Math 308, Midterm 1
April 18, 2016

Name: ______________________________

Instructions.

- Vectors are indicated with arrows, as in

  Let $\vec{v}$ be a vector in $\mathbb{R}^n$.

- You are allowed a two-sided sheet of notes in your own handwriting.
- No calculators.
- There are 7 problems on 6 pages. Make sure your exam is complete.
- **Any cheating observed during, or noticed afterwards when comparing exams, will result in a 0 on this exam. Moreover, such an exam cannot be dropped. This means you lose 28% percent of your total raw grade. It will be difficult to pass the class if this occurs.**

Question	Points	Score
1	10	
2	9	
3	4	
4	6	
5	5	
6	8	
7	8	
Total:	50	

1. Rita is a software engineer writing a script involving 6 tasks. Each must be done one after the other. Let t_i be the time for the ith task. These times have a certain structure:

   - Any 3 adjacent tasks will take half as long as the next two tasks.
   - The second task takes 1 second.
   - The fourth task takes 10 seconds.

[4 points] (a) Write a matrix for the system of equations describing the length of each task.

Solution:

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

[2 points] (b) She plugs this into a solver and gets:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 15 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \end{array}\right)$$

What variables are free?

Solution: t_5 and t_6

[2 points] (c) Suppose she knows additionally that the sixth task takes 20 seconds and the first three tasks will run in 50 seconds. Write the extra rows that you would add to your answer in part (a).

Solution: $\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 20 \\ 1 & 1 & 1 & 0 & 0 & 0 & 50 \end{array}\right]$

[2 points] (d) Using all of these constraints she plugs this into her computer and gets the matrix:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 44 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 & 90 \\ 0 & 0 & 0 & 0 & 0 & 1 & 20 \end{array}\right).$$

How long does the script take to run? (*It's okay to write your answer as a sum that you don't actually compute.*)

Solution:

$$5 + 1 + 44 + 10 + 90 + 20 = 170 \text{ seconds.}$$

2. Rose needs to know if $\left\{ \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \vec{w} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right\}$ have the same span.

[2 points] (a) Call $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a vector in the span of a set of vectors. Write down **two** augmented matrices. One that corresponds to the span of $\vec{u}$ and $\vec{v}$, and one the corresponds to the span of $\vec{w}$, and $\vec{x}$.

Solution:

$$\left(\begin{array}{cc|c} 1 & 0 & a \\ 2 & 2 & b \\ 2 & 1 & c \end{array} \right) \qquad \left(\begin{array}{cc|c} 4 & 3 & a \\ 4 & 0 & b \\ 6 & 3 & c \end{array} \right)$$

[2 points] (b) Rose plugs the matrix for the span of $\vec{u}$ and $\vec{v}$ into a computer and gets: $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$.

She sees this and realizes something is wrong. Explain why this output doesn't make sense.

Solution: This says that the span has no vectors in it, when we know that at least two vectors are in it.

[3 points] (c) She row reduces by hand and obtains:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -a + \frac{1}{2}b \\ 0 & 0 & -a - \frac{1}{2}b + c \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & \frac{1}{4}b \\ 0 & 1 & \frac{1}{3}a - \frac{1}{3}b \\ 0 & 0 & -6a - 3b + 6c \end{pmatrix}.$$

Does $\text{Span}\{\vec{u}, \vec{v}\} = \text{Span}\{\vec{w}, \vec{x}\}$? *Explain why or why not.*

Solution: Yes. We see from the bottom row (which are multiples of one another) that the span for both is the plane $-a - \frac{1}{2}b + c = 0$.

[2 points] (d) Suppose that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{Span}\{\vec{u}, \vec{v}\}$. Write a linear combination of $\vec{u}$ and $\vec{v}$ that equals $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Solution: $a\vec{u} + (-a + \frac{1}{2}b)\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

[4 points] 3. Put this matrix in reduced echelon form: $\begin{pmatrix} 2 & 1 & x \\ 2 & y & 0 \end{pmatrix}$

Solution:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2}x + \frac{x}{2\,(y-1)} \\ 0 & 1 & -\frac{x}{y-1} \end{pmatrix}$$

4. Answer the following multiple choice questions.

[2 points] (a) If $\text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}\}$ then we may write $\vec{v}_1$ as a linear combination of $\vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}$.

