(s) is made small, $S_c(s)$ becomes high and viceversa. Hence compromise is necessary, while designing $G_c(s)$.

To reduce the effect of $T_d(s)$ on error, L(s) must be large over the range of frequencies hence $\frac{1}{1+L(s)}$ i.e. S(s) is small.

To reduce the effect of N(s) on error, L(s) must be small over the range of frequencies hence $\frac{L(s)}{1+L(s)}$ i.e. $S_c(s)$ is small.

Effectively magnitude of $G_c(s)$ must be high to reduce effect of $T_d(s)$ while it should be small to reduce the effect of N(s).

Practically L(s) is made large at low frequencies while L(s) is made small at high frequencies.

Review Question

1. *How to reduce the effect of disturbances and noise on tracking error ? Explain in detail.*

8.3 Effect of Parameter Variations in an Open Loop Control System

Consider an open loop control system shown in the Fig. 8.3.1 The overall transfer function of the system is given by,

$$G(s) = \frac{C(s)}{R(s)}$$

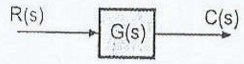


Fig. 8.3.1

$$\therefore \quad C(s) = G(s) \cdot R(s)$$

Let Δ G(s) be the change in G(s) due to the parameter variations. The corresponding change in the output be Δ C(s).

$$\therefore \quad C(s) + \Delta C(s) = [G(s) + \Delta G(s)] \; R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = G(s) \cdot R(s) + \Delta G(s) \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = C(s) + \Delta G(s) \cdot R(s)$$

$$\therefore \quad \boxed{\Delta C(s) = \Delta G(s) \cdot R(s)} \qquad \text{... (8.3.1)}$$

The equation (8.3.1), gives the effect of change in transfer function, due to the parameter variations, on the system output, in an open loop control system. Let us discuss now, the effect of such parameter variations in a closed loop control system.

Review Question

1. *Discuss the effect of parameter variations in an open loop control system.*

8.4 Effect of Parameter Variations in a Closed Loop System

Consider a closed loop system as shown in Fig. 8.4.1. The signal E(s) is the Laplace transform of the error signal e(t). The overall transfer function of the system is given by,

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

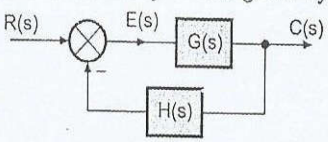


Fig. 8.4.1

Let Δ G(s) be the change in G(s) which is due to the parameter variations in the system. The corresponding change in the output be Δ C(s).

$$C(s) + \Delta C(s) = \frac{[G(s) + \Delta G(s)]}{1 + [G(s) + \Delta G(s)]\,H(s)} \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = \frac{[G(s) + \Delta G(s)]}{1 + G(s)H(s) + \Delta G(s)H(s)} \cdot R(s)$$

The term Δ G(s) H(s) is negligibly small as compared to G(s) H(s), as the change Δ G(s) is very small compared to G(s). Neglecting the term Δ G(s) H(s) from the denominator, we get,

$$C(s) + \Delta C(s) = \frac{G(s) + \Delta G(s)}{1 + G(s)H(s)} \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = \frac{G(s)}{1 + G(s)H(s)} \cdot R(s) + \frac{\Delta G(s)}{1 + G(s)H(s)} \cdot R(s)$$

$$\therefore \quad C(s) + \Delta C(s) = C(s) + \frac{\Delta G(s)}{1 + G(s)H(s)} \cdot R(s)$$

$$\therefore \quad \boxed{\Delta C(s) = \frac{\Delta G(s)}{1 + G(s)H(s)} \cdot R(s)} \qquad \text{... (8.4.1)}$$

The equation (8.4.1), gives the change in the output due to the parameter variations in G (s), in a closed loop system.

In practice, the magnitude of $1 + G(s)H(s)$ is very much greater than unity.

$$|G(s)H(s)| >> 1$$

Hence it can be observed from the equations (8.3.1) and (8.4.1), that in a closed loop system, due to the feedback, the change in the output, due to the parameter variations in G(s), is reduced by the factor $[1 + G(s)H(s)]$. In an open loop system, such a reduction does not exist, due to the absence of the feedback.

Review Question

1. *Discuss the effect of parameter variations in closed loop control system.*

8.5 Sensitivity of a Control System

GTU : May-05

The parameters of any control system cannot be constant throughout its entire life. There are always changes in the parameters due to environmetal changes and other disturbances. These changes are called parameter variations. Such parameter variations affect the system performance adversely. Such an effect, in the system performance due to parameter variations can be studied mathematically defining the term sensitivity of a control system. The change in a particular variable due to the parameter variations can be expressed in terms of sensitivity as below :

Let the variable in a system which is varying be 'T', due to the variations in the parameter 'K' of the system. The sensitivity of the system parameter T to the parameter K is expressed as,

$$\boxed{S = \frac{\%\ \text{change in T}}{\%\ \text{change in K}}} \qquad \text{... (8.5.1)}$$

Mathematically, it can be expressed as

$$S_K^T = \frac{d\,ln\,(T)}{d\,ln\,(K)} = \frac{\left(\frac{1}{T}\right)}{\left(\frac{1}{K}\right)} \cdot \frac{\partial T}{\partial K}$$

$$\therefore \quad S_K^T = \frac{(\partial T/T)}{(\partial K/K)} \qquad \text{... (8.5.2)}$$

The symbolic representation S_K^T represents the sensitivity of a variable T with respect to the variations in the parameter K. In practice, the variable T may be an output variable while the parameter K may be the gain, the feedback factor etc. The representation S_K^T is also called the sensitivity function of a system. For a good control system, the value of such a sensitivity function should be as minimum as possible.

Let T(s) be the overall transfer function of a control system. The forward path transfer function G(s) is varying. Then the sensitivity of overall transfer function with respect to the variations in G (s) is defined as,

$$S_G^T = \frac{\partial T(s)/T(s)}{\partial G(s)/G(s)} = \frac{G(s)}{T(s)} \cdot \frac{\partial T(s)}{\partial G(s)}$$

For the open loop system,

$$T(s) = G(s)$$

$$\therefore \quad S_G^T = \frac{G(s)}{G(s)} \cdot \frac{\partial G(s)}{\partial G(s)} = 1 \qquad \text{... (8.5.3)}$$

Thus the sensitivity of T(s) with respect to G(s) for an open loop system is unity.

Let us find out the sensitivity function for a closed loop system. For a closed loop system,

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\therefore \quad \frac{\partial T(s)}{\partial G(s)} = \frac{[1+G(s)H(s)]\cdot[1]-[G(s)][H(s)]}{[1+G(s)H(s)]^2} = \frac{1}{[1+G(s)H(s)]^2}$$

$$\therefore \quad S_G^T = \frac{G(s)}{T(s)} \cdot \frac{\partial T(s)}{\partial G(s)} = \frac{G(s)}{\left[\frac{G(s)}{1+G(s)H(s)}\right]} \cdot \frac{1}{[1+G(s)H(s)]^2}$$

$$\therefore \quad S_G^T = \frac{1}{1+G(s)H(s)} \qquad \text{... (8.5.4)}$$

Comparing the two equations (8.5.3) and (8.5.4), it can be observed that due to the feedback the sensitivity function gets reduced by the factor $1/[1+G(s)\,H(s)]$ compared to an open loop system. And less the value of sensitivity function, less sensitive is the system to the variations in the forward path transfer function G(s).

8.5.1 Sensitivity of T(s) with Respect to H(s)

Let us calculate the sensitivity function which indicates the sensitivity of the overall transfer function T(s) with respect to the feedback path transfer function H(s). Such a function can be expressed as,

$$S_H^T = \frac{H(s)}{T(s)} \cdot \frac{\partial T(s)}{\partial H(s)}$$

For a closed loop system,

$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$

$$\therefore \quad \frac{\partial T(s)}{\partial H(s)} = \frac{[1+G(s)H(s)][0]-[G(s)][G(s)]}{[1+G(s)H(s)]^2} = \frac{-[G(s)]^2}{[1+G(s)H(s)]^2}$$

$$\therefore \quad S_H^T = \frac{H(s)}{T(s)} \cdot \frac{-[G(s)]^2}{[1+G(s)\,H(s)]^2}$$

$$= \frac{H(s)}{\left[\frac{G(s)}{1+G(s)H(s)}\right]} \cdot \frac{-[G(s)]^2}{[1+G(s)H(s)]^2} = \frac{-G(s)H(s)}{1+G(s)H(s)} \qquad \text{... (8.5.5)}$$

It can be observed from the equations (8.5.4) and (8.5.5) that the closed loop system is more sensitive to variations in the feedback path parameters than the variations in the forward path parameters. Thus, the specifications of the feedback elements must be observed strictly as compared to the specifications of the forward path elements.

Example 8.5.1 *A position control system is shown in the Fig. 8.5.1. Evaluate the sensitivities S_K^T, S_a^T. K = 20 and a = 4.*

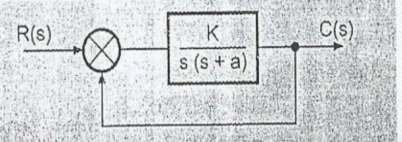


Fig. 8.5.1

Solution : For a given system,

$$G(s) = \frac{K}{s\,(s+a)} \quad \text{and} \quad H(s) = 1$$

Now $$S_K^T = \frac{K}{T(s)} \cdot \frac{\partial T(s)}{\partial K}$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+a)}}{1+\frac{K}{s(s+a)}} = \frac{K}{s^2+as+K}$$

$$\therefore \quad \frac{\partial T(s)}{\partial K} = \frac{(s^2+as+K)(1)-K(1)}{(s^2+as+K)^2} = \frac{s^2+as}{(s^2+as+K)^2}$$

$$\therefore \quad S_K^T = \frac{K}{\left(\frac{K}{s^2+as+K}\right)} \times \frac{(s^2+as)}{(s^2+as+K)^2} = \frac{s(s+a)}{s^2+as+K}$$

$$\therefore \quad \boxed{S_K^T = \frac{s(s+4)}{(s^2+4s+20)}}$$

Now $$S_a^T = \frac{a}{T(s)} \cdot \frac{\partial T(s)}{\partial a}$$

$$\frac{\partial T(s)}{\partial a} = \frac{(s^2+as+K)(0)-K(s)}{(s^2+as+K)^2} = \frac{-K}{(s^2+as+K)^2}$$

$$\therefore \quad S_a^T = \frac{a}{\left(\frac{K}{s^2+as+K}\right)} \times \frac{-Ks}{(s^2+as+K)^2} = \frac{-as}{s^2+as+K}$$

$$\therefore \quad \boxed{S_a^T = \frac{-4s}{s^2+4s+20}}$$

Example 8.5.2 *Find the sensitivity of the overall transfer function of the system shown in Fig. 8.5.2 with respect to i) Forward path transfer function, ii) Feedback path transfer function. The value of ω is 1.2 rad/sec.*

UPTU : 2012-13

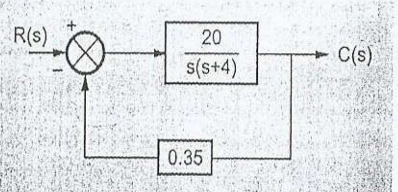


Fig. 8.5.2

Solution :

$$G(s) = \frac{20}{s(s+4)}, \quad H(s) = 0.35$$

$$\therefore \quad T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{20}{s(s+4)}}{1+\frac{20}{s(s+4)}\times 0.35} = \frac{20}{s^2+4s+7}$$

i) Sensitivity of T(s) with respect to G(s) is,

$$S_G^T = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\frac{20\times 0.35}{s(s+4)}} = \frac{s(s+4)}{s^2+4s+7}$$

Put s = jω to convert sensitivity function to frequency domain.

$$\therefore \quad S_G^T = \frac{j\omega(j\omega+4)}{(j\omega)^2+4j\omega+7} = \frac{-\omega^2+4j\omega}{(7-\omega^2)+4j\omega} \text{ and use } \omega = 1.2$$

$$\therefore \quad |S_G^T| = \frac{\sqrt{(\omega^2)^2+(4\omega)^2}}{\sqrt{(7-\omega^2)^2+(4\omega)^2}} = \frac{\sqrt{(1.2^2)^2+(4\times 1.2)^2}}{\sqrt{(7-1.2^2)^2+(4\times 1.2)^2}} = \mathbf{0.6822}$$

ii) Sensitivity of T(s) with respect to H(s) is,

$$S_H^T = \frac{-G(s)H(s)}{1+G(s)H(s)} = \frac{-\frac{20\times 0.35}{s(s+4)}}{1+\frac{20\times 0.35}{s(s+4)}} = \frac{-7}{s^2+4s+7}$$

$$\therefore \quad S_H^T = \frac{-7}{(7-\omega^2)+4j\omega}, \quad |S_H^T| = \frac{7}{\sqrt{(7-\omega^2)^2+(4\omega)^2}}$$

$$\therefore \quad |S_H^T| = \frac{7}{\sqrt{(7-1.2^2)^2+(4\times 1.2)^2}} = \mathbf{0.953}$$

Example 8.5.3 *Evaluate the sensitivity of T(s) to variation of K of,*

$$T(s) = \frac{(75s+6)K}{3s^2+21s+30+(25s^2+25)K}.$$

GTU : May-05, Marks 8

Solution : The sensitivity of T(s) to variation of K is given by,

$$\therefore \quad S_K^T = \frac{K}{T}\frac{\partial T(s)}{\partial K}$$

$$\frac{\partial T(s)}{\partial K} = \frac{[3s^2+21s+30+(25s^2+25)K][75s+6]-[75s+6]K[25s^2+25]}{[3s^2+21s+30+(25s^2+25)K]^2}$$

$$= \frac{[75s+6][3s^2+21s+30+25Ks^2+25K-25Ks^2-25K]}{[3s^2+21s+30+(25s^2+25)K]^2}$$

$$= \frac{3[75s+6][s^2+7s+10]}{[3s^2+21s+30+(25s^2+25)K]^2} = \frac{3\,[755+6]\,[s+1]\,[s+5]}{[3s^2+21s+30+(25s^2+25)K]^2}$$

$$\therefore \quad S_K^T = \frac{K}{\dfrac{[75s+6]K}{[3s^2+21s+30+(25s^2+25)K]}} \times \frac{3[75s+6][s+2][s+5]}{[3s^2+21s+30+(25s^2+25)K]^2}$$

$$\therefore \quad \boxed{S_K^T = \frac{3[s+2][s+5]}{3s^2+21s+30+(25s^2+25)K}}$$

Review Questions

1. *Define sensitivity of control system and hence obtain sensitivity of T(s) with respect to G(s) for open and closed loop systems.*
2. *Obtain the sensitivity of T(s) with respect to H's).*
3. *Discuss the effect of feedback on the sensitivity of a system.*

8.6 Effect of Feedback on the System Parameters

Let us study the effect of feedback on the system parameters such as time constant, overall gain, stability, bandwidth etc.

8.6.1 Effect of Feedback on Time Constant of a Control System

Consider an open loop system with overall transfer function as,

$$G(s) = \frac{K}{1+sT}$$

When this system is subjected to unit step input, its response can be obtained as,

$$\frac{C(s)}{R(s)} = \frac{K}{1+sT} \quad \text{..... As open loop system}$$

$$R(s) = \frac{1}{s} \quad \text{unit step input}$$

$$C(s) = \frac{K}{s(1+sT)}$$

$$\therefore \quad c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{K}{s} - \frac{K}{\left(s+\frac{1}{T}\right)}\right] = K\left[1-e^{-t/T}\right]$$

So T is the time constant of the open loop system.

Now the feedback is introduced in the system with feedback transfer function as $H(s) = h$. This is shown in the Fig. 8.6.1.

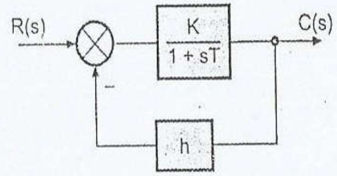


Fig. 8.6.1

Let us obtain the response of this closed loop system for unit step input. The overall transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{1+sT}}{1+\dfrac{K}{1+sT}\cdot h}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{K}{1+sT+Kh} = \frac{K}{1+Kh+sT} = \frac{K/T}{\left[s+\left(\dfrac{1+Kh}{T}\right)\right]}$$

$$R(s) = \frac{1}{s}$$

$$\therefore \quad C(s) = \frac{K/T}{s\left[s+\left(\dfrac{1+Kh}{T}\right)\right]}$$

$$\therefore \quad c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{K/T}{s\left[s+\left(\dfrac{1+Kh}{T}\right)\right]}\right] = L^{-1}\left[\frac{(K/1+Kh)}{s} - \frac{(K/1+Kh)}{s+\left(\dfrac{1+Kh}{T}\right)}\right]$$

$$= \frac{K}{1+Kh} - \frac{K}{1+Kh}\cdot e^{-\frac{t(1+Kh)}{T}} = \frac{K}{1+Kh}\left[1-e^{-\frac{t}{(T/1+Kh)}}\right]$$

Thus it can be observed that the new time constant due to the feedback is $(T/1+Kh)$. Thus for positive value of h and $K>1$, the time constant $(T/1+Kh)$ is less than T. Thus it can be concluded that the time constant of a closed loop system is less than the open loop system.

Key Point *Less the time constant faster is the response. Hence the feedback improves the time response of a system.*

8.6.2 Effect of Feedback on Overall Gain

Consider an open loop system with overall transfer function as G(s). The overall gain of such system is nothing but G(s).

If the feedback with transfer function H(s) is introduced in such a system, then its overall gain becomes $[G(s)/1 \pm G(s)H(s)]$. The positive or negative sign in the denominater gets decided by the sign of the feedback.

For a negative feedback, the gain G(s) reduces by a factor $[1/(1+G(s)H(s))]$.

Due to the negative feedback overall gain of the system reduces.

8.6.3 Effect of Feedback on Stability

It is discussed earlier that the feedback reduces the time constant and makes the system response more fast. Hence the transient response decays more quickly.

Consider an open loop system with overall transfer function as,

$$G(s) = \frac{K}{s+T}$$

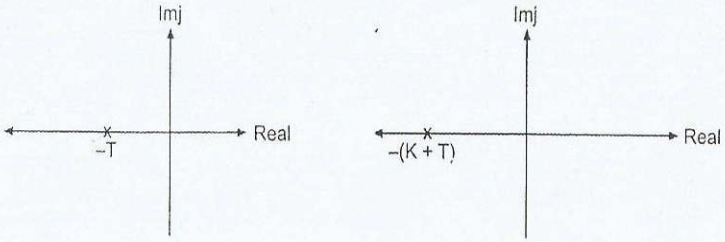
The open loop pole is located at $s=-T$.

Now let a unity negative feedback is introduced in the system. The overall transfer function of a closed loop system becomes,

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s+T}}{1+\frac{K}{s+T}} = \frac{K}{s+(K+T)}$$

Thus the closed loop pole is now located at $s=-(K+T)$. This is shown in the Fig. 8.6.2 (a) and (b).

Thus the feedback controls the time response i.e. dynamics of the system by adjusting location of its poles. The stability of a system depends on the location of poles in s-plane. Thus it can be concluded that the feedback affects the stability of the system.



(a) Open loop system (b) Closed loop system

Fig. 8.6.2

The feedback may improve the stability and also may be harmful to the system from stability point of view. The closed loop system may be unstable though the open loop system is stable.

Key Point *Thus the stability of the system can be controlled by proper design and application of the feedback.*

8.6.4 Effect of Feedback on Bandwidth

The gains of various control systems are the functions of the frequency ω. As the frequency increases, the gain decreases.

The frequency at which the gain of the system reduces by the factor $1/\sqrt{2}$ from the value of gain at d.c. condition is called cut-off frequency denoted as ω_b. And the range from d.c. ($\omega = 0$) upto $\omega = \omega_b$ is called **bandwidth** of the system.

More the bandwidth, large is the range of frequencies over which system response is accurate. It also indicates the higher speed of the control system response.

