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Analysis and finite element approximations of stochastic optimal control problems constrained by stochastic elliptic partial differential equations

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DEDICATION

I dedicate this thesis to the memory of my father, Jonghak Lee.

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ABSTRACT

In this thesis we study mathematically and computationally optimal control problems for stochastic elliptic partial differential equations. The control objective is to minimize the expectation of a tracking cost functional, and the control is of the deterministic, distributed type. The main analytical tool is the Wiener-Itô chaos or the Karhunen-Loève expansion. Mathematically, we prove the existence of an optimal solution; we establish the validity of the Lagrange multiplier rule and obtain a stochastic optimality system of equations; we represent the stochastic functions in their Wiener-Itô chaos expansions and deduce the deterministic optimality system of equations. Computationally, we approximate the optimality system through the discretizations of the probability space and the spatial space by the finite element method; we also derive error estimates in terms of both types of discretizations. Finally, we present some results of numerical experiments.

CHAPTER 1. INTRODUCTION

Rapid advances in computing technology in recent years have made it possible to obtain highly accurate numerical solutions for certain partial differential equations (PDEs). Thus, if we have an exact physical model and we can discretize the model precisely, then the solutions from numerical simulation must show us the physical phenomenon. That is, if we do not have any discretization errors in computation and if we assume that the modeling error is insignificant, then the output of our computational analysis should be what we expect in natural phenomenon. Therefore, if there remains a difference between simulation and observation when we solve deterministic PDEs numerically, then that must arise from uncertainty in our input data. For this reason, we use random variables to express uncertainty in our input data and reformulate traditional deterministic PDEs as stochastic partial differential equations (SPDEs) and then analyze new partial differential equations with uncertainty.

In fact, many physical and engineering models involve uncertain data or uncertain parameters; i.e., many realistic models are SPDEs. In the last decade, deterministic elliptic PDEs have been reformulated into stochastic elliptic PDEs based on the Karhunen-Loéve (K-L) expansion and there has been much progress in both the analysis and the finite element approximations for stochastic elliptic PDEs; see e.g., Deb, M. K., Babuska, I., and Oden, J. T. (2001), Babuska, I. and Chatzipantelidis, P. (2002), Babuska, I., Liu, K., and Tempone, R. (2003), Schwab, C. and Todor, R. A. (2003), Babuska, I., Tempone, R., and Zouraris, G. E. (2004), Babuska, I., Tempone, R., and Zouraris, G. E. (2005), and Frauenfelder, P., Schwab, C., and Todor, R. A. (2005).

In this thesis we consider first the following stochastic elliptic PDE:

$$\begin{aligned} -\text{div}\,[a(x,\omega)\nabla u(x,\omega)] &= f(x) \quad \text{in } D, \\ u(x,\omega) &= 0 \quad \text{on } \partial D, \end{aligned} \tag{1.0.1}$$

where $D \subset \mathbb{R}^d$ is a convex bounded polygonal domain, $a : D \times \Omega \to \mathbb{R}$ is a stochastic function, $u : \overline{D} \times \Omega \to \mathbb{R}$ is the unknown stochastic function, and $f \in L^2(D)$. Note that our stochastic elliptic PDE is based on the Wiener-Itô (W-I) chaos expansion instead of the K-L expansion in the literature and that we establish an error estimate on the solution under weaker regularity requirement in the spatial domain than in the literature. We then talk about the constrained minimization problem: find the minimizer of

$$\mathcal{J}_\beta(u,f) = E\left(\frac{1}{2}\int_D |u-U|^2\,dx + \frac{\beta}{2}\int_D |f|^2\,dx\right)$$

subject to the above stochastic elliptic PDE, where $U : \overline{D} \times \Omega \to \mathbb{R}$ is given as a target solution, β is a positive constant that measures the importance of the two terms in the above functional $\mathcal{J}_\beta(u,f)$, and f is a deterministic control function.