True False **Not enough information**

[2 points] (b) If $\{\vec{u}, \vec{v}\}$ is a linearly *dependent* set of vectors, then so is $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$.

True False Not enough information

[2 points] (c) One may choose α, β, γ so that there are exactly two quadratics $f(x) = ax^2 + bx + c$ containing the points $(-1, \alpha), (0, \beta), (1, \gamma)$.

True **False** Not enough information

5. You are modeling data that has 100 points, $(x_1, y_1), \ldots, (x_{100}, y_{100})$. The x_i are all distinct. A colleague suggests you fit them to a degree m polynomial:

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0.$$

[1 point] (a) What is the smallest m can be to ensure that the polynomial contains all 100 points?

Solution: $m \geq 99$.

[4 points] (b) Explain why m cannot be smaller than your answer in part (a). If you use a theorem explain why you can use it, if you argue without the theorem than justify your answer.

Solution: It's a theorem that: *If $m < n$ then $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$ do not span $\mathbb{R}^n$.* Finding the coefficients of our polynomial corresponds to a system that has m vectors in $\mathbb{R}^{100}$:

$$\left[\begin{array}{cccc:c} x_1^m & x_1^{m-1} & \cdots & x_1 & y_1 \\ x_2^m & x_2^{m-1} & \cdots & x_2 & y_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_{100}^m & x_{100}^{m-1} & \cdots & x_{100} & y_{100} \end{array}\right]$$

When $m < 100$, by the theorem, the vectors do not span $\mathbb{R}^{100}$, thus there are solutions vectors $\vec{y}$ for which there are no solutions.

6. In this question, the symbol $*$ indicates that any number might go in that spot.

[2 points] (a)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vec{x} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

Check off all possibilities for the number of solutions to this equation:

☑ No solutions
☑ One unique solution
☑ Infinitely many solutions

[2 points] (b)

$$x_1 \begin{bmatrix} * \\ * \end{bmatrix} + x_2 \begin{bmatrix} * \\ * \end{bmatrix} + x_3 \begin{bmatrix} * \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$$

Check off all possibilities for the number of solutions to this equation:

☑ No solutions
☐ One unique solution
☑ Infinitely many solutions

[2 points] (c)

$$x_1 \begin{bmatrix} * \\ * \\ * \end{bmatrix} + x_2 \begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check off all possibilities for the number of solutions to this equation:

☐ No solutions
☑ One unique solution
☑ Infinitely many solutions

[2 points] (d)

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check off all possibilities for the number of solutions to this equation:

☐ No solutions
☐ One unique solution
☑ Infinitely many solutions

7. Emily is designing a computer component and needs to know the equation for the line at the intersection of two planes. The first plane has equation $x + 2y - tz = 1$, the second plane has equation $x + 2y + 2z = 0$. Here t is an unknown parameter that Emily can change.

[2 points] (a) Write a matrix corresponding to a system of equations for a point (a, b, c) contained in both planes.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -t & 1 \\ 1 & 2 & 2 & 0 \end{array}\right)$$

[2 points] (b) After putting into a matrix and row-reducing her answer from the previous part Emily obtains:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -\frac{t}{t+2} + 1 \\ 0 & 0 & 1 & -\frac{1}{t+2} \end{array}\right).$$

What value of t ensures that $c = 1$?

Solution: $\frac{-1}{t+2} = 1$ so $t = -3$.

[2 points] (c) For the value of t in the previous part, write **two points** on the line in the intersection of the two planes.

Solution: When $t = -3$ the solution set is $(-2 - 2b, b, 1)$. Set $b = 0$ and $b = 1$ to obtain two points:

$$(-2, 0, 1) \text{ and } (-4, 1, 1).$$

[2 points] (d) For the value of t used in the previous two parts, write the equation of the line at the intersection of the two planes in the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + s\vec{v}$ where s is any real number, $\vec{v}$ is a vector in the direction of the line, and (a, b, c) is a point on the line.

Solution: A point on the line is $(-4, 1, 1)$. A vector on the line is

$$\vec{v} = \begin{bmatrix} -4 - (-2) \\ 1 - 0 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

So the line equation is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + s\vec{v}$.