For studying the effect of feedback on the bandwidth, consider the system as shown in the Fig. 8.6.3.

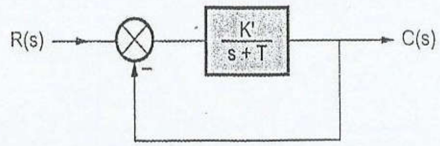


Fig. 8.6.3 Feedback control system

The open loop transfer function is $G(s) = \frac{K'}{s+T}$ while the closed loop transfer function is $T(s) = \frac{G(s)}{1+G(s)}$ as $H(s) = 1$.

Now, $$G(s) = \frac{K'}{s+T} = \frac{\left(\frac{K'}{T}\right)}{1+\frac{1}{T}s}$$

Let $$\frac{K'}{T} = K \text{ and } \frac{1}{T} = \tau = \text{Time constant}$$

$$\therefore \quad G(s) = \frac{K}{1+\tau_s}$$

i.e. $$G(j\omega) = \frac{K}{1+j\omega\tau} \quad \text{...Open loop transfer function}$$

and $$T(j\omega) = \frac{\frac{K}{1+j\omega\tau}}{1+\frac{K}{1+j\omega\tau}} = \frac{K}{1+K+j\omega\tau} = \frac{\left(\frac{K}{1+K}\right)}{1+j\left[\frac{\omega\tau}{1+K}\right]}$$

Let $$\frac{\tau}{1+K} = \tau_c$$

$$\therefore \quad T(j\omega) = \frac{\left(\frac{K}{1+K}\right)}{1+j\omega\tau_c} \quad \text{...Closed loop transfer function}$$

Now $|G(j\omega)| = \dfrac{K}{\sqrt{1+\omega^2\tau^2}}$

At $\omega = 0$, $|G(j\omega)| = K$

At $\omega = \omega_b$, $|G(j\omega)| = \dfrac{K}{\sqrt{2}}$

$$\therefore \quad \frac{K}{\sqrt{2}} = \frac{K}{\sqrt{1+\omega_b^2\tau^2}}$$

$$\therefore \quad (\sqrt{2})^2 = 1+\omega_b^2\tau^2$$

$$\therefore \quad \omega_b = \frac{1}{\tau} \qquad \text{...Bandwidth of open loop system}$$

While $|T(j\omega)| = \dfrac{\left(\dfrac{K}{1+K}\right)}{\sqrt{1+\omega^2\tau_c^2}}$

At $\omega = 0$, $|T(j\omega)| = \dfrac{K}{1+K}$

At $\omega = \omega_b$, $|T(j\omega)| = \dfrac{K}{\sqrt{2}(1+K)}$

$$\therefore \quad \frac{K}{\sqrt{2}(1+K)} = \frac{\left(\dfrac{K}{1+K}\right)}{\sqrt{1+\omega_b^2\tau_c^2}}$$

$$\therefore \quad \omega_b = \frac{1}{\tau_c} \qquad \text{...Bandwidth of closed loop system}$$

$$\therefore \quad \frac{\omega_b(OL)}{\omega_b(CL)} = \frac{\left(\dfrac{1}{\tau}\right)}{\left(\dfrac{1}{\tau_c}\right)} = \frac{\tau_c}{\tau} = \frac{\dfrac{\tau}{1+K}}{\tau} = \frac{1}{1+K}$$

$$\therefore \quad \boxed{\omega_b(CL) = (1+K)\,\omega_b(OL)}$$

Thus the bandwidth of closed loop system is (1 + K) times the bandwidth of open loop system.

Key Point *The feedback increases the bandwidth and hence increases the speed of the response of the system.*

8.6.5 Effect of Feedback on Disturbance

Every control system has some nonlinearities present in it. The dominant nonlinearities like friction, dead zone, saturation etc. affect the output of the system adversely. Some external disturbance signals also make the system output inaccurate. The examples of such external disturbances are high frequency noise in electronic applications, thermal noise in amplifier tubes, wind gusts on antenna of radar systems etc. The disturbance may be in the forward path, feedback path or output of a system.

8.6.5.1 Disturbance in the Forward Path

Let us assume that there is a disturbance in the forward path of a control system produced due to varying properties of forward path elements or due to effect of surrounding conditions. Fig. 8.6.4 shows the disturbance signal $T_d(s)$ produced in the forward path.

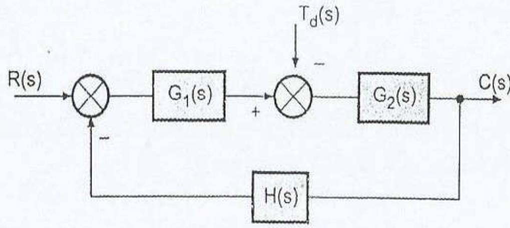


Fig. 8.6.4

Assuming R(s) to be zero, let us obtain the ratio $C(s)/T_d(s)$ to study the effect of disturbance on output. With $R(s) = 0$, system becomes.

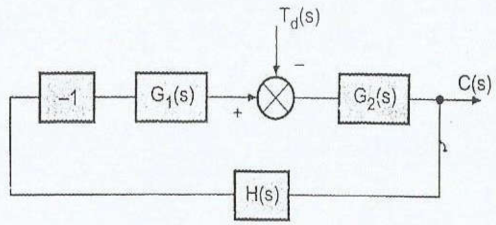


Fig. 8.6.5

The resultant elements are,

$$G(s) = G_2(s)$$

$$H'(s) = -G_1(s)H(s)$$

Positive feedback

Negative input

$$\therefore \quad \frac{C(s)}{-T_d(s)} = \frac{G_2(s)}{1-[G_2(s)(-G_1(s)H(s))]}$$

$$\therefore \quad \frac{C(s)}{T_d(s)} = \frac{-G_2(s)}{1+G_1G_2H(s)}$$

$$\therefore \quad C(s) = \frac{-T_d(s)\,G_2}{1+G_1G_2H}$$

In the denominator assume that $1 << G_1G_2H$ hence we get,

$$C(s) = \frac{-T_d(s)}{G_1H(s)}$$

Thus to make the effect of disturbance on the output as small as possible, the $G_1(s)$ must be selected as large as possible.

8.6.5.2 Disturbance in the Feedback Path

These are produced due to the nonlinear behaviour of the feedback path elements. The Fig. 8.6.6 shows the disturbance signal $T_d(s)$ produced in the feedback path.

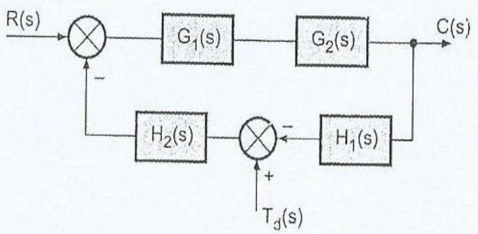


Fig. 8.6.6

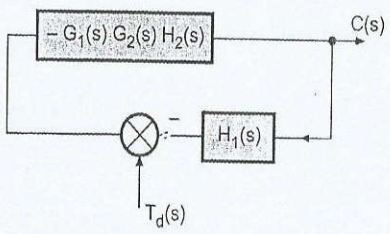


Fig. 8.6.6 (a)

With $R(s) = 0$, the effect of $T_d(s)$ on output can be obtained.

The system becomes,

$$\therefore \quad \frac{C(s)}{T_d(s)} = \frac{-G_1G_2H_2}{1+G_1G_2H_1H_2}$$

For large values of G_1, G_2, H_1, H_2, in the denominator 1 can be neglected.

$$\therefore \quad \frac{C(s)}{T_d(s)} = -\frac{1}{H_1(s)}$$

Thus designing proper feedback element $H_1(s)$, the effect of disturbance in feedback path on output can be reduced.

8.6.5.3 Disturbance at the Output

Consider that there is disturbance $T_d(s)$ affecting the output directly as shown in the Fig. 8.6.7.

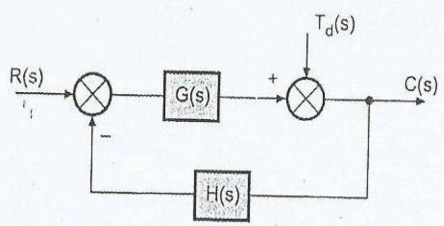


Fig. 8.6.7

With $R(s) = 0$, we get

$$\therefore \quad \frac{C(s)}{T_d(s)} = \frac{1}{1-[-G(s)H(s)]} = \frac{1}{1+G(s)H(s)}$$

For large values of $G(s)H(s)$, 1 in denominator can be neglected.

$$\therefore \quad C(s) = \frac{T_d(s)}{G(s)H(s)}$$

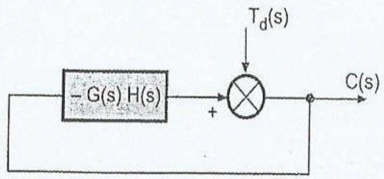


Fig. 8.6.8

Thus if disturbance is affecting the output directly then by changing the values of G(s), H(s) or both the effect of disturbance can be minimised.

The feedback minimizes the effect of disturbance signals occurring in the control system.

8.6.6 Advantages of Feedback

Thus the various advantages of feedback can be summarized as :

1. It minimizes the effect of disturbances.
2. It reduces the sensitivity of the system to parameter variations.
3. It reduces the effect of noise on the system performance.
4. It reduces the steady state error of the system.
5. It reduces the time constant of the system making the system response fast.
6. The transient response of the system can be easily controlled as per the requirements.
7. It increases the bandwidth of the system and improves the frequency response of the system.

The only limitation is that it reduces the overall gain of the system. While the system must be designed properly from its stability point of view when the feedback is used.

Review Questions

1. *Discuss the effect of feedback on : i) Time constant ii) Overall gain iii) Stability iv) Bandwidth v) Disturbance.*
2. *State the various advantages and disadvantages of feedback.*

8.7 Introduction to Time Response of Control Systems

Time response of a control system means, how output behaves with respect to time. So it can be defined as below.

Definition : Time response : *The response given by the system which is function of the time, to the applied excitation is called time response of a control system.*

In any practical system, output of the system takes some finite time to reach to its final value. This time varies from system to system and is dependent on different factors. Similarly final value achieved by the output also depends on the different factors like friction, mass or inertia of moving elements, some nonlinearities present etc.

For example consider a simple ammeter as a system. It is connected in a system so as to measure current of magnitude 5 A. Ammeter pointer hence must deflect to show us 5 A reading on it. So 5 A is its ideal value that it must show. Now pointer will take some finite time to stabilise to indicate some reading and after stabilising also, it depends on various factors like friction, pointer inertia etc. whether it will show us accurate 5 A or not.

Based on this example, we can classify the total output response into two parts. First is the part of output during the time, it takes to reach to its final value. And second is the final value attained by the output which will be near to its desired value if system is stable and accurate.

This can be further explained by considering another practical example. Suppose we want to travel from city A to city B. So our final desired position is city B. But it will take some finite time to reach to city B. Now this time depends on whether we travel by a bus or a train or a plane. Similarly whether we will reach to city B or not depends on number of factors like vehicle condition, road condition, weather condition etc. So in short we can classify the output as,

i) Where to reach ?

ii) How to reach ?

Successfulness and accuracy of system depends on the final value reached by the system output which should be very close to what is desired from that system. While reaching to its final value, in the mean time, output should behave smoothly.

Thus final state achieved by the output is called **steady state** while output variations within the time it takes to achieve the steady state is called **transient response** of the system.

Definition : Transient response :

*The output variation during the time, it takes to achieve its final value is called as **transient response**. The time required to achieve the final value is called **transient period**.*

This can also be defined as that part of the time response which decays to zero after some time as system output reaches to its final value.

Key Point *The transient response may be exponential or oscillatory in nature. Symbolically it is denoted as $c_t(t)$.*

To get the desired output, system must pass satisfactorily through transient period. Transient response must vanish after some time to get the final value closer to the desired value. Such systems in which transient response dies out after some time are called **Stable Systems.**

Mathematically for **stable** operating systems,

$$\lim_{t \to \infty} c_t(t) = 0$$

From transient response we can get following information about the system,

i) When the system has started showing its response to the applied excitation ?

ii) What is the rate of rise of output. From this, parameters of system can be designed which can withstand such rate of rise. It also gives indication about speed of the system.

iii) Whether output is increasing exponentially or it is oscillating.

iv) If output is oscillating, whether it is over shootting its final value.

v) When it is settling down to its final value ?

All this information matters much at the time of designing the systems.

Definition : Steady state response :

It is that part of the time response which remains after complete transient response vanishes from the system output.

This also can be defined as **response of the system as time approaches infinity from the time at which transient response completely dies out.** The steady state response is generally the final value achieved by the system output. Its significance is that it tells us how far away the actual output is from its desired value.

Key Point *The steady state response indicates the accuracy of the system. The symbol for steady state output is C_{ss}.*

From steady state response we can get following information about the system :

i) How much away the system output is from its desired value which indicates error.

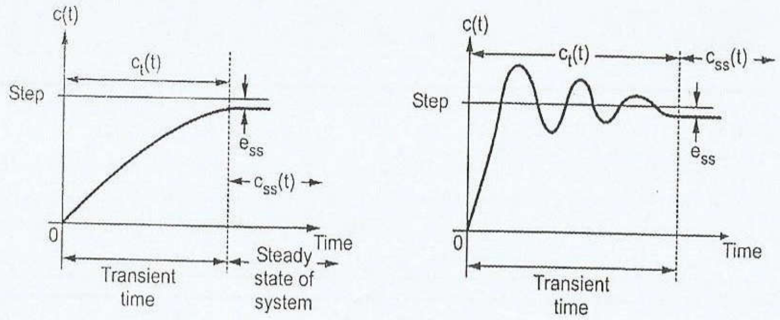
ii) Whether this error is constant or varying with time. So the entire information about system performance can be obtained from transient and steady state response.

Hence total time response c(t) we can write as,

$$c(t) = C_{ss} + c_t(t)$$

The difference between the desired output and the actual output of the system is called steady state error which is denoted as e_{ss}. This error indicates the accuracy and plays an important role in designing the system.

The above definitions can be shown in the waveform as in the Fig. 8.7.1 (a), (b) where input applied to the system is step type of input.



(a) $c_t(t)$ is exponential (b) $c_t(t)$ is oscillatory

Fig. 8.7.1

Review Questions

1. *What is the difference between steady state response and transient response of a control system ?*
2. *Define steady state response and steady state error.*

8.8 Derivation of Steady State Error

GTU : May-14

Consider a simple closed loop system using negative feedback as shown in the Fig. 8.8.1.

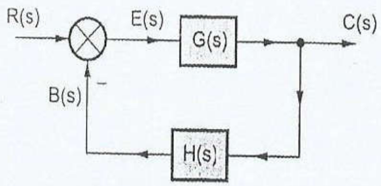


Fig. 8.8.1

where $E(s)$ = Error signal, and $B(s)$ = Feedback signal

Now, $E(s) = R(s) - B(s)$

But $B(s) = C(s)H(s)$

$\therefore$ $$E(s) = R(s) - C(s)H(s)$$

and $$C(s) = E(s)G(s)$$

$\therefore$ $$E(s) = R(s) - E(s)\,G(s)H(s)$$

$\therefore$ $$E(s) + E(s)G(s)H(s) = R(s)$$

$\therefore$ $$E(s) = \frac{R(s)}{1+G(s)H(s)} \quad \text{for nonunity feedback}$$

$$E(s) = \frac{R(s)}{1+G(s)} \quad \text{for unity feedback}$$

This E(s) is the error in Laplace domain and is expression in 's'. We want to calculate the error value. In time domain, corresponding error will be e(t). Now steady state of the system is that state which remains as $t \to \infty$.

$\therefore$ $$\text{Steady state error, } e_{ss} = \lim_{t \to \infty} e(t)$$

Now we can relate this in Laplace domain by using **final value theorem** which states that,

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} sF(s) \qquad \text{where } F(s) = L\{ F(t) \}$$

Therefore, $$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \quad \text{where } E(s) \text{ is } L\{ e(t) \}.$$

Substituting E(s) from the expression derived, we can write

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)}$$

For negative feedback systems use positive sign in denominator while use negative sign in denominator if system uses positive feedback.

From the above expression it can be concluded that **steady state error depends on,**

i) R(s) i.e. reference input, its type and magnitude.

ii) G(s)H(s) i.e. open loop transfer function.

iii) Dominant nonlinearities present if any.

Now we will study the effect of change in input and product G(s)H(s) on the value of steady state error. As transfer function approach is applicable to only linear systems, the effect of nonlinearities is not discussed.

Example 8.8.1 *The open loop transfer function of a thermal control system is given by,* $G(s) = \frac{20}{s+5}$. *It uses unity feedback. Find its steady state error if it is excited by unit step input.*

Solution : $G(s)=\left(\frac{20}{s+5}\right), H(s)=1, r(t)=1 \quad \text{i.e.} \quad R(s)=\frac{1}{s}$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\times\frac{1}{s}}{1+\frac{20}{s+5}}$$

$$= \lim_{s\to 0} \frac{s+5}{s+25} = 0.2$$

Review Question

1. *Derive an expression to find steady state error of a closed loop control system.* **GTU : May-14, Marks 4**

8.9 The Cost of Feedback

Inspite of the advantages of feedback, there are certain issues related to the use of feedback. The cost of feedback includes,

1. The feedback makes the system design complicated by increasing number of components in the system. The sensors required for the feedback are costly and introduce noise and inaccuracies.
2. Due to feedback, the overall gain of the system reduces.
3. Due to feedback, the system stability decreases The stable open loop system may become unstable when used as closed loop system. Hence while designing feedback control system, the stability analysis plays an important role.

But if the feedback system is designed properly then its advantages are greater than disadvantages and hence the feedback is always preferred in control systems.

Review Question

1. *Write a note on the cost of feedback used in control systems.*

□□□

9 The Performance of Feedback Control Systems

Syllabus

Test input signals, Performance of second-order systems, Effects of a third pole and a zero on the second-order system response, The s-plane root location and the transient response, The steady-state error of feedback control systems, Performance indices, The simplification of linear systems.

Contents

Example 13.14.7 *Test the stability of the unity feedback system* $G(s) = \dfrac{Ks}{(s-1)^2 (s+5)}$ *when* $K = 10$ *using Nyquist criterion and then find the range of K for stability.*

AU : April-05, Marks 10

[Ans. : $10.667 < K < \infty$, System is unstable]

13.15 Advantages of Nyquist Plot

1) It gives same information about absolute stability as provided by Routh's criterion.
2) Useful for determining the stability of the closed loop system from open loop transfer function without knowing the roots of characteristic equation.
3) It also indicates relative stability giving the values of G.M. and P.M.
4) It indicates reality, the manner in which system should be compensated to yield desired response.
5) Information regarding frequency response can be obtained.
6) Very useful for analyzing conditionally stable systems.

Review Question

1. *State the advantages of nyquist plot.*

□□□

14 The Design of Feedback Control Systems

Syllabus

Approaches to system design, Cascade compensation networks, Phase-lead design using the Bode diagram, Phase-lead design using the root locus, System design using integration networks, Phase-lag design using the root locus, Phase-lag design using the Bode diagram, Design on the Bode diagram using analytical methods.

Contents

14.1 Introduction

AU : May-07, 09

All the control systems are designed to achieve specific objectives. The certain requirements are defined for the control system. A good control system has less error, good accuracy, good speed of response, good relative stability, good damping which will not cause undue overshoots etc. For satisfactory performance of the system, gain is adjusted first. In practice, adjustment of gain alone can not provide satisfactory results. This is because when gain is increased, steady state behaviour of the system improves but results into poor transient response, in some cases may even instability. In such cases it is necessary to redesign the entire system. Thus the design of control systems is a challenging job. Practically the design specifications are provided interms of precise numerical values according to which the system is designed. The set of such specifications include peak overshoot, peak time, damping ratio, natural frequency of oscillations, error coefficients, gain margin, phase margin etc.