Briefly, solving a stochastic optimal control problem involves finding the solution of a constraint equation with flexible input data (the control) such that a certain objective function of the solution is minimized. To find the optimal solution of a minimization problem, we here derive the optimality system of equations by using the method of Lagrange multipliers and use the finite element methods to solve that system of equations. Usually, the optimality systems arising from an optimal control problem are coupled, which is the main difficulty in establishing the error estimates for the solution of the optimal control problem. Here, we use the crucial technique called the theory of Brezzi-Rappaz-Raviart (B-R-R), which plays an important role in uncoupling the optimality system of equations. For an abstract framework for using the B-R-R theory in deterministic optimal control problems, we refer the reader to Gunzburger, M. D. and Hou, L. S. (1996) and for applications, Gunzburger, M. D. , Hou, L. S. , and Svobodny,

T. (1991), Gunzburger, M. D. , Hou, L. S. , and Svobodny, T. (1991), and Hou, L. S. and Lavindran, S. S. (1998).

In our stochastic optimal control problems, we use SPDEs as constraint equations that involve stochastic functions. For this reason, we first talk about a stochastic function as our input data in Chapter 2. The plan of this chapter is as follows. In Section 2.1, we express our stochastic function using the Wiener-Itô chaos expansion. In Section 2.2, we derive the Karhunen-Loève expansion of our coefficient $a(x, \omega)$. In Section 2.3, we compare the Wiener-Itô chaos expansion with the Karhunen-Loève expansion.

Once we know how to represent uncertainty in our SPDE constraint as an infinite sum of deterministic functions and random variables, we are ready to analyze the SPDE. In Chapter 3 we study this constraint equation, a linear stochastic elliptic PDE, in detail. Chapter 3 is organized as follows. In Section 3.1, we first introduce our model problem, the original linear stochastic elliptic PDEs. Second, we define stochastic function spaces and give some notation. Finally, we show the existence and uniqueness of the solution of elliptic boundary problems. To use finite element techniques, we need to transform our SPDE constraint into a deterministic PDE constraint. For this reason, in Section 3.2 we assume that we have finite dimensional information as input data; i.e., we suppose that our coefficient $a(s, \omega)$ can be expressed as a finite sum of deterministic functions and random variables (in fact, the source of randomness in realistic models can be expanded by a finite number of random variables and, hence, this assumption is reasonable). We then transform the SPDE into the high-dimensional deterministic PDE under appropriate assumptions. Second, we introduce other function spaces that are used for high-dimensional deterministic problems. Third, we study finite element spaces on both a deterministic domain and a stochastic domain. Finally, we look for the finite element approximation of the solution of the high-dimensional deterministic elliptic PDE and derive the error estimates for the solution of our constraint equation.

After analyzing a linear elliptic equation in Chapter 4, we study stochastic optimal control problems. That is, we consider minimization problems constrained by the SPDE. In Section 4.1, we first introduce a stochastic objective functional of the solution of our constraint equation. We then prove the existence and uniqueness of the solution of our constrained minimization problem. In Section 4.2, to prepare for using the method of Lagrange multipliers for solving constrained minimization problems, we first show the existence of a Lagrange multiplier. We then use Lagrange's method to derive the stochastic optimality system of equations. After finding the optimality system, we need to study discrete approximations for the optimality system of equations and present the error estimates for the solution of the optimality system. For this, in Section 4.3 we note that the B-R-R theory can be used for finding the error estimate for the solution of the optimality system from the error estimate for a high-dimensional, deterministic PDE. We then prove the assumptions we have made to find the error estimates for the optimality system of equations.

In Chapter 5, we give some computational results. We use finite element techniques to calculate SPDEs and minimization problems constrained by SPDEs.

CHAPTER 2. REPRESENTATION OF STOCHASTIC FUNCTIONS

To use the finite element method, it is important to represent stochastic functions by a countable set of deterministic functions and random variables . We first construct the Wiener-Itô (W-I) chaos expansion of stochastic functions; e.g., see Holden, H., Øksendal, B., Ubøe, J., and Zhang, T. (1996) and Melnikova, I. V., Filinkov, A. I., and Anufrieva, U. A. (2003). We then derive the Karhunen-Loève (K-L) expansion of stochastic functions; see Ghanem, R. G. and Spanos, P. D. (1991), Deb, M. K., Babuska, I., and Oden, J. T. (2001), Babuska, I. and Chatzipantelidis, P. (2002), Babuska, I., Liu, K., and Tempone, R. (2003), Babuska, I., Tempone, R., and Zouraris, G. E. (2004), Babuska, I., Tempone, R., and Zouraris, G. E. (2005), Frauenfelder, P., Schwab, C., and Todor, R. A. (2005), and Luo, W. (2006). Finally, we compare these two expansions.