Key Point *In practice, if a system is to be redesigned so as to meet the required specifications, it is necessary to alter the system by adding an external device to it. Such a redesign or alteration of system using an additional suitable device is called* ***compensation of a control system.***

While an external device which is used to alter the behaviour of the system so as to achieve given specifications is called **compensator**. The compensator provides whatever is missing in a system, so as to achieve required performance.

14.2 Types of Compensation

An external device, compensator can be introduced in a system anywhere as per the convenience and the requirement. Depending upon where the compensator is introduced in a system, the various types of compensation are,

1. Series compensation
2. Parallel compensation
3. Series-parallel compensation

14.2.1 Series Compensation

The compensator is a physical device whose transfer function is denoted as $G_c(s)$. If the compensator is placed in series with the forward path transfer function of the plant, the scheme is called **series compensation.** The arrangement is shown in the Fig. 14.2.1.

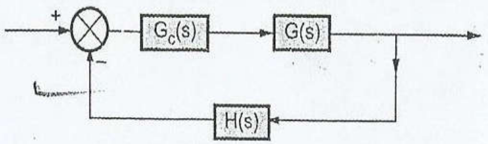


Fig. 14.2.1 Series compensation

This scheme is also called cascade compensation. The flow of signal in such a series scheme is from lower energy level towards higher energy level. This requires additional amplifiers to increase the gain and also to provide necessary isolation. The number of components required in series scheme is more than in parallel scheme.

14.2.2 Parallel Compensation

In some cases, the feedback is taken from some internal element and compensator is introduced in such a feedback path to provide an additional internal feedback loop. Such compensation is called **feedback compensation** or **parallel compensation.** The arrangement is shown in the Fig. 14.2.2.

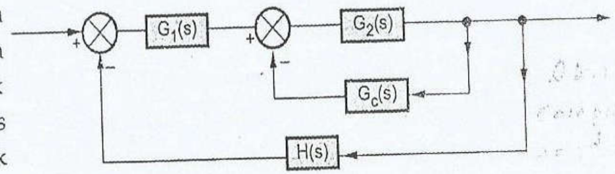


Fig. 14.2.2 Parallel compensation

The energy transfer in such parallel scheme is from higher energy level towards lower energy level point. Hence in such scheme the additional amplifiers are not required. Thus the number of components required are less than required in the series scheme.

14.2.3 Series-Parallel Compensation

In some cases, it is necessary to provide both types of compensations, series as well as feedback. Such a scheme is called **series-parallel compensation.** The arrangement is shown in the Fig. 14.2.3.

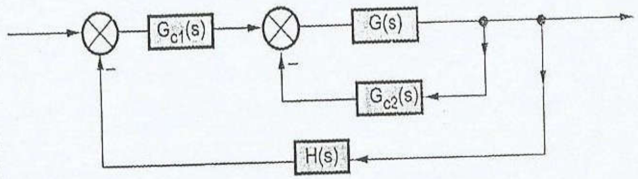


Fig. 14.2.3 Series-parallel compensation

The selection of the proper compensation scheme depends on the nature of the signals available in the system, the power levels at the various points, available components, the economic considerations, and the designer's experience.

Review Question

1. *What is compensation ? What is compensator ? Which are the various compensation schemes used in practice ?*

14.3 Compensating Networks

The compensator is a physical device. It may be an electrical network, mechanical unit, pneumatic, hydraulic or combinations of various types of devices. In this chapter we are going to study the electrical networks which are used for series compensation.

The commonly used electrical compensating networks are,

1. Lead network or Lead compensator
2. Lag network or Lag compensator
3. Lag-lead network or Lag-lead compensator

When a sinusoidal input is applied to a network and it produces a sinusoidal steady state output having a phase lead with respect to input then the network is called **lead network**. If the steady state output has phase lag then the network is called **lag network**. In the lag-lead network both phase lag and lead occur but in the different frequency regions.

Key Point *The phase lag occurs in the low frequency region while the phase lead occurs in the high frequency region.*

Let us discuss in detail the characteristics of these three compensating networks.

Review Question

1. *Which are the important electrical networks used practically for the compensation of the control systems ?*

14.4 Lead Compensator

Consider an electrical network which is a lead compensating network, as shown in the Fig. 14.4.1.

Let us obtain the transfer function of such an electrical lead network. Assuming unloaded circuit and applying KCL for the output node we can write,

$$I_1 + I_2 = I$$

$$C\frac{d(e_i - e_o)}{dt} + \frac{1}{R_1}(e_i - e_o) = \frac{1}{R_2} e_o$$

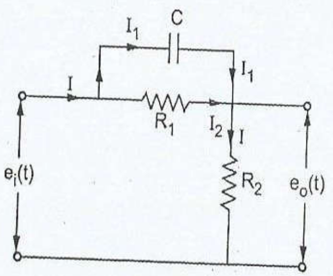


Fig. 14.4.1 Lead network

$i = C\cdot\frac{dv}{dt}$ or $V = \frac{1}{C}\int i\,dt$

lead → for high freq.
lag → for low freq.

Taking Laplace transform of the equation,

$$sC\,E_i(s) - sC\,E_o(s) + \frac{1}{R_1}E_i(s) - \frac{1}{R_1}E_o(s) = \frac{1}{R_2}E_o(s)$$

$$\therefore E_i(s)\left[sC + \frac{1}{R_1}\right] = E_o(s)\left[sC + \frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{R_1 R_2}{R_1 + R_2 + R_1 R_2 sC}\cdot\frac{1 + sCR_1}{R_1}$$

$$\therefore \frac{E_o(s)}{E_i(s)} = \frac{\left(s + \frac{1}{R_1 C}\right)}{s + \frac{(R_1 + R_2)}{R_1 R_2 C}} = \frac{\left(s + \frac{1}{R_1 C}\right)}{\left[s + \frac{1}{\left(\frac{R_2}{R_1 + R_2}\right)R_1 C}\right]}$$

This is generally expressed as,

$$\frac{E_o(s)}{E_i(s)} = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

where $T = R_1 C$ and $\alpha = \frac{R_2}{R_1 + R_2} < 1$

The lead compensator has zero at $s = -\frac{1}{T}$ and a pole at $s = -\frac{1}{\alpha T}$.

As $0 < \alpha < 1$, the zero is always located to the right of the pole. The pole zero plot is shown in the Fig. 14.4.2. The minimum value of α is generally taken as 0.05.

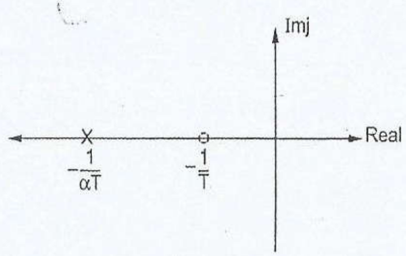


Fig. 14.4.2

14.4.1 Maximum Lead Angle ϕ_m and α

Let us see what is the maximum lead angle ϕ_m which lead compensator can provide and at what frequency it provides this angle.

$$\frac{E_o(s)}{E_i(s)} = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = \frac{\alpha(1 + Ts)}{(1 + \alpha Ts)}$$

Replace s by jω,

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{\alpha(1+j\omega T)}{(1+j\omega\alpha T)}$$

$$\therefore \quad \left|\frac{E_o(j\omega)}{E_i(j\omega)}\right| = M = \frac{\alpha\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2\alpha^2T^2}} \quad \text{...(14.4.1)}$$

While the phase angle is given by,

$$\therefore \quad \phi = \tan^{-1}\omega T - \tan^{-1}\omega\alpha T \quad \text{...(14.4.2)}$$

Let us find ω_m for which the angle ϕ is at its maximum.

$$\therefore \quad \frac{d\phi}{d\omega} = 0$$

$$\therefore \quad \frac{d}{d\omega}[\tan^{-1}\omega T - \tan^{-1}\omega\alpha T] = 0 \quad \text{i.e.} \quad \frac{\frac{1}{T}}{\omega^2+\left(\frac{1}{T}\right)^2} - \frac{\left(\frac{1}{\alpha T}\right)}{\omega^2+\left(\frac{1}{\alpha T}\right)^2} = 0$$

$$\therefore \quad \frac{T}{1+\omega^2T^2} - \frac{\alpha T}{1+\alpha^2\omega^2T^2} = 0 \quad \text{i.e.} \quad T(1+\alpha^2\omega^2T^2) - \alpha T(1+\omega^2T^2) = 0$$

$$\therefore \quad (1+\alpha^2\omega^2T^2 - \alpha - \alpha\omega^2T^2 = 0 \quad \text{i.e.} \quad \omega^2\alpha T^2(\alpha-1) + (1-\alpha) = 0$$

$$\therefore \omega^2\alpha T^2(\alpha-1) = -(1-\alpha) \quad \text{i.e.} \quad \omega^2\alpha T^2 = 1$$

$$\omega^2 = \frac{1}{\alpha T^2}$$

$$\omega_m = \frac{1}{T\sqrt{\alpha}} = \sqrt{\frac{1}{T}\cdot\frac{1}{\alpha T}} \quad \text{...(14.4.3)}$$

This is the frequency at which phase lead is at its maximum. From equation (14.4.3) we can say that the ω_m is the geometric mean of the two corner frequencies of the compensator which are $\omega_{C1} = \frac{1}{T}$ and $\omega_{C2} = \frac{1}{\alpha T}$.

Taking tan of both sides of equation (14.4.2) we get,

$$\tan\phi = \tan[\tan^{-1}\omega T - \tan^{-1}\alpha\omega T]$$

$$\therefore \quad \tan\phi = \frac{\omega T - \alpha\omega T}{1+\omega T\cdot\alpha\omega T} = \frac{\omega T(1-\alpha)}{1+\omega^2T^2\alpha}$$

At $\quad \omega = \omega_m = \frac{1}{T\sqrt{\alpha}}$,

$$\tan\phi_m = \frac{(1-\alpha)}{\sqrt{\alpha}(1+1)} = \frac{1-\alpha}{2\sqrt{\alpha}} \quad \text{... (14.4.4)}$$

$$\therefore \quad \sin\phi_m = \frac{1-\alpha}{1+\alpha} \quad \text{...(14.4.5)}$$

The equation (14.4.5) is also used to get the relation between α and the maximum lead angle ϕ_m.

14.4.2 Polar Plot of Lead Compensator

From equation (14.4.1) and (14.4.2) it is easy to obtain polar plot of lead compensator.

When $\quad \omega = 0, \quad M = \alpha \quad$ and $\quad \phi = 0°$

When $\quad \omega = \infty, \quad M = 1 \quad$ and $\quad \phi = 0°$

So both the points, starting as well as terminating points are on positive real axis. For any value of ω between 0 to ∞, magnitude is always positive while for $\alpha < 1$, $\tan^{-1}\omega T > \tan^{-1}\alpha\omega T$ hence ϕ will be always positive. Hence all the points of polar plot are always in the first quadrant of complex plane.

Thus the polar plot is as shown in the Fig. 14.4.3.

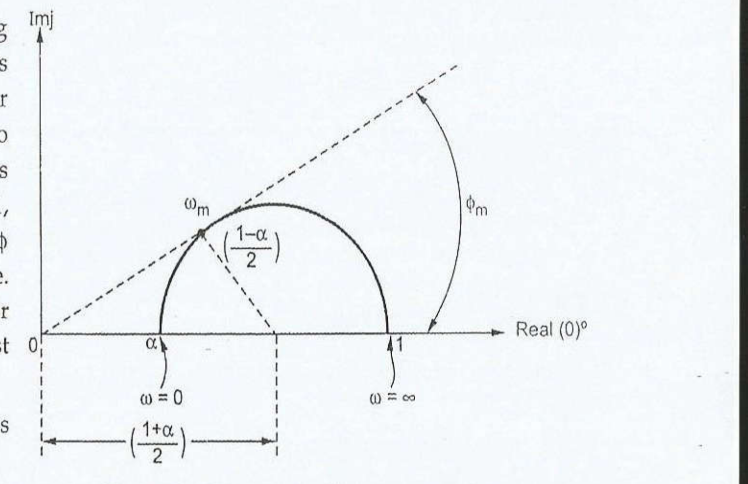


Fig. 14.4.3 Polar plot of lead compensator

14.4.3 Bode Plot of Lead Compensator

The corner frequencies of the lead compensators are,

$$\omega_{C1} = \frac{1}{T} \text{ for a zero at } s = -\frac{1}{T}$$

and $$\omega_{C2} = \frac{1}{\alpha T} \text{ for a pole at } s = -\frac{1}{\alpha T}$$

and $$K = \alpha$$

So a line of $20 \log \alpha$ dB with zero slope till $\omega_{C1} = \frac{1}{T}$. Then a line of slope + 20 dB/dec till $\omega_{C2} = \frac{1}{\alpha T}$. And again a line of zero slope from ω_{C2} to ∞. This is the nature of the resultant magnitude plot. As $\alpha < 1$, the straight line $20 \text{ Log } \alpha$ is below 0 dB.

While angle contribution increases initially, achieves maximum ϕ_m at ω_m and then again decreases.

The Bode plot is shown in the Fig. 14.4.4.

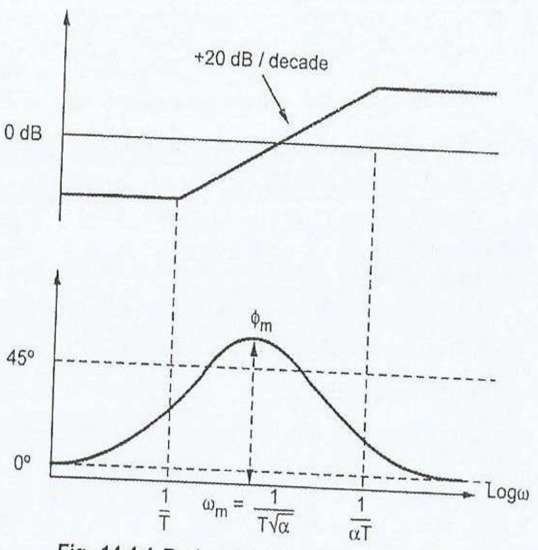


Fig. 14.4.4 Bode plot of lead compensator

It can be noted that at $\omega = \omega_m$,

$$M = \frac{\alpha\sqrt{1+\omega_m^2 T^2}}{\sqrt{1+\omega_m^2 \alpha^2 T^2}} = \frac{\alpha\sqrt{1+\frac{1}{\alpha}}}{\sqrt{1+\alpha}}$$

$$\therefore \quad \boxed{M = \sqrt{\alpha} \quad \text{at } \omega = \omega_m}$$

Thus at $\omega = \omega_m$ the magnitude in dB is,

$$= 20 \text{ Log } \sqrt{\alpha} = 20 \text{ Log } (\alpha)^{\frac{1}{2}} = 10 \text{ Log } \alpha$$

$$\boxed{M = -10 \text{ Log}\left(\frac{1}{\alpha}\right) \text{ dB at } \omega = \omega_m} \qquad \ldots(14.4.6)$$

14.4.4 Steps to Design Lead Compensator

Step 1 : At zero frequency the lead compensator has gain α. But as $\alpha < 1$ it provides an attenuation. To cancel this attenuation, the practical lead compensator is realised with an amplifier having gain K_c in series with basic lead network. Hence the practical transfer function of a lead compensator from the design point of view is assumed to be,

$$G_c(s) = K_c \alpha \frac{(1+Ts)}{(1+\alpha Ts)}$$

where $\quad K_c \alpha = \text{d.c. gain} = K \qquad \ldots(14.4.7)$

$$\therefore \quad \boxed{G_c(s) = \frac{K(1+Ts)}{(1+\alpha Ts)}} \qquad \ldots(14.4.8)$$

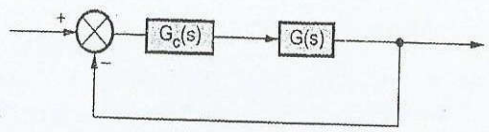


Fig. 14.4.5

The open loop transfer function of the compensated system thus becomes,

$$G_c(s)\, G(s) = \frac{K(1+Ts)}{(1+\alpha Ts)} \cdot G(s) = \frac{(1+Ts)}{(1+\alpha Ts)} \cdot K\, G(s) = \frac{(1+Ts)}{(1+\alpha Ts)} G_1(s)$$

where $\quad \boxed{G_1(s) = KG(s)}$

Key Point *Generally in such design problems one of the error constant is given as specification.*

From the above result, determine the value of K satisfying the given error constant.

Step 2 : Using the value of K determined above, draw the Bode plot of $G_1(j\omega)$. This is the Bode plot of, gain adjusted but uncompensated system. Obtain the phase margin.

Step 3 : Generally P.M. is specified for the design problem.

Let $\quad \phi_s = \text{P.M. specified}$

$\quad \phi_1 = \text{P.M. obtained in the step 2}$

Determine necessary phase lead ϕ_m required to be added. For this use the relation,

$$\boxed{\phi_m = \phi_s - \phi_1 + \varepsilon}$$

where ε = Margin of safety as cross-over frequency may shift due to compensation

$$\therefore \quad \boxed{\varepsilon = 5° \text{ to } 15°}$$

Step 4 : Using the equation,

$$\boxed{\sin \phi_m = \frac{1-\alpha}{1+\alpha}}$$

determine the value of α.

Step 5 : Determine the frequency ω_m at which the magnitude of the uncompensated system is $-10 \text{ Log}\left(\frac{1}{\alpha}\right)$ dB. **Select this frequency as new gain crossover frequency.** This frequency ω_m is,

$$\boxed{\omega_m = \frac{1}{T\sqrt{\alpha}}}$$

as α is known, determine $\frac{1}{T}$.

Step 6 : Determine the two frequencies of the lead compensator.

$$\omega_{C1} = \frac{1}{T} \text{ and}$$
$$\omega_{C2} = \frac{1}{\alpha T}$$

Step 7 : As $K = K_c \alpha$,

determine the value of K_c.

Step 8 : Check the gain margin of the compensated system. If it is not satisfactory repeat the design by modifying the pole-zero location of the compensator till a satisfactory result is obtained.

14.4.5 Effects of Lead Compensation

The various effects of a lead compensation are,

1. The lead compensator adds a dominant zero and a pole. This increases the damping of the closed loop system.
2. The increased damping means less overshoot, less rise time and less settling time. Thus there is improvement in the transient response.
3. It improves the phase margin of the closed loop system.
4. The slope of the magnitude plot in Bode diagram of the forward path transfer function is reduced at the gain cross over frequency. This improves gain and phase margins improving the relative stability.
5. It increases bandwidth of the closed loop system. More the bandwidth, faster is the response.
6. The steady state error does not get affected.

14.4.6 Limitations of Lead Compensation

The various limitations of the lead compensation are,

1. Lead compensation requires an additional increase in gain to offset the attenuation inherent in the lead network. Larger gain requirement means in most of the cases implies larger space, more elements, greater weight and higher cost.
2. More bandwidth is sometimes not desirable. This is because the noise entering the system at the input may become objectionable due to large bandwidth. It makes system more susceptible to the noise signals because of increase in the high frequency gain.

3. The compensated system may have a larger undershoot than overshoot. So tendency to over compensate system may lead to a conditionally stable system.
4. The maximum lead angle available from a single lead network is about 60°. Thus if lead of more than 70° to 90° is required a multistage lead compensators are required.

Example 14.4.1 *For the system shown in the Fig. 14.4.6, design a lead compensator such that the closed loop system will satisfy the following specifications :*

Static velocity error constant = 20 sec^{-1}

Phase margin = 50°

Gain margin ≥ 10 dB

AU : Dec.-11

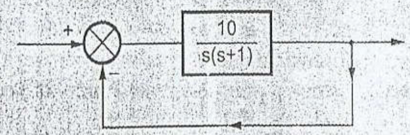


Fig. 14.4.6

Solution : Step 1 : Assume a lead compensator as,

$$G_c(s) = K_c \alpha \frac{(1+Ts)}{(1+\alpha Ts)} = \frac{K(1+Ts)}{(1+\alpha Ts)}$$

and $$G_1(s) = KG(s) = \frac{10K}{s(s+1)}$$

$$K_v = 20 = \lim_{s \to 0} s\, G(s)H(s) = \lim_{s \to 0} s \cdot \frac{10K(1+Ts)}{s(s+1)(1+\alpha Ts)}$$

$\therefore$ $20 = 10\,K$ i.e. $K = 2$

$\therefore$ $$G_1(s) = \frac{20}{s(s+1)}$$

Step 2 : Sketch the Bode plot of $G_1(s)$ which is shown in the Fig. 14.4.7.

Factors : 20 Log 20 = 26 dB

1 pole at origin

1 simple pole with corer frequency $\omega_C = 1$.

Thus line of slope −20 dB/dec till $\omega_C = 1$ and line of slope −40 dB/dec from 1 onwards.

Phase angle table : $G_1(j\omega) = \frac{20}{j\omega(1+j\omega)}$

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	ϕ_R
0.1	− 90°	− 5.71°	− 95.71°
1	− 90°	− 45°	− 135°
2	− 90°	− 63.4°	− 153.4°

10	– 90°	– 84.2°	– 174.2°
∞	– 90°	– 90°	– 180°

From the Fig. 14.4.7,

$$\phi_1 = \text{P.M.} = 15^\circ, \quad \omega_{gc} = 4 \text{ rad/sec}, \quad \text{G.M.} = +\infty \text{ dB}$$

Step 3 : $\phi_s = 50^\circ$

$$\therefore \quad \phi_m = \phi_s - \phi_1 + \varepsilon, \quad \text{let } \varepsilon = 5^\circ$$

$$= 50^\circ - 15^\circ + 5^\circ = 40^\circ$$

Step 4 : $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha}$ i.e. $\sin 40^\circ = \dfrac{1-\alpha}{1+\alpha} = 0.6427$

$$\therefore \quad 1-\alpha = 0.6427\,(1+\alpha) \quad \text{i.e.} \quad \alpha = 0.2174$$

Choose $\alpha = 0.21$

Step 5 : $-10 \text{ Log}\left(\dfrac{1}{\alpha}\right) = -6.78 \text{ dB for } \alpha = 0.21$

Refer Fig. 14.4.7 and find frequency at which gain of the uncompensated system is – 6.78 dB, this is ω_m.

From Fig. 14.4.7, $\omega_m = 6$ rad/sec

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \text{i.e.} \quad \frac{1}{T} = 2.7495$$

Step 6 : Two corner frequencies of the lead compensator are,

$$\omega_{C1} = \frac{1}{T} = 2.7495 \quad \text{and} \quad \omega_{C2} = \frac{1}{\alpha T} = 13.09$$

Step 7 : $K = K_c\,\alpha$

$$\therefore \quad K_c = \frac{K}{\alpha} = \frac{2}{0.21} = 9.523$$

Step 8 : $G_c(s) = 9.523 \times 0.21 \dfrac{(1+0.3637\,s)}{(1+0.0763\,s)} = \dfrac{2(1+0.3637\,s)}{(1+0.0763\,s)}$

This is the designed lead compensator.

$$\therefore \quad G_c(s)\,G(s) = \frac{20\,(1+0.3637\,s)}{s(1+s)\,(1+0.0763\,s)}$$

Draw the Bode plot for this transfer function and obtain the values of G.M. and P.M. The plot is drawn on the same semilog paper shown in the Fig. 14.4.7.

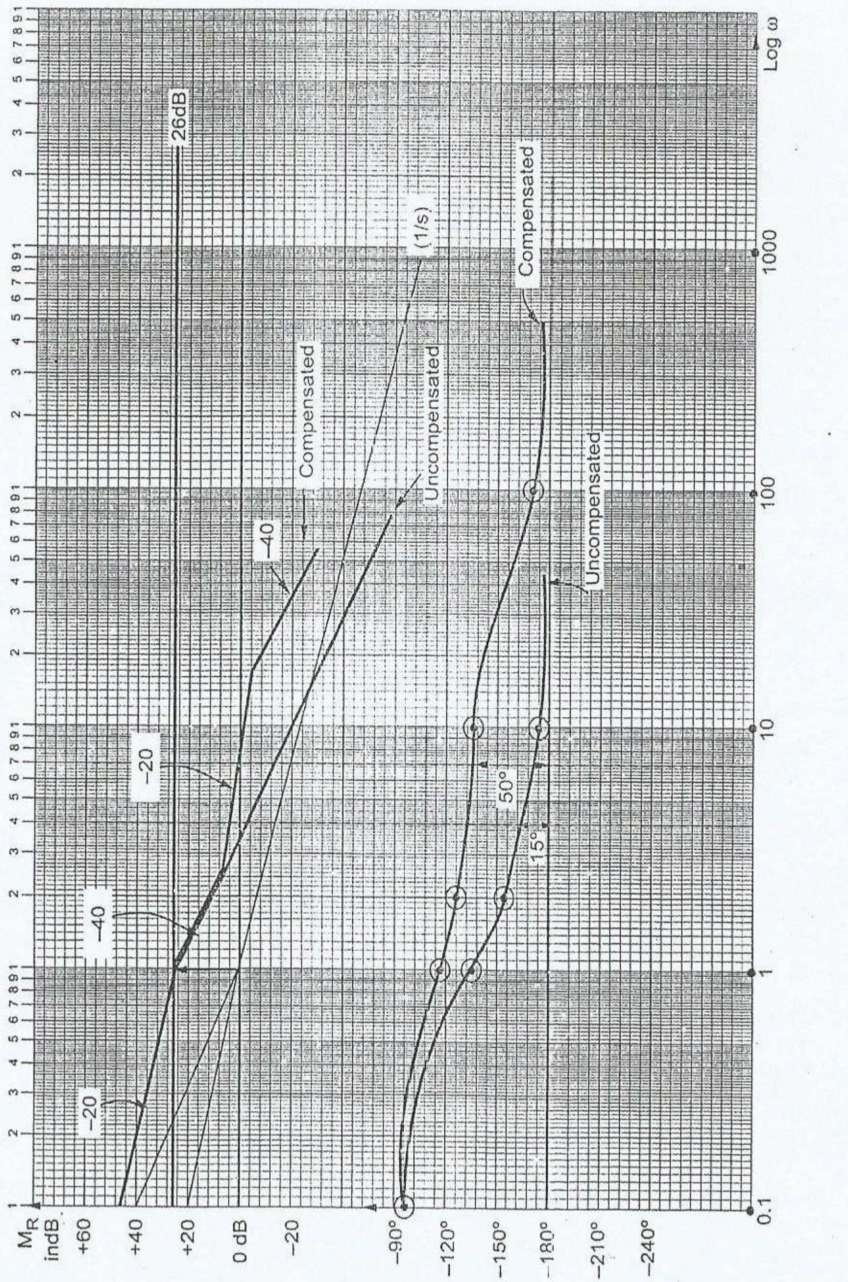


Fig. 14.4.7

Phase angle table for the compensated system :

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	$+\tan^{-1} 0.3637\,\omega$	$-\tan^{-1} 0.0763\,\omega$	ϕ
0.1	− 90°	− 5.71°	+ 2.08°	− 0.43°	− 94.06°
1	− 90°	− 45°	+ 20°	− 4.36°	− 119.36°
2	− 90°	− 63.4°	+ 36°	− 8.67°	− 126.07°
10	− 90°	− 84.2°	+ 74°	− 37.3°	− 137.5°
100	− 90°	− 89.4°	+ 88°	− 82.53°	− 173.9°

For magnitude plot, K = 20

$\therefore \quad 20 \text{ Log } 20 = 26 \text{ dB}$

One pole at origin, straight line of slope − 20 dB/dec.

$\omega_{C1} = 1$, slope becomes − 40 dB/dec due to simple pole.

$\omega_{C2} = \frac{1}{T} = 2.75$, slope becomes − 20 dB/dec due to simple zero.

$\omega_{C3} = \frac{1}{\alpha T} = 13.09$, slope becomes − 40 dB/dec due to simple pole.

From the Fig. 14.4.7, for compensated system

$$\omega_{gc} = 7 \text{ rad/sec, P.M.} = +50°, \text{ G.M.} = +\infty \text{ dB}$$

Thus the compensated system satisfies all the specifications.

Example 14.4.2 *Consider a type 1 unit feedback system with an OLTF $G_f = \frac{K}{s(s+1)}$. It is specified that $K_v = 12 \text{ sec}^{-1}$ and $\phi_{PM} = 40°$. Design lead compensator to meet the specifications.* **AU : April-04, 05, Nov.-04, 05, 06**

Solution :

Step 1 : Lead compensator $G_c(s) = K_c\,\alpha\,\frac{(1+Ts)}{(1+\alpha Ts)} = \frac{K(1+Ts)}{(1+\alpha Ts)}$

$$\therefore \quad G_c(s)G(s) = \frac{(1+Ts)}{(1+\alpha Ts)} \times \frac{K}{s(s+1)} \quad \text{where } G(s) = \frac{1}{s(s+1)}$$

$$\text{Now} \quad K_v = \lim_{s\to 0} sG_c(s)\,G(s) = \lim_{s\to 0} \frac{K}{s(s+1)}\,\frac{(1+Ts)}{(1+Ts)} = K$$

$$\therefore \quad K = 12$$

$$\therefore \quad G_1(s) = \frac{12}{s(s+1)}$$

Step 2 : Sketch the Bode plot of $G_1(s)$ i.e. uncompensated system as shown in the Fig. 14.4.8 (a). (Fig. 14.4.8(a) see on next page).

Factors : K = 12 i.e. 20 Log K = 21.58 ≈ 22 dB

1. Pole at the origin i.e. − 20 dB/dec

1 simple pole with $\omega_{C1} = 1$ i.e. − 20 dB/dec for $\omega > 1$.

So resultant starting slope − 20 dB/dec and then − 40 dB/dec for $\omega > 1$.

Phase angle table : $G_1(j\omega) = \frac{12}{j\omega(1+j\omega)}$

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	ϕ_R
0.1	− 90°	− 5.71°	− 95.71°
1	− 90°	− 45°	− 135°
2	− 90°	− 63.43°	− 153.43°
∞	− 90°	− 90°	− 180°

From the plot, ϕ_1 = P.M. = 15°, ω_{gc} = 3.5 rad/sec, G.M. = + ∞ dB

Step 3 : $\phi_s = 40°$ (given) i.e. given P.M.

$$\therefore \quad \phi_m = \phi_s - \phi_1 + \varepsilon, \text{ Let } \varepsilon = 8°$$

$$= 40° - 15° + 8° = 33°$$

Step 4 : $\sin \phi_m = \frac{1-\alpha}{1+\alpha} = \sin 33° = 0.5446$

$$\therefore \quad 1-\alpha = 0.5446\,(1+\alpha) \quad \text{i.e.} \quad \alpha = 0.2948$$

$$\text{Choose } \alpha = \mathbf{0.3}$$

Step 5 : $-10 \text{ Log}\left(\frac{1}{\alpha}\right) = -5.23$ dB

Find the frequency from the Fig. 14.4.8 (a), which gain of uncompensated system is − 5.23 dB, which is ω_m.

$$\therefore \quad \omega_m = \mathbf{5.8 \text{ rad/sec}}$$

$$\text{But} \quad \omega_m = \frac{1}{T\sqrt{\alpha}}$$

$$\therefore \quad T = 0.3147 \quad \text{i.e.} \quad \frac{1}{T} = 3.176$$

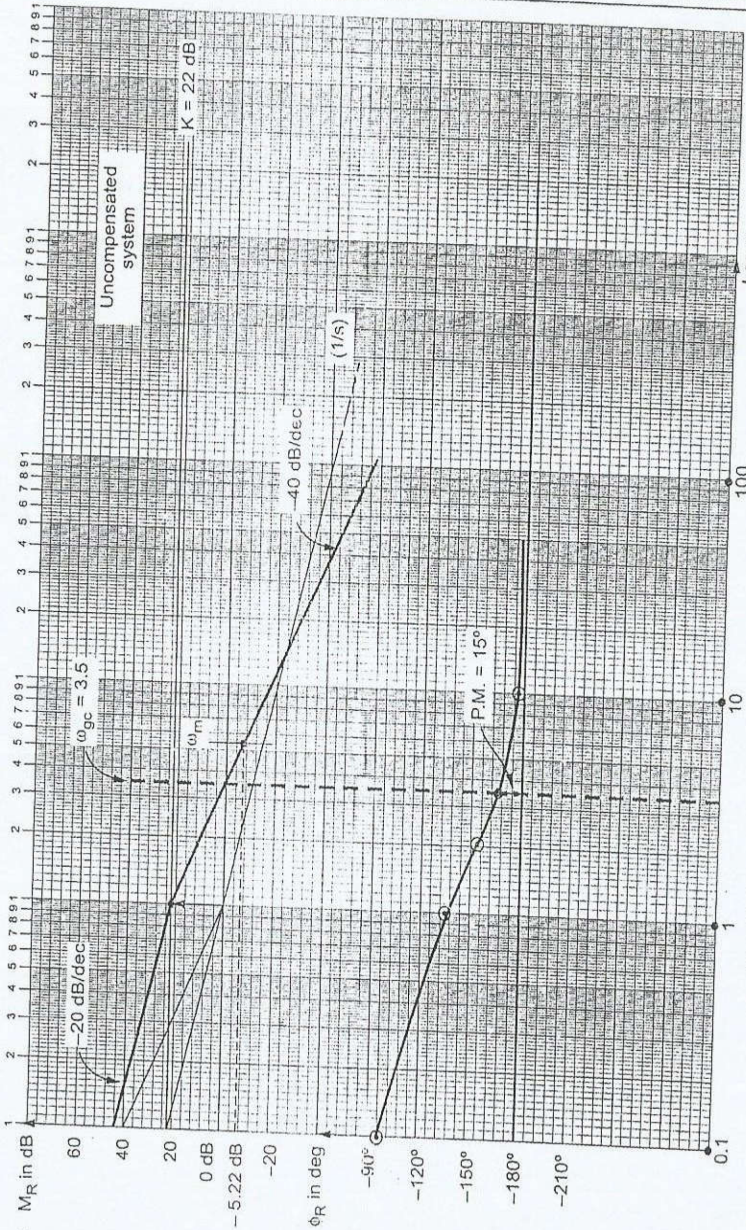


Fig. 14.4.8 (a) Bode plot of uncompensated system in Example 14.4.2

Step 6 : Two corner frequencies of lead compensator are,

$$\omega_{C1} = \frac{1}{T} = 3.176, \quad \omega_{C2} = \frac{1}{\alpha T} = \frac{1}{0.0944} = 10.6$$

Step 7 : $K = K_c \alpha$

$$\therefore \quad K_c = \frac{K}{\alpha} = \frac{12}{0.3} = 40$$

Step 8 : $G_c(s) = \dfrac{40 \times 0.3 \times (1+0.3147\ s)}{(1+0.0944\ s)}$

$$\therefore \quad G_c(s)G(s) = \frac{12\ (1+0.3147\ s)}{s\ (1+0.0944\ s)\ (1+s)}$$

Draw the Bode plot for new specifications.

Factors : $K = 12$ i.e. $20 \text{ Log } K = 22$ dB

	Resultant slopes
One pole at origin, – 20 dB/dec	– 20 dB/dec
Simple pole, $\omega_{C1} = 1$, – 20 dB/dec	– 40 dB/dec
Simple zero, $\omega_{C2} = \frac{1}{0.3147} = 3.176$, + 20 dB/dec	– 20 dB/dec
Simple pole, $\omega_{C3} = \frac{1}{0.0944} = 10.6$, – 20 dB/dec	– 40 dB/dec

Phase angle table : $G_c(j\omega)G(j\omega) = \dfrac{12\ (1+0.3147\ j\omega)}{j\omega\ (1+0.0944\ j\omega)\ (1+j\omega)}$

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	$+\tan^{-1} 0.3147\ \omega$	$-\tan^{-1} 0.0944\ \omega$	ϕ_R
0.5	– 90°	– 26.56°	+ 8.94°	– 2.702°	– 110.32°
1	– 90°	– 45°	+ 17.46°	– 5.39°	– 122.93°
3	– 90°	– 71.56°	+ 43.35°	– 15.81°	– 134.02°
5	– 90°	– 78.56°	+ 57.56°	– 25.26°	– 136.26°
12	– 90°	– 85.23°	+ 75.16°	– 48.56°	– 148.63°
∞	– 90°	– 90°	+ 90°	– 90°	– 180°

The Bode plot is shown in the Fig. 14.4.8 (b) (See on next page) from which,

ω_{gc} = **5 rad/sec** $\omega_{pc} = \infty$, G.M. = + ∞ dB, P.M. = **+ 44°**

Thus the compensated system satisfies required specifications.

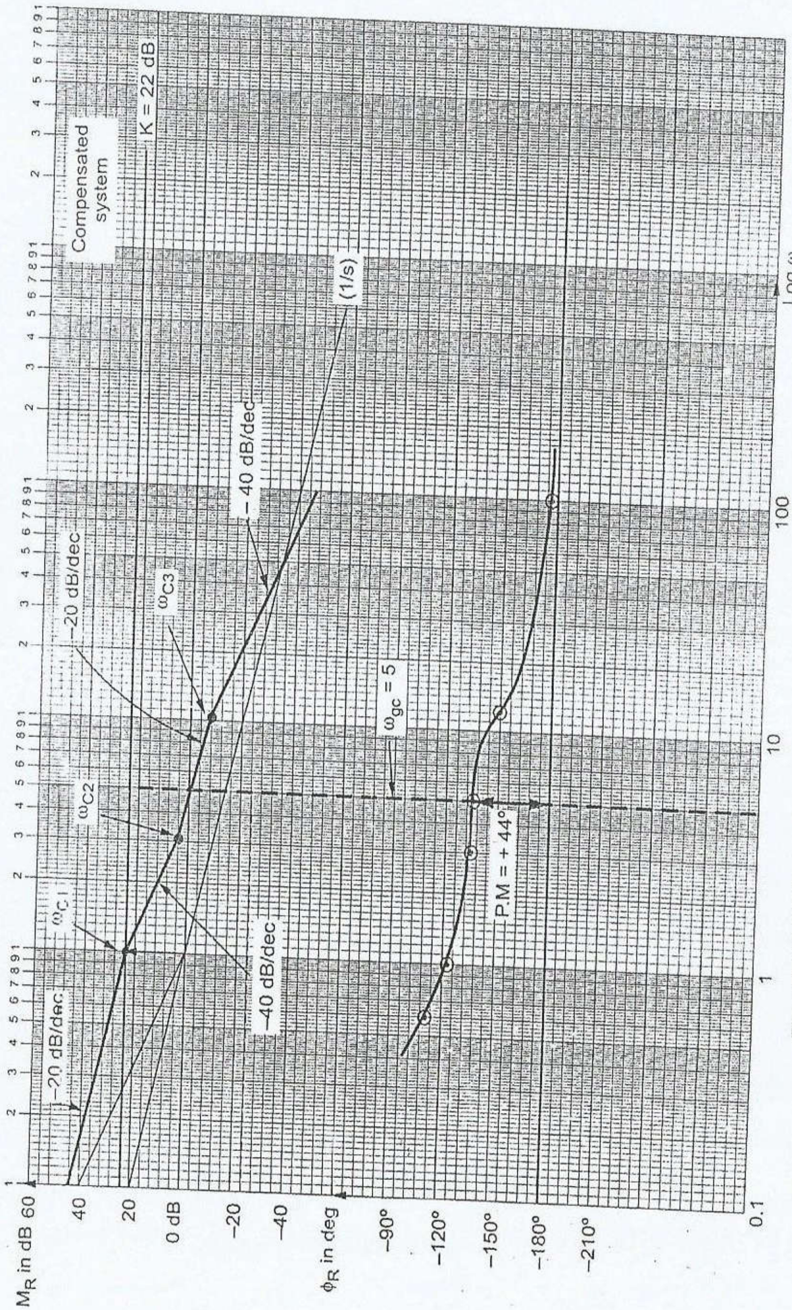


Fig. 14.4.8 (b) Bode plot of compensated system in Example 14.4.2

Review Questions

1. *Derive the T.F. of lead network.*
2. *Draw and explain polar and Bode plot of lead compensator.*
3. *Derive the relation between ϕ_m and α for the lead compensator.*
4. *Explain the design in frequency domain of lead compensator.*
5. *Discuss in detail about lead network.*

Examples for Practice

Example 14.4.3 : *Find the maximum phase shift that can be obtained from lead compensator :* $G_c(s) = \dfrac{1+0.12\,s}{1+0.04\,s}$ [Ans. : $\phi_m = 30°$]

Example 14.4.4 : *The forward path transfer function of a certain unity negative feedback control system is given as,*

$$G(s) = \frac{K}{s(s+2)(s+30)}.$$

The system has to satisfy the following specifications :
Phase margin ≥ 35°
Gain margin ≥ 15 dB
Steady state error for unit ramp input ≤ 0.04 rad.
Design a suitable series lead compensator. **AU : April-04**

Example 14.4.5 : *Design a lead compensator for a type-2 system with an open loop transfer function* $G(s)\ \dfrac{K}{s^2(0.2s+1)}$. *Assume that the system is required to be compensated to meet the following specifications :*
1) Acceleration error constant $K_a = 10$
2) Phase margin = 35° **AU : May-10**

14.5 Lag Compensator

Consider an electrical network which is a lag compensating network, as shown, in the Fig. 14.5.1.