2.1 Wiener-Itô chaos expansions

In this section, we first discuss the classical W-I chaos expansion of elements of the space of square-integrable functions defined on the space of tempered distributions in terms of stochastic Hermite polynomials. Then we represent stochastic functions by a countably infinite sum of the products of deterministic functions and random variables.

Let d be a fixed positive number. Consider the Schwartz space of rapidly decreasing smooth real valued functions on $\mathbb{R}^d$, which we denote by $S(\mathbb{R}^d)$; see e.g., Holden, H., Øksendal, B., Ubøe, J., and Zhang, T. (1996). We consider the dual space $S^*(\mathbb{R}^d)$ of

$S(\mathbb{R}^d)$, which is called the space of tempered distributions. We let $\Omega := S^*(\mathbb{R}^d)$ and denote the Borel σ-algebra over Ω by $\mathcal{F}$. Then there is a unique probability measure P on $\mathcal{F}$ satisfying the condition that

$$Ee^{i\langle\omega,\phi\rangle} := \int_\Omega e^{i\langle\omega,\phi\rangle}\, dP(\omega) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^d)}\right) \quad \forall\phi \in S(\mathbb{R}^d), \tag{2.1.1}$$

where $\langle\omega,\phi\rangle = \omega(\phi)$ is the action of $\omega \in \Omega$ on $\phi \in S(\mathbb{R}^d)$ (The Bochner-Minlos theorem). Here, P is called the white noise measure or the Gaussian measure on $S(\mathbb{R}^d)$.

Recall that the Hermite polynomials $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}); \quad n = 0, 1, 2, \cdots . \tag{2.1.2}$$

Define the Hermite functions $\xi_n(x)$ as follows:

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2}e^{-x^2/2}h_{n-1}(x); \quad n = 1, 2, 3, \cdots . \tag{2.1.3}$$

Observe that the Hermite functions are orthogonal with the weight $e^{-x^2/2}$ and are in $S(\mathbb{R})$ for all n. Note that $\{\xi_n\}_{n=1}^\infty$ constitutes an orthonormal basis for $L^2(\mathbb{R})$.

Let $\delta^j = (\delta_1^j, \delta_2^j, \cdots, \delta_d^j)$, where $\delta_i^j \in \mathbb{N}$, and define the tensor products

$$\xi_{\delta^j} := \xi_{\delta_1^j} \otimes \xi_{\delta_2^j} \otimes \cdots \otimes \xi_{\delta_d^j}; \quad j = 1, 2, 3, \cdots ,$$

where

$$i < j \Rightarrow \delta_1^i + \delta_2^i + \cdots + \delta_d^i \le \delta_1^j + \delta_2^j + \cdots + \delta_d^j. \tag{2.1.4}$$

It follows that the family of tensor products $\{\xi_{\delta^j}\}_{j=1}^\infty$ forms an orthogonal basis for $L^2(\mathbb{R}^d)$.

Let us define

$$\mathcal{I} = \{\alpha = (\alpha_1, \alpha_2, \cdots) \mid \alpha_i \in \mathbb{N} \cup \{0\} \text{ and there are only finitely many } \alpha_i \ne 0\}. \tag{2.1.5}$$

Note that the index set $\mathcal{I}$ is countable.

We define stochastic Hermite polynomials $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ by

$$H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \xi_{\delta^i}\rangle); \quad \omega \in \Omega. \tag{2.1.6}$$

It follows that $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ forms an orthogonal basis for $L^2(\Omega)$. The norm $\|H_\alpha\|$ satisfies

$$\|H_\alpha\|^2_{L^2(\Omega)} = \alpha! = \alpha_1!\alpha_2! \cdots .$$

Now we redefine H_α by $\frac{1}{\sqrt{\alpha!}}H_\alpha$. Then $\{H_\alpha\}_{\alpha\in\mathcal{I}}$ forms an orthonormal basis for $L^2(\Omega)$. Hence every $f(\omega) \in L^2(\Omega)$ has a unique expansion

$$f(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega),$$

where

$$c_\alpha = E(f(\omega)H_\alpha(\omega)) = \int_\Omega f(\omega)H_\alpha(\omega)\ dP(\omega). \tag{2.1.7}$$

We call this expansion the W-I chaos expansion of a function $f \in L^2(\Omega)$ in terms of stochastic Hermite polynomials.