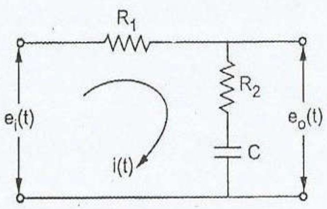


Fig. 14.5.1 Lag network

Let us obtain the transfer function of such an electrical lag network.

Assuming unloaded circuit and applying KVL to the loop we can write,

$$e_i(t) = i(t)\,R_1 + i(t)\,R_2 + \frac{1}{C}\int i(t)\,dt$$

Taking Laplace transform of the equation,

$$E_i(s) = I(s)\left[R_1 + R_2 + \frac{1}{sC}\right] \quad \ldots(14.5.1)$$

Now the output equation is,

$$e_o(t) = i(t)\,R_2 + \frac{1}{C}\int i(t)\,dt$$

Taking Laplace transform,

$$E_o(s) = I(s)\left[R_2 + \frac{1}{sC}\right] \quad \ldots(14.5.2)$$

Substituting I(s) from (14.5.2) in (14.5.1) we get,

$$E_i(s) = \frac{E_o(s)}{\left[R_2 + \frac{1}{sC}\right]}\left[R_1 + R_2 + \frac{1}{sC}\right]$$

$$\therefore \quad E_i(s) = \frac{E_o(s)[(R_1+R_2)sC+1]}{(1+R_2sC)}$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{1+sR_2C}{1+s\cdot(R_1+R_2)C} = \left(\frac{R_2}{R_1+R_2}\right)\cdot\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}}$$

This is generally expressed as,

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{\beta}\cdot\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}} \quad \ldots(14.5.3)$$

where $T = R_2C, \quad \beta = \frac{R_1+R_2}{R_2} > 1$

The lag compensator has zero at $s = -\frac{1}{T}$ and a pole at $s = -\frac{1}{\beta T}$. As $\beta > 1$, the pole is always located to the right of the zero. The pole-zero plot is shown in the Fig. 14.5.2. Usually β is choosen greater than 10.

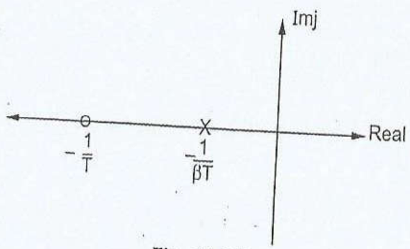


Fig. 14.5.2

14.5.1 Maximum Lag Angle and β

Let us see what is the maximum lag angle the compensator can provide.

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{\beta}\cdot\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \frac{1+Ts}{1+\beta Ts}$$

In frequency domain we get,

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{1+j\omega T}{1+j\omega\beta T} \quad \ldots(14.5.4)$$

$$\therefore \quad \boxed{M = \left|\frac{E_o(j\omega)}{E_i(j\omega)}\right| = \frac{\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2\beta^2T^2}}} \quad \ldots(14.5.5)$$

While the phase angle is given by,

$$\boxed{\phi = \tan^{-1}\omega T - \tan^{-1}\omega\beta T} \quad \ldots(14.5.6)$$

The equation is exactly similar to the lead network, only $\beta > 1$. To find ϕ_m, let us find ω_m which maximises ϕ.

$$\therefore \quad \frac{d\phi}{d\omega} = 0$$

$$\therefore \quad \frac{d}{d\omega}[\tan^{-1}\omega T - \tan^{-1}\omega\beta T] = 0$$

Solving we get, $\boxed{\omega_m = \frac{1}{T\sqrt{\beta}} = \sqrt{\frac{1}{T}\cdot\frac{1}{\beta T}}}$

This is the frequency at which the phase lag is at its maximum.

The two corner frequencies of lag compensator are,

$$\boxed{\omega_{C1} = \frac{1}{T} \text{ and } \omega_{C2} = \frac{1}{\beta T}}$$

Thus ω_m is the geometric mean of the two corner frequencies.

Key Point *Note that the primary function of a lag compensator is to provide attenuation in the high frequency range to give a system sufficient phase margin. The phase lag angle does not play a role in the lag compensation.*

14.5.2 Polar Plot of Lag Compensator

From the equations (14.5.5) and (14.5.6), polar plot of lag compensator can be achieved.

When $\omega = 0$, $M = 1$ and $\phi = 0°$

When $\omega = \infty$, $M = \frac{1}{\beta}$ and $\phi = 0°$

So both the points, starting and terminating are on positive real axis.

For any value of ω between 0 to ∞, the magnitude is always positive while for $\beta > 1$, $\tan^{-1}\omega T < \tan^{-1}\omega\beta T$. Thus the resultant ϕ will be always negative for any value of ω between 0 to ∞. Hence all the points of polar plot are always in the third quadrant of the complex plane.

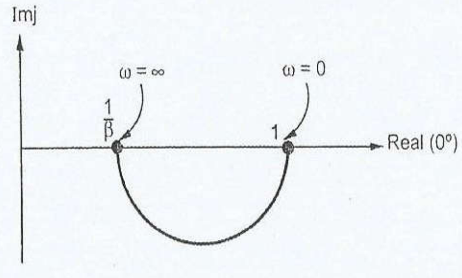


Fig. 14.5.3 Polar plot of lag compensator

Thus the polar plot is as shown in the Fig. 14.5.3.

14.5.3 Bode Plot of Lag Compensator

The corner frequencies of the lag compensator are,

$$\omega_{C1} = \frac{1}{\beta T} \text{ for a pole at } s = -\frac{1}{\beta T}$$

$$\omega_{C2} = \frac{1}{T} \text{ for a zero at } s = -\frac{1}{T}$$

The pole is more dominating than zero. The transfer function of basic lag network is,

$$T(s) = \frac{1+sT}{1+s\beta T}$$

Thus as K = 1, the initial line is 0 dB line till $\omega_{C1} = \frac{1}{\beta T}$ where pole occurs. From $\omega_{C1} = \frac{1}{\beta T}$ to $\omega_{C2} = \frac{1}{T}$ there is a line of slope

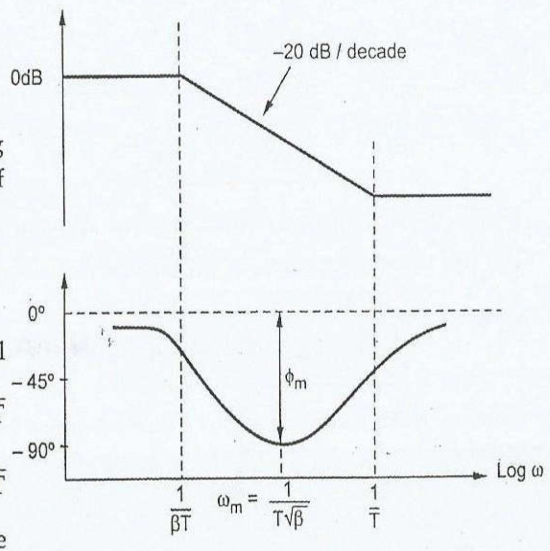


Fig. 14.5.4 Bode plot of lag compensator

– 20 dB/dec. From $\omega_{C2} = \frac{1}{T}$ onwards again the net slope is zero due to existence of zero with $\omega_{C2} = \frac{1}{T}$.

The Bode plot is shown in the Fig. 14.5.4.

14.5.4 Steps to Design Lag Compensator

Step 1 : Assume a lag compensator having a transfer function,

$$G_c(s) = \frac{1}{\beta}\frac{\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\beta T}\right)} = \frac{(1+Ts)}{(1+\beta Ts)}$$

Assume $G_1(s) = K\,G(s)$

From the given error constant, determine the value of K which satisfies the steady state performance.

Step 2 : Using the value of K determined above, draw the Bode plot of $G_1(j\omega)$. **Obtain the phase margin.** This is say ϕ_1, for uncompensated system.

Step 3 : Generally phase margin is specified.

Let ϕ_s = P.M. specified

Then determine, $\boxed{\phi_2 = \phi_s + \varepsilon}$

where ε = margin of safety = 5° to 15°.

The ε compensates for the phase lag of the lag compensator.

Step 4 : Find the frequency ω_2 corresponding to phase margin of ϕ_2 degrees i.e. the frequency at which phase angle of open loop transfer function is $-180° + \phi_2$. Choose this as new gain crossover frequency.

Step 5 : To have ω_2 as the new gain crossover frequency, determine the attenuation necessary to shift the magnitude curve up or down to 0 dB. This shift is due to the contribution of β which is $20 \text{ Log } \frac{1}{\beta}$.

$\therefore$ Shift to have ω_2 as new gain crossover $= 20 \text{ Log } \frac{1}{\beta} = -20 \text{ Log } \beta$

Key Point *Down shift must be taken negative while up shift positive. Hence determine the value of β.*

Step 6 : Choose the upper corner frequency $\frac{'1'}{T}$ which is $\frac{1}{2}$ or $\frac{1}{10}$ below the ω_2 determined in step 4.

$$\therefore \quad \omega_{C2} = \frac{1}{T} = \frac{\omega_2}{2} \text{ or } \frac{\omega_2}{10}$$

Thus determine the value of T.

The other corner frequency for the lag compensator is,

$$\omega_{C1} = \frac{1}{\beta T}$$

Step 7 : Thus once transfer function of the lag compensator is known, draw the Bode plot of compensated system and check the specifications. If specifications are not satisfied, repeat the design by modifying the pole-zero locations of the compensator till a satisfactory result is obtained.

14.5.5 Effects and Limitations of Lag Compensator

The various effects and limitations of lag compensator are,

1. Lag compensator allows high gain at low frequencies thus it is basically a low pass filter. Hence it improves the steady state performance.
2. In lag compensation, the attenuation characteristics is used for the compensation. The phase lag characteristics is of no use in the compensation.
3. The attenuation due to lag compensator shifts the gain crossover frequency to a lower frequency point. Thus the bandwidth of the system gets reduced.
4. Reduced bandwidth means slower response. Thus rise time and settling time are usually longer. The transient response lasts for longer time.
5. The system becomes more sensitive to the parameter variations.
6. Lag compensator approximately acts as proportional plus integral controller and thus tends to make system less stable.

Example 14.5.1 *For a certain system,*

$$G(s) = \frac{0.025}{s(1+0.5s)(1+0.05s)}$$

Design a suitable lag compensator to give, velocity error constant = 20 sec⁻¹ and phase margin = 40°.

Solution : Step 1 : Assume

$$G_1(s) = KG(s) = \frac{0.025K}{s(1+0.5s)(1+0.05s)}$$

$$\therefore \quad K_v = 20 = \lim_{s\to 0} s\, G_1(s) G_c(s)$$

$$\therefore \quad 20 = \lim_{s\to 0} s \cdot \frac{(1+Ts)}{(1+\beta Ts)} \cdot \frac{0.025K}{s(1+0.5s)(1+0.05s)}$$

$$\therefore \quad 20 = 0.025K \quad \text{i.e.} \quad K = 800$$

$$\therefore \quad G_1(s) = \frac{0.025 \times 800}{s(1+0.5s)(1+0.05s)} = \frac{20}{s(1+0.5s)(1+0.05s)}$$

Step 2 : Draw the Bode plot of $G_1(j\omega)$

Factors : 20 Log 20 = 26 dB

1 pole at origin, straight line of slope – 20 dB/dec

$$\omega_{C1} = \frac{1}{0.5} = 2, \text{ simple pole, slope } -20 \text{ dB/dec}$$

$$\omega_{C2} = \frac{1}{0.05} = 20, \text{ simple pole, slope } -20 \text{ dB/dec}$$

$$G_1(j\omega) = \frac{20}{j\omega(1+0.5j\omega)(1+0.05j\omega)}$$

Phase angle table :

ω	$\frac{1}{j\omega}$	$-\tan^{-1} 0.5\omega$	$-\tan^{-1} 0.05\omega$	φ
0.2	– 90°	– 5.71°	– 0.57°	– 96.28°
2	– 90°	– 45°	– 5.71°	– 140.7°
5	– 90°	– 68.19°	– 14.03°	– 172.22°
10	– 90°	– 78.69°	– 26.56°	– 195.25°

The Bode plot is shown in the Fig. 14.5.5. From it the specifications of uncompensated system are, (See Fig. 14.5.5 on next page).

$$\omega_{gc} = 6.4 \text{ rad/sec}, \ \omega_{pc} = 5.6 \text{ rad/sec}, \ \text{G.M.} = -2 \text{ dB}, \ \text{P.M.} = -5°$$

Uncompensated system is unstable

Step 3 : $\phi_s = 40°$ and $\varepsilon = 5°$

$$\therefore \quad \phi_2 = \phi_s + \varepsilon = 45°$$

Step 4 : From Fig. 14.5.5, find ω_2 which gives 45° phase margin.

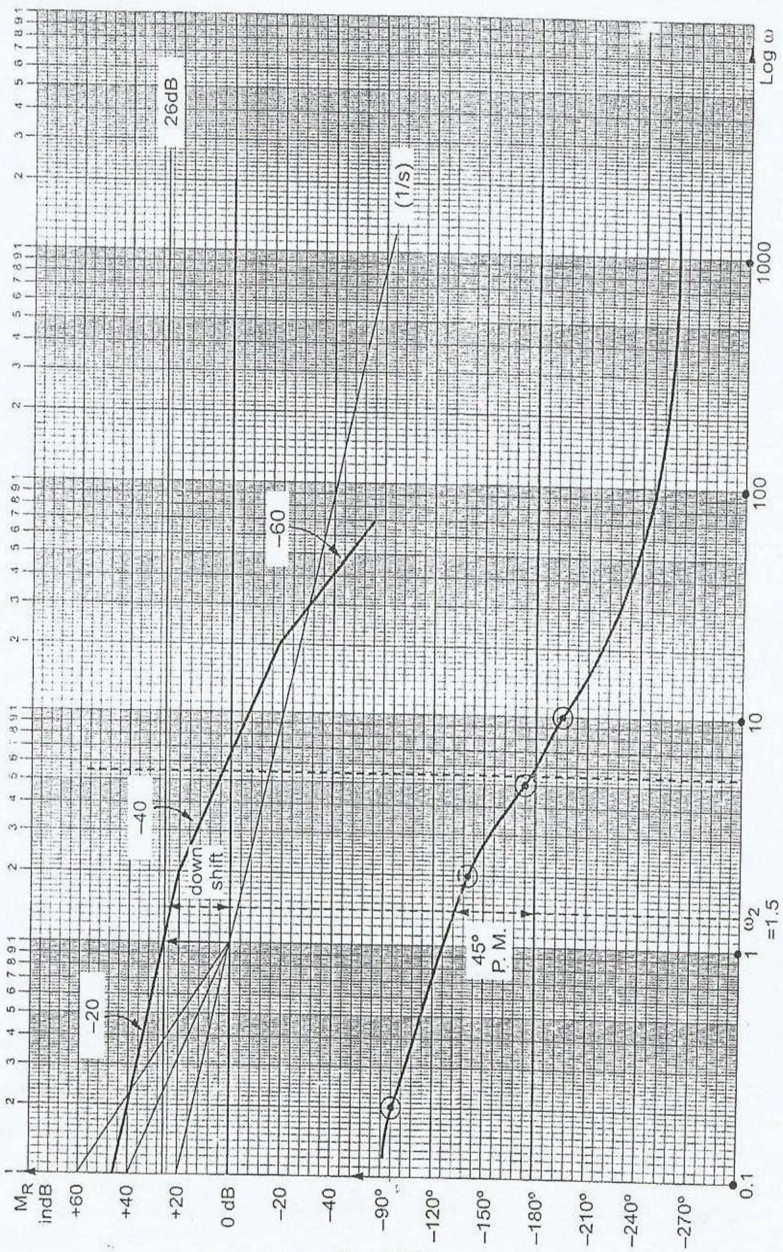


Fig. 14.5.5

$\omega_2 = 1.5$ rad/sec

Step 5 : Let $\omega_2 = 1.5$ be the new gain crossover.

For this it is necessary to bring the magnitude curve down by 23 dB.

$\therefore \quad -20 \text{ Log } \beta = -23, \quad$ - ve as down shift

$\therefore \quad \beta = 14.12$

Assuming $\quad \beta = 14$

Step 6 : Choose $\omega_{C2} = \frac{\omega_2}{10} = \frac{1.5}{10} = 0.15$ rad/sec

Now $\quad \omega_{C2} = \frac{1}{T} = 0.15 \quad$ i.e. $\quad T = 6.66$ and $\omega_{C1} = \frac{1}{\beta T} = 0.0107$

Step 7 : Thus the lag compensator is,

$$G_c(s) = \frac{(1+6.66\,s)}{(1+93.46\,s)}$$

Thus the transfer function of the compensated system is,

$$G_c(s)G(s) = \frac{20(1+6.66\,s)}{s\,(1+0.5\,s)\,(1+0.05\,s)\,(1+93.46\,s)}$$

To check for the specifications, draw the Bode plot of compensated system as shown in the Fig. 14.5.6. It is drawn on separate semilog paper as the starting frequency required for this system is 0.001.

Factors : 20 Log 20 = 26 dB

1 pole at origin

$\omega_{C1} = \frac{1}{93.46} = 0.01$, simple pole, $\omega_{C2} = \frac{1}{6.66} = 0.15$, simple zero

$\omega_{C3} = \frac{1}{0.5} = 2$, simple pole, $\omega_{C4} = \frac{1}{0.05} = 20$, simple pole

$$G_c(j\omega)G(j\omega) = \frac{20(1+6.66\,j\omega)}{j\omega\,(1+0.5j\omega)\,(1+0.05\,j\omega)\,(1+93.46\,j\omega)}$$

Phase angle table for compensated system :

ω	$\frac{1}{j\omega}$	$-\tan^{-1} 93.46\,\omega$	$+\tan^{-1} 6.66\,\omega$	$-\tan^{-1} 0.5\,\omega$	$-\tan^{-1} 0.05\,\omega$	ϕ_R
0.01	- 90°	- 43.06°	+ 3.81°	- 0.28°	- 0.028°	- 129.5°
0.05	- 90°	- 77.92°	+ 18.41°	- 1.43°	- 0.14°	- 151.08°
0.1	- 90°	- 84°	+ 33.66°	- 2.86°	- 0.28°	- 143.48°
1	- 90°	- 89.38°	+ 81.46°	- 26.56°	- 2.86°	- 127.34°
2	- 90°	- 89.7°	+ 85.7°	- 45°	- 5.71°	- 144.7°
10	- 90°	- 90°	+ 89°	- 78.6°	- 26.56°	- 195.5°

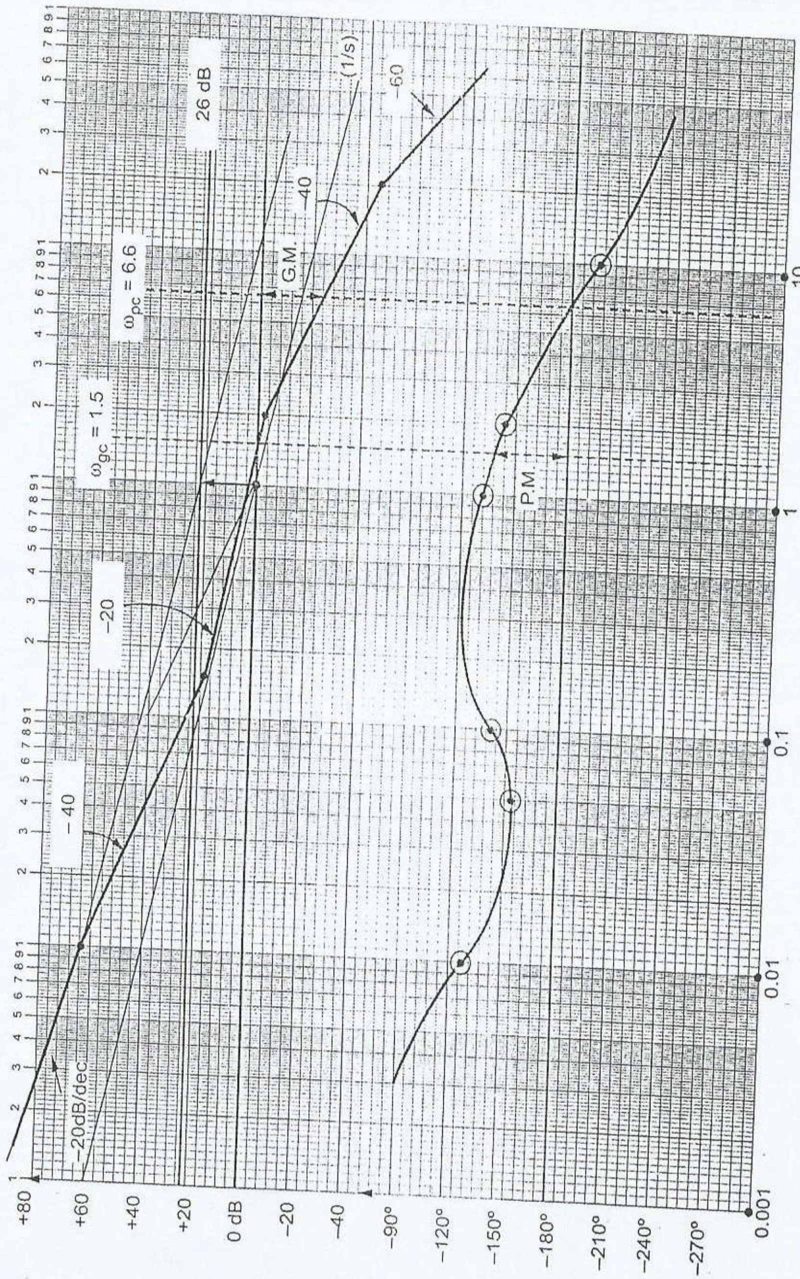


Fig. 14.5.6 Bode plot of compensated system in Example 14.5.1

From the Fig. 14.5.6, the various specifications are,

ω_{gc} = 1.5 rad/sec, ω_{pc} = 6.6 rad/sec, G.M. = + 24 dB, P.M. = 48°

Thus the compensated system satisfies all the specifications.

Example 14.5.2 *The open loop transfer function of a uncompensated system is* $G(s) = \frac{5}{s(s+2)}$. *Design a suitable lag compensator for the system so that the static velocity error constant* K_v *is 20* sec^{-1}, *the phase margin is atleast 55° and the gain margin is atleast 12 dB.*

AU : April-05

Solution : Step 1 : Assume $G_1(s) = KG(s) = \frac{5K}{s(s+2)}$

$$K_v = \lim_{s\to 0} s\, G_1(s)G_c(s) = 20 \quad \text{where } G_c(s) = \frac{1+Ts}{1+\beta Ts}$$

$$\therefore \quad \lim_{s\to 0} s\cdot\frac{5K}{s(s+2)}\times\frac{(1+Ts)}{(1+\beta Ts)} = 20 \quad \ldots G_c(s) = \text{T.F. of lag compensator}$$

$$\therefore \quad \frac{5K}{2} = 20 \quad \text{i.e. } K = 8$$

$$\therefore \quad G_1(s) = \frac{40}{s(s+2)} = \frac{20}{s(1+0.5s)}$$

Step 2 : Draw the Bode plot of $G_1(j\omega) = \frac{20}{j\omega(1+0.5j\omega)}$

Factors : K = 20 i.e. 20 Log K = 26 dB

One pole at origin so straight line of slope – 20 dB/dec. One simple pole with $T_1 = 0.5$ i.e. $\omega_{C1} = \frac{1}{T_1} = 2$. Thus Bode plot is straight line of slope – 20 dB/dec for $\omega > \omega_{C1}$. So initial slope – 20 dB/dec and will become – 40 dB/dec at ω_{C1}.

Phase angle table

ω	$\frac{1}{j\omega}$	$-\tan^{-1}(0.5\omega)$	ϕ_R
0.2	– 90°	– 5.71°	– 95.71°
2	– 90°	– 45°	– 135°
4	– 90°	– 63.53°	– 153.43°
10	– 90°	– 78.69°	– 168.69°

The Bode plot of uncompensated system is shown in the Fig. 14.5.7(a) from which,

ω_{gc} = 6 rad/sec, ω_{pc} = ∞ rad/sec, G.M. = + ∞ dB, P.M. = + 18°

Though uncompensated sytem is stable, P.M. is less than 55° as required.

Step 3 : ϕ_s = 55°, ε = 5° hence $\phi_2 = \phi_s + \varepsilon$ = 60°

Step 4 : Find new $\omega_{gc} = \omega_2$ for which P.M. = 60° from Bode plot of uncompensated system

$\therefore \quad \omega_2 = 0.9$ rad/sec

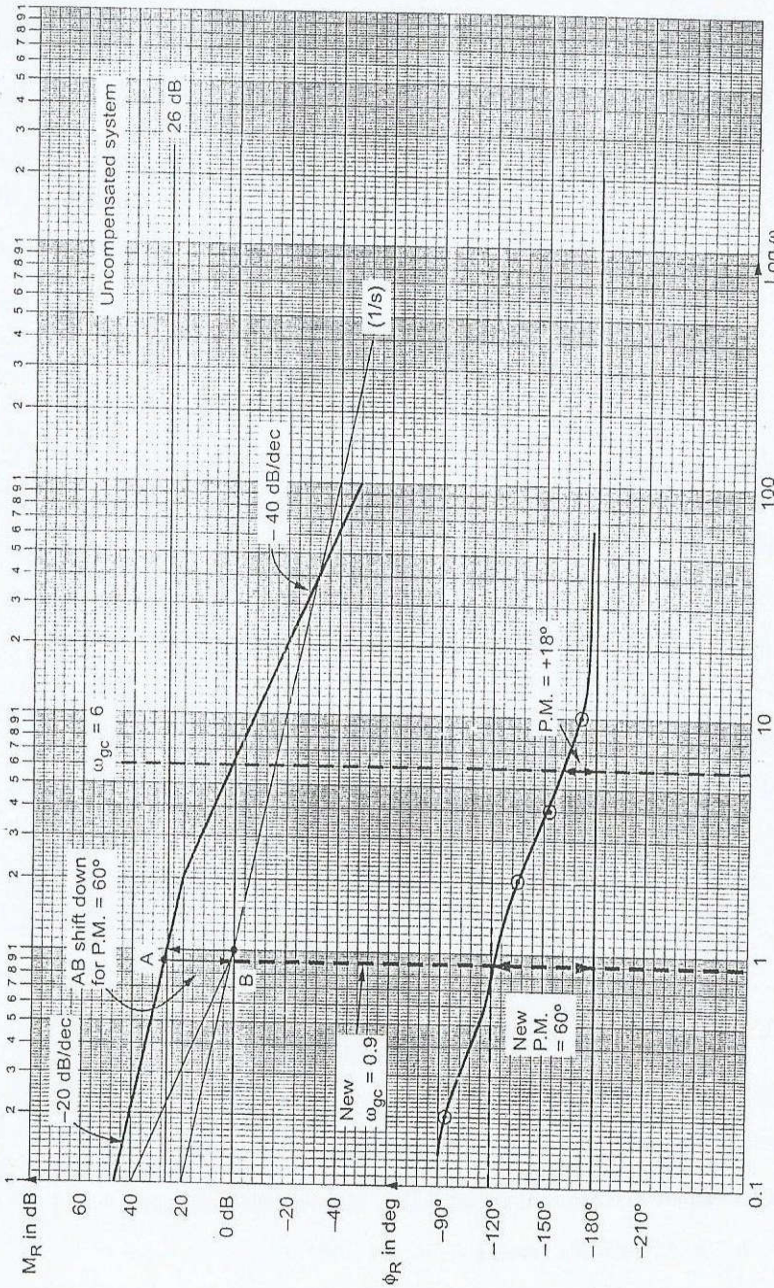


Fig. 14.5.7 (a) Bode plot of uncompensated system in Example 14.5.2

Step 5 : For ω_2 to be new ω_{gc}, the Bode plot must be shifted down from A to B i.e. – 28 dB (Negative because downward shift).

$$\therefore \quad -20 \text{ Log } \beta = -28$$

$$\therefore \quad \beta = 25.11 \approx 25$$

Step 6 : Choose $\omega_{C2} = \frac{\omega_2}{10} = \frac{0.9}{10} = 0.09$ rad/sec

But $\quad \omega_{C2} = \frac{1}{T}$ hence T = 11.11 ≈ 12

and $\quad \omega_{C1} = \frac{1}{\beta T} = \frac{1}{25 \times 12} = \frac{1}{300} = 0.00333$ rad/sec

Thus T.F. of compensated system is,

$$G_c(s)G_1(s) = \frac{20(1+12\,s)}{s(1+300\,s)(1+0.5\,s)}$$

Step 7 : Draw the Bode plot of compensated system and check for the specifications.

Factors : K = 20 i.e. 20 Log K = 26 dB

$\frac{1}{s}$ i.e. one pole at origin so – 20 dB/dec

$\omega_{C1} = 0.0033$, simple pole i.e. – 20 dB/dec ↲ – 40 dB/dec

$\omega_{C2} = 0.09$, simple zero i.e. + 20 dB/dec ↲ – 20 dB/dec

$\omega_{C3} = 2$, simple pole i.e. – 20 dB/dec ↲ – 40 dB/dec

$$G_c(j\omega)G_1(j\omega) = \frac{20(1+j\omega 12)}{j\omega(1+j\,300\,\omega)(1+0.5\,j\omega)}$$

Phase angle table :

ω	$\frac{1}{j\omega}$	$-\tan^{-1} 300\,\omega$	$+\tan^{-1} 12\,\omega$	$-\tan^{-1} 0.5\,\omega$	ϕ_R
0.002	– 90°	– 30.96°	+ 1.37°	– 0.057°	– 119.64°
0.05	– 90°	– 86.18°	+ 30.96°	– 1.43°	– 146.65°
0.2	– 90°	– 89.04°	+ 67.38°	– 5.71°	– 117.37°
2	– 90°	– 89.9°	+ 87.61°	– 45°	– 137.29°
10	– 90°	– 89.9°	+ 89.9°	– 78.6°	– 168.6°

The Bode plot of compensated system is shown in the Fig. 14.5.7 (b) from which,

(See Fig. 14.5.7 (b) on next page).

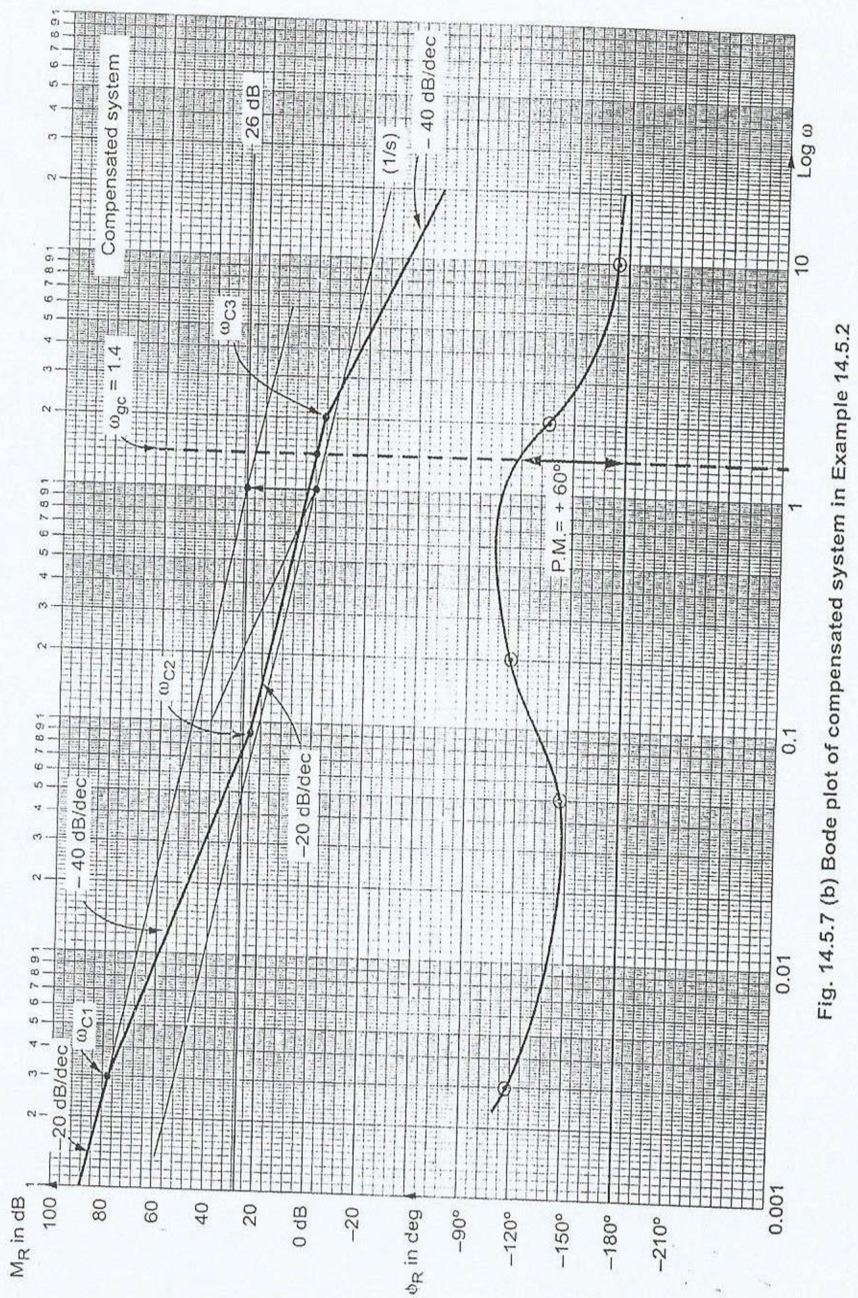


Fig. 14.5.7 (b) Bode plot of compensated system in Example 14.5.2

ω_{gc} = 1.4 rad/sec, $\omega_{pc} = \infty$ rad/sec, G.M. = + ∞ dB, P.M. = 60°

Thus the compensated system satisfies all the specifications

Review Questions

1. *Derive the transfer functions of lag network.*
2. *Draw and explain the polar plot of lag network.*
3. *Draw and explain the Bode plot of lag network.*
4. *Explain the design in frequency domain of lag compensator.*
5. *Discuss the limitations and effects of phase lead compensation.*

Examples for Practice

Example 14.5.3 : *Design a phase lag compensator so the system* $G(s)H(s) = \dfrac{100}{s(s+1)}$ *will have phase margin of 15°.* **AU : Dec.-07**

Example 14.5.4 : *A unity feedback system has an open loop transfer function*

$$G(s)H(s) = \frac{K}{s(1+s)(1+0.2s)}$$

Design a phase lag compensator to achieve the following specifications. Velocity error constants $K_v = 5$, *Phase margin = 45°.* **AU : Dec.-06**

14.6 Lag-Lead Compensator

A combination of a lag and lead compensators is nothing bit a lag-lead compensator. Consider an electrical network which is lag-lead network, as shown in the Fig. 14.6.1.

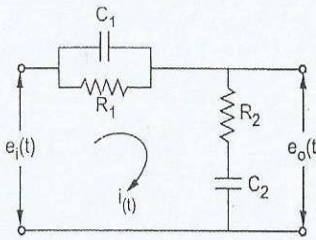


Fig. 14.6.1 Lag-lead network

Let us obtain the transfer function of the electrical lag-lead network.

Now sum of the current through R_1 and C_1 is nothing but current i(t).

$$\therefore \qquad \frac{e_i - e_o}{R_1} + C_1 \frac{d\,(e_i - e_o)}{dt} = i(t)$$

Taking Laplace transform we get,

$$\frac{1}{R_1} E_i(s) - \frac{1}{R_1} E_o(s) + sC_1 E_i(s) - sC_1 E_o(s) = I(s) \qquad ...(14.6.1)$$

The output equation is,

$$i(t) R_2 + \frac{1}{C_2} \int i(t)\, dt = e_o(t)$$

Taking Laplace transform,

$$I(s)\left[R_2+\frac{1}{sC_2}\right] = E_o(s) \quad \ldots(14.6.2)$$

Substituting I (s) from (14.6.1) in (14.6.2) we get,

$$\left\{E_i(s)\left[\frac{1}{R_1}+sC_1\right]-E_o(s)\left[\frac{1}{R_1}+sC_1\right]\right\}\left[R_2+\frac{1}{sC_2}\right] = E_o(s)$$

$$\therefore \quad \frac{E_i(s)(1+sR_1C_1)-E_o(s)(1+sR_1C_1)}{R_1}\cdot\frac{(1+sR_2C_2)}{sC_2} = E_o(s)$$

$$\therefore \quad E_i(s)\frac{(1+sR_1C_1)(1+sR_2C_2)}{sR_1C_2} = E_o(s)\left[1+\frac{(1+sR_1C_1)(1+sR_2C_2)}{sR_1C_2}\right]$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{(1+sR_1C_1)(1+sR_2C_2)}{sR_1C_2+(1+sR_1C_1)(1+sR_2C_2)}$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{(1+sR_1C_1)(1+sR_2C_2)}{s^2R_1R_2C_1C_2+s[R_1C_1+R_2C_2+R_1C_2]+1}$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{R_1R_2C_1C_2\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{R_1R_2C_1C_2\left[s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}\right]}$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \quad \ldots(14.6.3)$$

where $T_1 = R_1C_1, \quad T_2 = R_2C_2$

$$\frac{\beta}{T_1}+\frac{1}{\beta T_2} = \frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}$$

$$\alpha\beta T_1T_2 = R_1R_2C_1C_2$$

$$\alpha\beta = 1$$

It also can be expressed as,

$$\frac{E_o(s)}{E_i(s)} = \frac{(1+T_1s)(1+T_2s)}{\left(1+\frac{T_1}{\beta}s\right)(1+T_2\beta s)} \quad \ldots(14.6.4)$$

where $\beta > 1$

The phase lead portion involving T_1, adds phase lead angle while the phase lag portion involving T_2 provide attenuation near and above the gain crossover frequency.

The pole are $s = -\frac{\beta}{T_1}, -\frac{1}{\beta T_2}$ while the zeros are at $s = -\frac{1}{T_1}, -\frac{1}{T_2}$. The pole-zero plot for the lag-lead compensator is shown in the Fig. 14.6.2.