From the above argument, for a stochastic function $f(x,\omega)$ ($f(x,\cdot) \in L^2(\Omega)$ a.e. x), we have

$$f(x,\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha(x)H_\alpha(\omega), \tag{2.1.8}$$

where

$$c_\alpha(x) = E(f(x,\omega)H_\alpha(\omega)).$$

This expansion is called the W-I chaos expansion of a stochastic function $f(x,\omega)$.

Because our index set $\mathcal{I}$ is countable, for $n \in \mathbb{N}$, we may rewrite $f(x,\omega)$ as

$$f(x,\omega) = \sum_{n\geq 1} c_n(x)H_n(\omega), \tag{2.1.9}$$

where

$$c_n(x) = E(f(x,\omega)H_n(\omega)).$$

We will discuss later the SPDEs based on this expansion and a stochastic optimal control problem constrained by those SPDEs.

2.2 Karhunen-Loève expansions

In this section, we consider the K-L expansion, which is well known as a theoretical tool for approximating stochastic functions.

Recall that the coefficient $a(x, \omega)$ is an input datum for our SPDE; this coefficient is a stochastic function from $D \times \Omega$ to $\mathbb{R}$. We assume that $a(x, \omega)$ has a continuous and bounded covariance function. Here, the covariance function $C(x_1, x_2)$ is given by

$$C(x_1, x_2) = E(a(x_1, \omega)a(x_2, \omega)) - Ea(x_1, \omega)Ea(x_2, \omega). \qquad (2.2.10)$$

Remark 2.2.1 *The covariance function is zero if the coefficient is constant and is strictly positive definite otherwise. Also, clearly, it is symmetric.*

Because the covariance function is bounded, positive definite, and symmetric (see Loeve, M. (1978)), we have the following result (see Courant, R. and Hilbert, D. (1953)).

Theorem 2.2.2 *The covariance function has the following decomposition:*

$$C(x_1, x_2) = \sum_{n \geq 0} \lambda_n \phi_n(x_1)\phi_n(x_2),$$

where the eigenpairs $(\lambda_n, \phi_n(x))$ *are the solution to the integral equation*

$$\int_D C(x_1, x_2)\phi_n(x_1)\, dx_1 = \lambda_n \phi_n(x_2) \qquad (2.2.11)$$

and the eigenfunctions $\{\phi_n(x)\}$ *are orthogonal and form a complete set.*

We clearly see that $a(x, \omega)$ can be written as

$$a(x, \omega) = \bar{a}(x) + a_0(x, \omega),$$

where $\bar{a}(x) = Ea(x, \omega)$ and $a_0(x, \omega)$ is a stochastic function with zero mean and covariance function $C(x_1, x_2)$. This covariance function is the same as for the $a(x, \omega)$'s. Note that $a_0(x, \omega)$ can be expanded in terms of the eigenfunctions $\{\phi_n(x)\}$ as

$$a_0(x, \omega) = \sum_{n \geq 0} \sqrt{\lambda_n}\phi_n(x)X_n(\omega), \qquad (2.2.12)$$

where $(\lambda_n, \phi_n(x))$ are eigenpairs of (2.2.11).