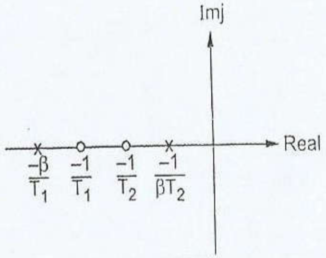


Fig. 14.6.2

14.6.1 Polar Plot of Lag-lead Compensator

The transfer function of Lag-lead compensator in the frequency domain can be obtained as,

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{(1+T_1j\omega)(1+T_2j\omega)}{\left(1+\frac{T_1}{\beta}j\omega\right)(1+jT_2\beta\omega)} \quad \ldots(14.6.5)$$

The magnitude of the transfer functions,

$$M = \frac{\sqrt{1+\omega^2T_1^2}\cdot\sqrt{1+\omega^2T_2^2}}{\sqrt{1+\frac{T_1^2\omega^2}{\beta^2}}\cdot\sqrt{1+\beta^2T_2^2\omega^2}} \quad \ldots(14.6.6)$$

The phase angle is given by,

$$\phi = +\tan^{-1}\omega T_1+\tan^{-1}\omega T_2-\tan^{-1}\frac{T_1\omega}{\beta}-\tan^{-1}\beta T_2\omega \quad \ldots(14.6.7)$$

Now for $\omega = 0, \quad M = 1$ and $\phi = 0°$

While for $\omega = \infty, \quad M = 1$ and $\phi = 0°$

Thus the starting and terminating points of polar plot is same as $1\angle 0°$. But there is certain frequency ω_1 upto which it acts as lag network giving polar plot in third quadrant. While for $\omega > \omega_1$ till ∞ it acts as lead network giving polar plot in the first quadrant. Hence the entire polar plot, is circular in nature. The polar plot is as shown in the Fig. 14.6.3.

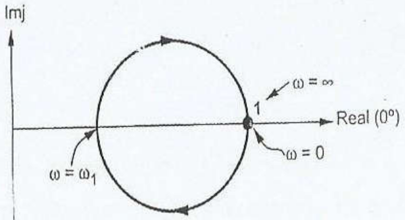


Fig. 14.6.3 Polar plot of lag-lead compensator

In the lag-lead network the frequency ω_1 is given by,

$$\omega_1 = \frac{1}{\sqrt{T_1 T_2}} \qquad ...(14.6.8)$$

14.6.2 Bode Plot of Lag-lead Compensator

The various corner frequencies of the lag-lead compensator are,

$$\omega_{C1} = -\frac{\beta}{T_1}, \text{ for a simple pole}$$

$$\omega_{C2} = -\frac{1}{T_1}, \text{ for a simple zero}$$

$$\omega_{C3} = -\frac{1}{T_2}, \text{ for a simple zero}$$

$$\omega_{C4} = -\frac{1}{-\beta T_2}, \text{ for a simple pole}$$

So line of zero slope continues till ω_{C1}. At ω_{C1} the slope becomes – 20 dB/decade. At ω_{C2} the slope again becomes zero and continues till ω_{C3} . At ω_{C3} the slope becomes + 20 dB/decade and continues till ω_{C4}. And at ω_{C4} it again becomes zero. The Bode plot is shown in the Fig. 14.6.4.

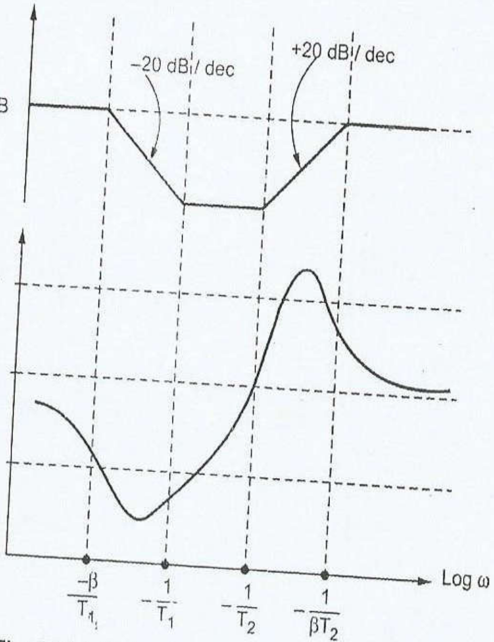


Fig. 14.6.4 Bode plot of lag-lead compensator

14.6.3 Effects of Lag-lead Compensator

Lag-lead compensator is used when both fast response and good static accuracy are desired. Use of lag-lead compensator increases the low frequency gain which improves the steady state. While at the same time it increases bandwidth of the system, making the system response very fast.

In general, the phase lead portion of this compensator is used to achieve large bandwidth and hence shorter rise time and settling time. While the phase lag portion provides the major damping of the system.

The design of lag-lead compensator is illustrated with an example.

Example 14.6.1 *Consider the unity feedback system whose open loop transfer function is,*

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Design suitable lag-lead compensator so as to achieve, static velocity error constant = 10 sec^{-1}, phase margin = 50°, gain margin ≥ 10 dB. **AU : May-11**

Solution : Let the transfer function of the compensator be,

$$G_c(s) = \frac{(1+T_1 s)(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)(1+\beta T_2 s)}$$

$$K_v = \lim_{s\to 0} s\, G_c(s) G(s)$$

$$\therefore \quad 10 = \lim_{s\to 0} \frac{s\cdot(1+T_1 s)(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)(1+\beta T_2 s)} \cdot \frac{K}{s(s+1)(s+2)}$$

$$\therefore \quad 10 = \frac{K}{2} \quad \text{i.e.} \quad K = 20$$

The uncompensated system is,

$$G_1(s) = \frac{20}{s(s+1)(s+2)} = \frac{10}{s(s+1)(1+0.5s)}$$

The Bode plot of uncompensated system is shown in the Fig. 14.6.5. From it the various specifications can be obtained.

Factors : 20 Log 10 = 20 dB

1 pole at origin, – 20 dB/dec

ω_{C1} = 1, simple pole, – 20 dB/dec

ω_{C2} = 2, simple pole, – 20 dB/dec

$$G_1(j\omega) = \frac{10}{j\omega(1+j\omega)(1+0.5j\omega)}$$

Phase angle table :

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	$-\tan^{-1}0.5\,\omega$	ϕ_R
0.1	− 90°	− 5.71°	− 2.86°	− 98.5°
1	− 90°	− 45°	− 26.56°	− 161.5°
2	− 90°	− 63.4°	− 45°	− 198.4°

From Fig. 14.6.5, the various values are, (See Fig. 14.6.5 on next page).

$$\omega_{gc} = 2.7 \text{ rad/sec}, \ \omega_{pc} = 1.5 \text{ rad/sec, G.M.} = -12 \text{ dB, P.M.} = -35°$$

The uncompensated system is unstable in nature.

For uncompensated system, $\omega_{pc} = 1.5$ i.e. $\angle G(j\omega)$ at $\omega = 1.5$ is $-180°$.

To have P.M. of 50°, it is necessary to add phase lead of 50° at $\omega = 1.5$ and at the same time $\omega = 1.5$ must be the new gain cross-over frequency.

Choose new gain crossover frequency as 1.5 rad/sec. The angle of phase lead required is + 50°.

$$\phi_m = 50°$$

Now $\quad \sin\phi_m = \frac{1-\alpha}{1+\alpha} \quad$ and $\quad \alpha\beta = 1$

$$\therefore \quad \alpha = \frac{1}{\beta}$$

$$\therefore \quad \sin\phi_m = \frac{1-\frac{1}{\beta}}{1+\frac{1}{\beta}} = \frac{\beta-1}{\beta+1} \quad \text{i.e.} \quad 0.766 = \frac{\beta-1}{\beta+1}$$

$$\therefore \quad \beta + 1 = 1.3054\,(\beta - 1) \quad \text{i.e.} \quad \beta = 7.54$$

Choosing $\beta = 10$ as practically minimum β is generally 10.

As the new gain cross-over frequency is 1.5, the corner frequency of phase lag portion is generally $\frac{1}{10}$th of the gain crossover frequency.

$$\therefore \text{ Corner frequency } \omega = \frac{1}{T_2} = \frac{1.5}{10} = 0.15 \text{ rad/sec}$$

As T_2 is known, the another corner frequency is,

$$\text{Corner frequency } \omega = \frac{1}{\beta T_2} = 0.015 \text{ rad/sec}$$

Hence the transfer function of the phase lag portion is,

$$\frac{(1+T_2 s)}{(1+\beta T_2 s)} = \frac{(1+6.67\,s)}{(1+66.67\,s)}$$

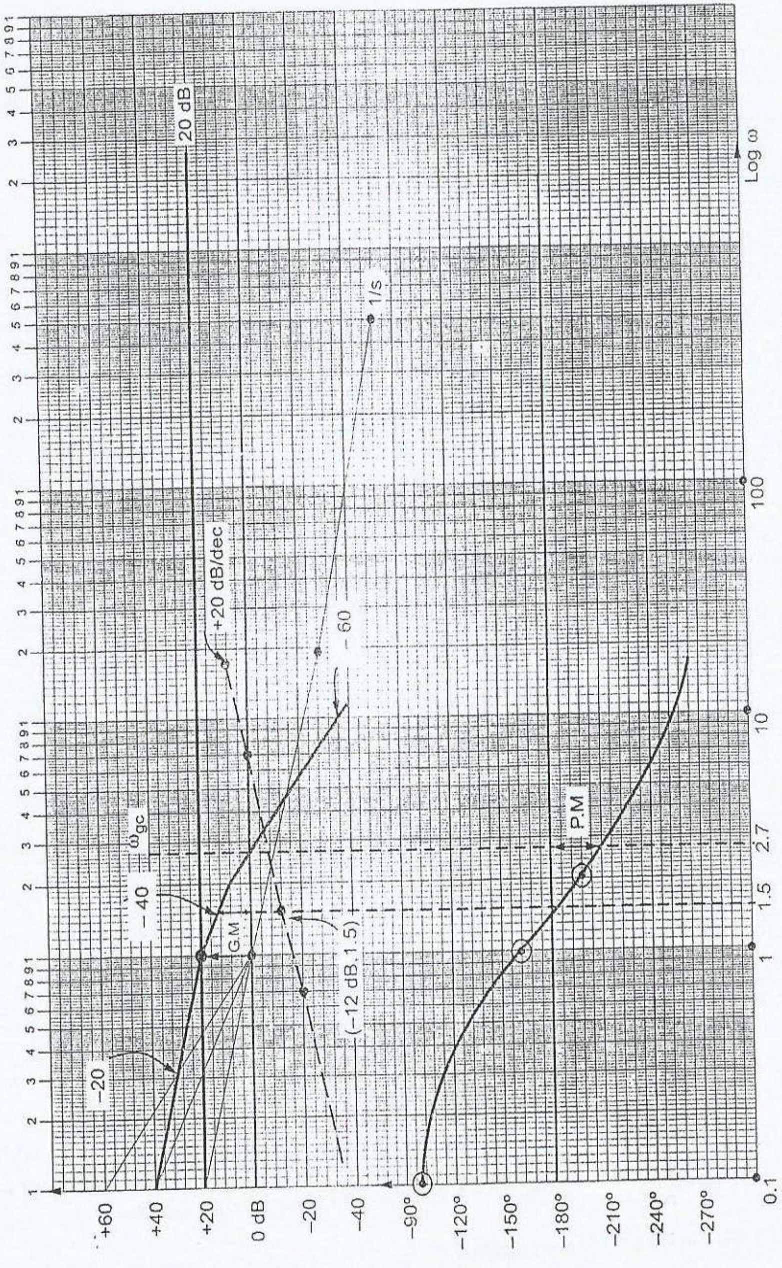


Fig. 14.6.5 Bode plot of uncompensated system in Example 14.6.1

The phase lead portion can be obtained as below :

From the Fig. 14.6.5, it can be observed that $|G(j\omega)|$ at $\omega = 1.5$ is + 12 dB.

So to have $\omega = 1.5$ as new gain crossover frequency, the lag-lead compensator must contribute − 12 dB at $\omega = 1.5$ rad/sec. Thus draw a straight line from the point having co-ordinates (1.5 rad/sec, − 12 dB) of slope + 20 dB/dec. Its intersection with 0 dB line is at 7 rad/sec and its intersection with − 20 dB line is at 0.7 rad/sec. These are the corner frequencies for the lead portion. Hence the transfer function of the lead portion is,

$$\frac{(1+T_1 s)}{\left(1+\frac{T_1}{\beta}s\right)} = \frac{(1+1.43\,s)}{(1+0.143\,s)}$$

as $$T_1 = \frac{1}{\omega_{C1}} = \frac{1}{0.7} = 1.43 \text{ and } \frac{T_1}{\beta} = \frac{1.43}{10} = 0.143$$

Hence the transfer function of lag-lead compensator is,

$$G_c(s) = \frac{(1+1.43\,s)\,(1+6.67\,s)}{(1+0.143\,s)\,(1+66.67\,s)}$$

Thus the transfer function of the compensated system is,

$$G_c(s)G(s) = \frac{10(1+1.43\,s)\,(1+6.67\,s)}{s(1+0.143\,s)\,(1+66.67\,s)(1+s)\,(1+0.5\,s)}$$

Let us obtain the Bode plot of this compensated system and check the specifications.

Factors : K = 10, 20 Log 10 = 20 dB

1 Pole at origin pole, − 20 dB/dec

ω_{C1} = 0.015, simple pole, − 20 dB/dec, ω_{C2} = 0.15, simple zero, + 20 dB/dec

ω_{C3} = 0.7, simple zero, + 20 dB/dec, ω_{C4} = 1, simple pole, − 20 dB/dec

ω_{C5} = 2, simple pole, − 20 dB/dec, ω_{C6} = 7, simple pole, − 20 dB/dec

$$G_c(j\omega)G(j\omega) = \frac{10(1+1.43\,j\omega)\,(1+6.67\,j\omega)}{j\omega(1+0.143\,j\omega)(1+66.67\,j\omega)(1+j\omega)\,(1+0.5\,j\omega)}$$

Phase angle table for the compensated system :

ω	$\frac{1}{j\omega}$	$-\tan^{-1}66.67\omega$	$+\tan^{-1}6.67\,\omega$	$+\tan^{-1}1.43\,\omega$	$-\tan^{-1}\omega$	$-\tan^{-1}0.5\omega$	$-\tan^{-1}0.143\omega$	ϕ_R
0.02	− 90°	− 58.13°	+ 7.6°	+ 1.63°	− 1.1°	− 0.57°	− 0.16°	− 135.7°
0.1	− 90°	− 81.46°	+ 33.7°	+ 8.13°	− 5.7°	− 2.86°	− 0.81°	− 139°
0.5	− 90°	− 88.2°	+ 73.3°	+ 35.5°	− 26.56°	− 14.03°	− 4.08°	− 114.1°
1	− 90°	− 89.1°	+81.47°	+ 55°	− 45°	− 26.56°	− 8.13°	− 122.3°

Fig. 14.6.6 Bode plot of compensated system in Example 14.6.2

2	− 90°	− 89.5°	+ 85.7°	+ 70.1°	− 63.4°	− 45°	− 15.96°	− 148°
4	− 90°	− 89.7°	+ 87.8°	+ 80°	− 75.9°	− 63.4°	− 29.8°	− 181°

The Bode plct of compensated system is shown in the Fig. 14.6.6.

The specifications are

$$\omega_{gc} = 1.3 \text{ rad/sec}, \omega_{pc} = 3.9 \text{ rad/sec, G.M.} = +16 \text{ dB, P.M.} = +50^\circ$$

Thus compensated system satisfies the specifications.

Review Questions

1. *With a help of the frequency plot, explain how is phase lag-lead compensation obtained in control systems.*
2. *Draw the circuit of a lag-lead compensator and derive its transfer function. What are the effects?*
3. *Draw and explain the polar plot of lag-lead network.*
4. *Draw and explain the Bode plot of lag-lead network.*
5. *Compare the characteristics of three types of compensators.*

14.7 Compensation using Root Locus

Uptill now we have seen the compensation using Bode Plot where the specifications are given in the frequency domain. When the specifications are given in the time domain the root locus approach of design is very powerful.

The damping factor and undamped natural frequency are the two main specifications used in the root locus compensation. These two specifications decide the dominant closed loop poles near jω axis. Thus compensation using root locus means to reshape the root locus near jω axis and origin in order to place the dominant closed loop poles at the desired locations.

The root locus compensation also can be achieved using series compensation with lead, lag or lag-lead network.

Let us revise the relation between damping ratio ξ and angle θ.

The dominant complex conjugate poles are expressed as $-\xi\omega_n \pm j\omega_d$. Then in the complex plane we have $\xi = \sin\theta$ where θ is measured from jω axis or $\xi = \cos\theta$ if θ is measured from negative real axis.

Thus lines of constant ξ are radial lines passing through the origin as shown in the Fig. 14.7.1 (b).

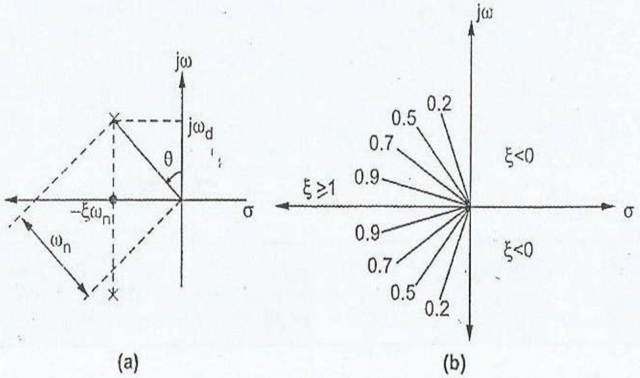


Fig. 14.7.1

For the sake of design purpose we will use,

$$\xi = \sin\theta, \ \theta \text{ measured from } j\omega \text{ axis}$$

With this background let us see the design procedures for lead, lag and lag-lead compensator.

14.8 Designing Lead Compensator using Root Locus

The procedure to design lead compensator is,

Step 1 : From the given specifications, find the desired locations of the dominant closed loop poles.

Step 2 : Assume the lead compensator as,

$$G_c(s) = K_c\,\alpha \frac{(1+Ts)}{(1+\alpha Ts)}, \quad \alpha < 1$$

K_c is determined from the requirement of open loop gain.

Step 3 : Find the sum of the angles at the desired location of one of the dominant closed loop poles with the open loop poles and zeros of the original system. This angle must be an odd multiple of 180°. If it is not, calculate the necessary angle ϕ to be added to get the sum as an odd multiple of 180°. This ϕ must be contributed by lead compensator. If ϕ is more than 60° then two or more lead networks may be needed. This ϕ, helps to determine values of α and T.

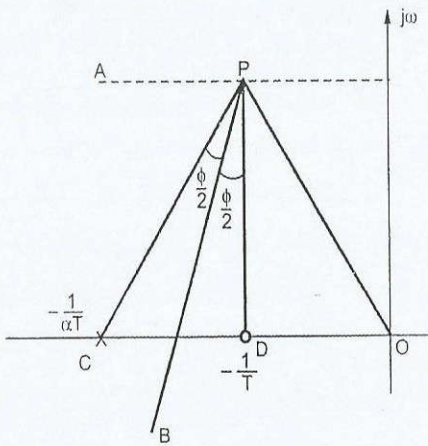


Fig. 14.8.1

Step 4 : To determine α and T for known ϕ, draw the horizontal line from one of the dominant closed loop pole say P. Join origin to P, as shown in the Fig. 14.8.1. Bisect the angle between the lines PA and PO. Draw the two line PC and PD that makes angle $\pm\frac{\phi}{2}$ with the bisector PB. The intersection of PC and PD with the negative real axis gives the necessary pole and zero of compensator.

Step 5 : The open loop gain can be determined by applying the magnitude condition at point P.

Step 6 : Check that the compensated system satisfies all the specifications. If not, adjust the compensator pole and zero till all the specifications are satisfied.

Example 14.8.1 *Design a suitable lead compensator for a system with unity feedback and having open loop transfer function :*

$$G(s) = \frac{K}{s(s+1)(s+4)}$$

to meet the specifications : 1. Damping ratio $\xi = 0.5$ 2. Undamped natural frequency $\omega_n = 2$ rad/sec.

AU : May-07,09

Solution : Let us sketch the root locus of the uncompensated system.

$$P = 3,\ Z = 0,\ N = P = 3,\ P - Z = 3$$

Starting points : 0, – 1, – 4

Terminating points : ∞, ∞, ∞

The angles of asymptotes : 60°, 180°, 300°

Centroid : $\frac{0-1-4}{3} = -1.67$

Breakaway point : 1 + G(s)H(s) = 0

$$\therefore \quad 1 + \frac{K}{s(s+1)(s+4)} = 0 \quad \text{i.e.} \quad s^3 + 5s^2 + 4s + K = 0$$

$$\therefore \quad K = -s^3 - 5s^2 - 4s \quad \text{i.e.} \quad \frac{dK}{ds} = -3s^2 - 10s - 4 = 0$$

$$\therefore \quad s^2 + 3.33\,s + 1.33 = 0 \quad \text{i.e.} \quad s = -0.464,\ -2.86$$

The point s = – 0.464 is valid breakaway point.

Intersection with imaginary axis : The Routh's array is,

s^3	1	4
s^2	5	K
s^1	$\frac{20-K}{5}$	0
s^0	K	

$$20 - K = 0$$

$$\therefore \quad K_{mar} = 20$$

$$\therefore \quad 5s^2 + K_{mar} = 0 \text{ is auxillary equation}$$

$$\therefore \quad 5s^2 = -20$$

$$\therefore \quad s = \pm j2 \quad \text{are intersection points}$$

The root locus is as shown in the Fig. 14.8.2.

Step 1 : $\xi = 0.5$ and $\omega_n = 2$

The desired dominant closed loop poles are,

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -1 \pm j\,1.73$$

It can be seen that dominant poles are not on the root locus shown in the Fig. 14.8.2.

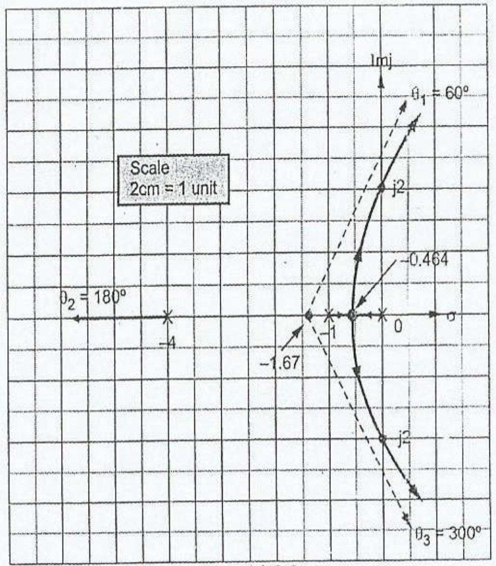


Fig. 14.8.2

Step 2 : Assume lead compensator with

$$G_c(s) = K_c\,\alpha\frac{(1+Ts)}{(1+\alpha Ts)}$$

Step 3 : ∠G(s)H(s) at dominant pole

$$\therefore \quad \angle\frac{K}{s(s+1)(s+4)} \text{ at } s = -1 + j\,1.73 \text{ is,}$$

$$\frac{\angle K}{\angle -1+j\,1.73 \angle -1+j\,1.73+1 \angle -1+j\,1.73+4} = \frac{\angle K}{\angle -1+j\,1.73\ \angle j\,1.73\ \angle 3+j\,1.73}$$

$$= \frac{0°}{120.029°\ 90°\ 29.97°} = -240°$$

∴ Angle to be contributed by lead compensator is, – 180° – (– 240°) = 60°

Step 4 : Find locations of pole and zero.

∴ Zero at s = – 1 will cancel the pole at s = –1 of the given system hence we will select the zero closer to s = –1 but slightly to left of s = – 1.

∴ Zero of compensator at

s = – 1.2

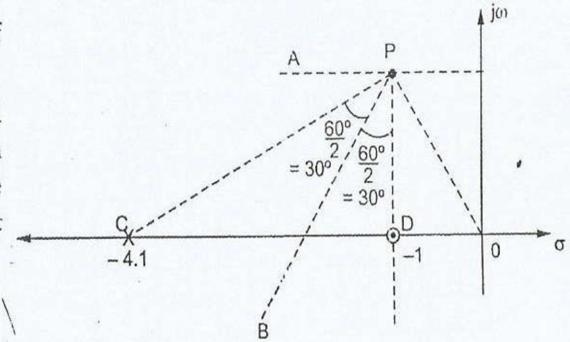


Fig. 14.8.3

and pole of compensator at s = − 4.1 which is very close to pole s = −4 of original system hence we will select it at s = − 4.6.

$$\therefore \quad \frac{1}{T} = 1.2 \quad \therefore\ T = 0.833$$

and

$$\frac{1}{\alpha T} = 4.6 \quad \therefore\ \alpha T = 0.2173$$

$$\therefore \quad \alpha = 0.26 < 1$$

The transfer function of the compensated system is,

$$G_c(s)G(s) = \frac{K(1+0.833\,s)}{s(1+s)(s+4)(1+0.2173\,s)}$$

Step 5 : Use the magnitude condition to obtain value of K at s = −1 + j 1.73.

$$|G_c(s)\,G(s)|_{s=-1+j1.73} = 1$$

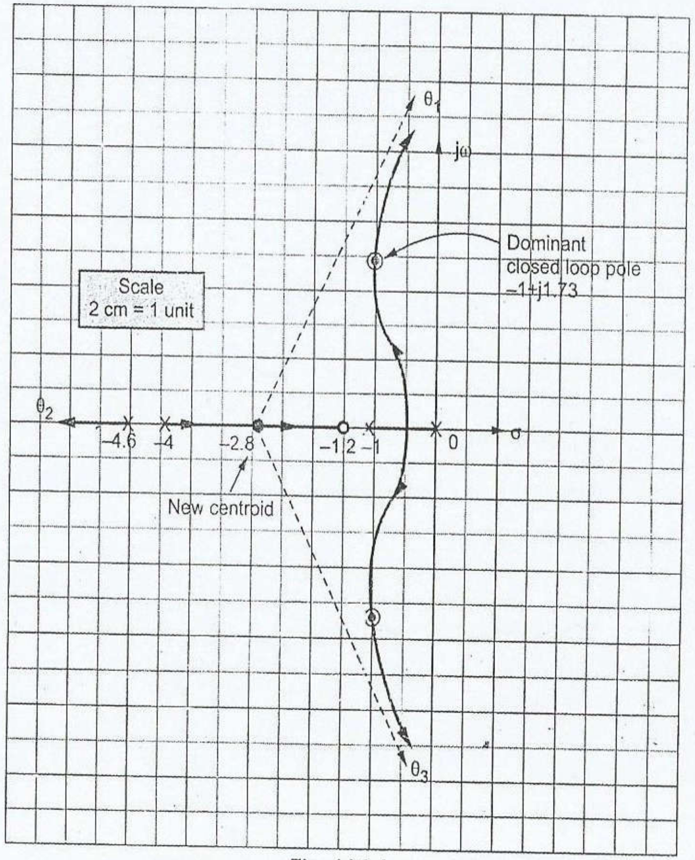


Fig. 14.8.4

$$\therefore \quad \frac{K[1+0.833(-1+j1.73)]}{(-1+j1.73)(j1.73)(3+j1.73)[1+0.213(-1+j1.73)]} = 1$$

$$\therefore \quad \frac{K|0.167+j1.44|}{|-1+j1.73||0+j1.73||3+j1.73||0.7827+j0.3759|} = 1$$

$$\therefore \quad \frac{K\times 1.449}{1.998\times 1.73\times 3.46\times 0.8682} = 1$$

$$\therefore \quad K = 7.166$$

$$\therefore \quad G_c(s)G(s) = \frac{7.166(1+0.833\,s)}{s(1+s)(s+4)(1+0.2173\,s)}$$

$$\therefore \quad \boxed{G_c(s)G(s) = \frac{27.47(s+1.2)}{s(s+1)(s+4)(s+4.6)}}$$

The root locus of the compensated system is shown in the Fig. 14.8.4. Students are expected to calculate breakaway points and intersection with the imaginary axis.

Review Question

1. *Explain the design of lead compensator using root locus.*

Example for Practice

Example 14.8.2 : *A unity feedback system with a forward transfer function -* $G(s) = \frac{K}{s(s+7)}$ *is operating with a closed loop response that has 15 % overshoot. Find*

i) The settling time

ii) Design a lead compensator to decrease the settling time by three times choose the compensator's zero to be at − 10.

14.9 Designing Lag Compensator using Root Locus

The lag compensator is generally used when the system is showing good transient response which is satisfactory but unsatisfactory steady state characteristics. This compensation needs not to change the root locus appreciably but to increase the open loop gain.

Thus lag compensator essentially does not change the root locus appreciably near the dominant closed loop poles but increases the open loop gain as per the requirement. The angle contribution by the lag compensator is very small of about 5° and to achieve this the pole and zero of the lag compensator are placed close together and near the origin of the s-plane.

Increase in gain means increase in static error constant. So when it is necessary to satisfy static error constant specification using root locus, lag compensator is used.

Steps to design the lag compensator are,

Step 1 : Draw the root locus of the uncompensated system and locate the dominant closed loop poles on the root locus.

Step 2 : Assume the lag compensator having transfer function,

$$G_c(s) = \hat{K}_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}, \beta > 1$$

Step 3 : Calculate the static error constant specified in the problem.

Step 4 : Determine the amount of increase in static error constant necessary to satisfy the specification.

Step 5 : Determine pole and zero of the compensator, such that they do not produce appreciable change in the original root locus but produce necessary increase in static error constant.

Note that the ratio of value of gain required in the specifications and the gain found in the uncompensated system is the required ratio between the distance of the zero from origin and that of pole from the origin.

Step 6 : Draw the root locus of compensated system. Locate the dominant closed loop poles.

Step 7 : Adjust $\hat{K}_c$ from the magnitude condition so as to place the dominant closed loop poles at the desired location.

Example 14.9.1 *Design a lag compensator for a system with open loop transfer function as,*

$$G(s) = \frac{K}{s(s+1)(s+4)}$$

to meet the following specifications :

1. Damping ratio = 0.5 *2. Velocity error constant ≥ 5 sec^{-1}*

3. Settling time = 10 sec

AU : Dec.-03, Nov.-06

Solution : Let us draw the root locus of the uncompensated system. This is already obtained in Example 14.8.1 which is shown again in the Fig. 14.9.1.

From the given specifications, $\xi = 0.5$

and $T_s = \frac{4}{\xi\omega_n} = 10$

$\therefore \quad \omega_n = 0.8$ rad/sec

Hence the dominant closed loop poles are,

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -0.4 \pm j\,0.693$$

The gain K at dominant pole $-0.4 + j\,0.693$ can be obtained by magnitude condition.

$$|G(s)H(s)|_{s=-0.4+j0.693} = 1$$

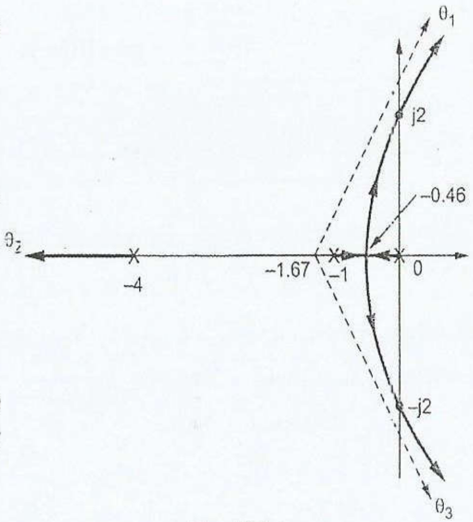


Fig. 14.9.1

$$\therefore \quad \frac{|K|}{|-0.4+j\,0.693|\;|0.6+j\,0.693|\;|3.6+j\,0.693|} = 1$$

$$\therefore \quad \frac{K}{0.8 \times 0.9167 \times 3.667} = 1 \quad \text{i.e.} \quad K = 2.688$$

The transfer function of uncompensated system is,

$$G(s) = \frac{2.688}{s(s+1)(s+4)}$$

$$\therefore \quad K_v = \lim_{s\to 0} s\,G(s) = \lim_{s\to 0} \frac{s \cdot 2.688}{s(s+1)(s+4)} = 0.672$$

It is desired to have K_v of 5 sec^{-1}. The factor by which static error constant is to be increased is,

$$\text{Factor} = \frac{K_v \text{ desired}}{K_v \text{ of uncompensated system}} = \frac{5}{0.672} = 7.44$$

Let us choose this factor be 10, hence $\beta = 10$

Place the zero and pole of the lag compensator very close to the origin.

Let zero of compensator at s = – 0.1 and

Pole of compensator at s = – 0.01

So transfer function of the lag compensator is,

$$G_c(s) = \hat{K}_c \frac{s+0.1}{s+0.01}$$

Hence the transfer function of the compensated system becomes,

$$G_c(s)G(s) = \hat{K}_c \frac{s+0.1}{s+0.01} \cdot \frac{2.688}{s(s+1)(s+4)}$$

$$\therefore \quad G_c(s)G(s) = \frac{K(s+0.1)}{s(s+1)(s+4)(s+0.01)} \quad \text{where } K = 2.688\ \hat{K}_c$$

Draw the root locus of the compensated system.

$$P = 4,\ Z = 1,\ P - Z = 4 - 1 = 3,\ N = 4$$

Starting points → 0, – 1, – 4, – 0.01, Terminating points → – 0.1, ∞, ∞, ∞

Angles of asymptotes = 60°, 180°, 300°

Centroid = – 1.64

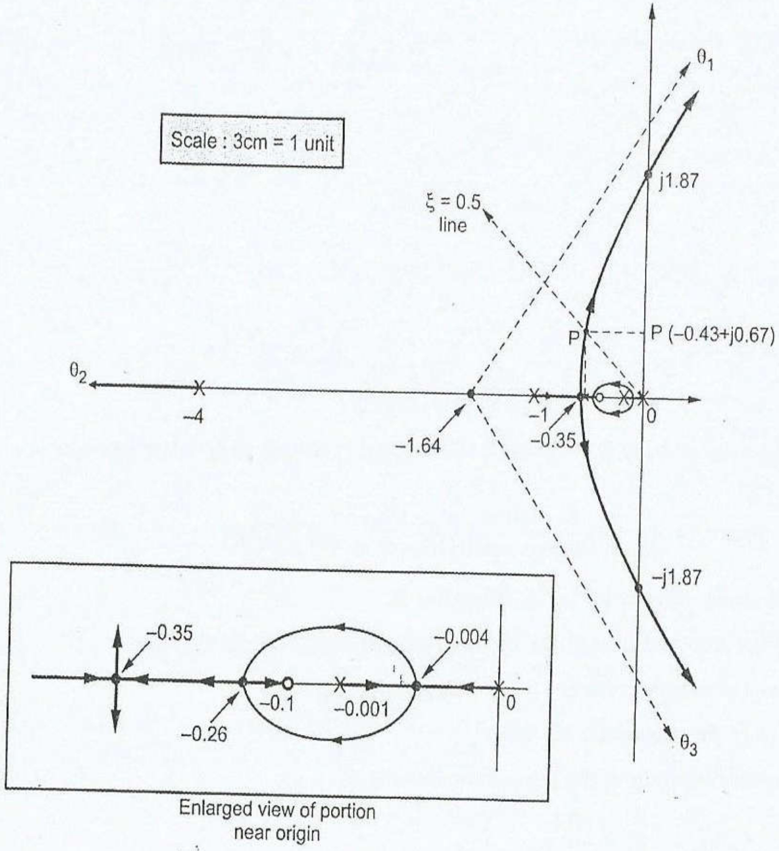


Fig. 14.9.2

For breakaway point get $\frac{dK}{ds} = 0$. This gives the equation,

$$s^4 + 3.47\, s^3 + 1.85\, s^2 + 0.27\, s + 0.0013 = 0$$

This gives three approximate valid breakaway points as,

s = – 0.004, – 0.26 and – 0.3511

Hence the root locus of compensated system is as shown in the Fig. 14.9.2.

The new dominant pole is – 0.43 + j 0.67.

Apply magnitude condition at this new location of dominant closed loop pole.

$$|G_c(s)H(s)|_{s=-0.43+j0.67} = 1$$

$$\therefore \quad \frac{|K|\,|-0.33+j\,0.67|}{|-0.43+j\,0.67|\,|0.57+j\,0.67|\,|3.57+j\,0.67|\,|-0.42+j\,0.67|} = 1$$

$$\therefore \quad \frac{K \times 0.7468}{0.7961 \times 0.8786 \times 3.6323 \times 0.7907} = 1$$

$$\therefore \quad K = 2.69$$

But $\quad K = 2.688$ i.e. $\hat{K}_c = 1.0019$

Hence the compensated system has the transfer function,

$$G_c(s)G(s) = \frac{2.69(s+0.1)}{s(s+0.01)\,(s+1)\,(s+4)}$$

This gives static error constant of 6.725 which is greater than 5.

Review Question

1. *Explain the design of lag compensator using root locus.*

14.10 Designing Lag-Lead Compensator using Root Locus

It is known that the lead compensator increases the speed of the response and improves stability. The lag compensator improves the steady state response. If improvement in both transient as well as the steady state response is desired then the lag-lead compensator is used. Lag-lead compensator combines the advantages of both lead and lag compensators.

Assume the transfer function of the lag-lead compensator as,

$$G_c(s) = K_c \frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{\gamma}{T_1}\right)} \cdot \frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_2}\right)}, \quad \beta > 1, \gamma > 1$$

Let us consider two cases to design such compensator.

Case 1 : $\gamma \neq \beta$

In this design, determine the locations of dominant closed loop poles from the given specifications.

From uncompensated G(s), calculate the angle deficiency ϕ which must be contributed by lead portion of the compensator.

Choose T_2 sufficiently large so that magnitude of lag portion is unity.

$$\left| \frac{\left(s_1+\frac{1}{T_2}\right)}{\left(s_1+\frac{1}{\beta T_2}\right)} \right| = 1$$

Where $s = s_1$ is one of the dominant closed loop pole. From the angle deficiency ϕ determine values of T_1 and γ.

$$\angle \frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}} = \phi$$

The determine value of K_c from the magnitude condition.

$$\left| K_c \frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}} G(s_1) \right| = 1$$

If the static velocity error constant K_V is given then

$$K_V = \lim_{s \to 0} s\, G_c(s)\, G(s)$$

$$= \lim_{s \to 0} sK_c \frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{\gamma}{T_1}\right)} \cdot \frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_2}\right)} \cdot G(s) = \lim_{s \to 0} s\, K_c \cdot \frac{\beta}{\gamma} G(s)$$

As K_c and γ are known, β can be obtained.

Using this value of β, choose T_2 such that the magnitude of the lag portion is unity.

Case 2 : $\gamma = \beta$

Determine the locations of dominant closed loop poles according to given specifications.

As $\gamma = \beta$, the transfer function of compensator becomes,

$$G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{\beta}{T_1}\right)} \cdot \frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_2}\right)}, \quad \beta > 1$$

To have the dominant closed loop poles at desired location, calculate the angle deficiency ϕ.

Choose T_2 very large at the end so that

$$\left| \frac{\left(s_1+\frac{1}{T_2}\right)}{\left(s_1+\frac{1}{\beta T_2}\right)} \right| = 1$$

Where $s = s_1$ is one of the dominant closed loop pole.

$$\text{Now} \quad \left| K_c \frac{\left(s_1+\frac{1}{T_1}\right)}{\left(s_1+\frac{\beta}{T_1}\right)} G(s_1) \right| = 1 \text{ and } \angle \frac{s_1+\frac{1}{T_1}}{s_1+\frac{\beta}{T_1}} = \phi$$

These two equations give the values of T_1 and β.

Once β is known, choose large T_2 so that the magnitude of the lag portion is approximately unity and

$$-5^\circ < \angle \frac{s_1+\frac{1}{T_2}}{s_1+\frac{1}{\beta T_2}} < 0^\circ$$

βT_2 should be high but should be physically realizable.

Review Question

1. *Explain the design of lag-lead compensator using root locus.*

□□□