We now claim that random variables in (2.2.12) are orthogonal. From (2.2.10) and (2.2.12), we have

$$C(x_1, x_2) = \sum_{n,m\geq 0} \sqrt{\lambda_n \lambda_m} \phi_n(x_1)\phi_m(x_2) E(X_n(\omega)X_m(\omega)). \tag{2.2.13}$$

By substituting (2.2.13) into (2.2.11) and by using the orthogonality of eigenfunctions, we have

$$\lambda_k \phi_k(x_1) = \sum_{n\geq 0} \sqrt{\lambda_n \lambda_k} \phi_n(x_1) E(X_n(\omega)X_k(\omega)). \tag{2.2.14}$$

If we multiply (2.2.14) by $\phi_l(x_1)$ and integrate over the deterministic domain D we have, again by orthogonality of eigenfunctions,

$$\lambda_k \delta_{lk} = \sqrt{\lambda_l \lambda_k} E(X_l(\omega)X_k(\omega)). \tag{2.2.15}$$

The last equation implies that

$$E(X_l(\omega)X_k(\omega)) = \delta_{lk},$$

as we desired.

We now define the K-L expansion of a stochastic function $a(x, \omega)$ that has continuous and bounded covariance function $C(x_1, x_2)$.

Definition 2.2.3 *If $a(x, \omega)$ is a stochastic function, as defined above, it can be represented by*

$$a(x, \omega) = \bar{a}(x) + \sum_{n\geq 0} \sqrt{\lambda_n} \phi_n(x) X_n(\omega), \tag{2.2.16}$$

where $\bar{a}(x) = Ea(x, \omega)$, $EX_n(\omega) = 0$, $E(X_n(\omega)X_m(\omega)) = \delta_{nm}$, and $(\lambda_n, \phi_n(x))$ are the solution to the eigenvalue problem (2.2.11). *We call this expansion the K-L expansion of $a(x, \omega)$.*

Remark 2.2.4 *By multiplying* (2.2.12) *by $\phi_n(x)$ and integrating over the deterministic domain D, we can find an explicit expression of $X_n(\omega)$:*

$$X_n(\omega) = \lambda_n^{-1/2} \int_D \alpha(x, \omega) \phi_n(x) \, dx.$$

2.3 Wiener-Itô chaos expansions versus Karhunen-Loève expansions

From the orthogonality relations for Hermite polynomials with respect to the Gaussian measure, for redefined H_α's, we see that

$$E(H_\alpha H_\beta) = \delta_{\alpha\beta}$$

for any multi-indices α and β; see Holden, H., Øksendal, B., Ubøe, J., and Zhang, T. (1996).

Also, by using the property of an even function $e^{-\frac{1}{2}x^2}$ with respect to the Gaussian measure, we have

$$EH_\alpha = 0$$

for any multi index α.

Thus, the variance of H_α equals 1 and the covariance between H_α and H_β is equal to zero; i.e., for any $\alpha \neq \beta$, we obtain

$$E(H_\alpha H_\beta) = EH_\alpha EH_\beta.$$

If we choose the orthonormal eigenfunction in the K-L expansion as a basis for $L^2(D)$ and assume that $c_n(x) \in L^2(D)$ in (2.1.9), then $c_n(x)$ can be represented by the infinite sum of orthonormal basis functions for $L^2(D)$. Thus, from the definition of the K-L expansion and all the above properties of stochastic Hermite polynomials, we can think of the W-I chaos expansion of a stochastic function as a special case of the K-L expansion of it in terms of their random variables.

In this thesis, we focus on SPDEs based on the W-I chaos expansion consisting of known stochastic Hermite polynomials. Note that although we only study our stochastic optimal control problems with SPDE constraint equations based on the W-I chaos expansion, problems with the K-L expansion can be treated similarly and analysis of those can be obtained very easily from our results.

CHAPTER 3. STOCHASTIC ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we study a linear stochastic elliptic PDE that is a constraint equation of our main problem, a stochastic optimal control problem. We first prove the existence and uniqueness of the solution of a linear stochastic elliptic PDE. We then turn the original stochastic elliptic PDE into a high-dimensional deterministic PDE under appropriate assumptions. After that, we consider the finite element approximations of the solution of the resulting high-dimensional deterministic problem. Finally we establish the error estimates on the solution of elliptic equations (cf. see Babuska, I., Tempone, R., and Zouraris, G. E. (2004)).

3.1 The original stochastic elliptic partial differential equation

3.1.1 The model problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω is a set of outcomes, $\mathcal{F}$ is a σ-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure.

We consider the following stochastic linear elliptic boundary value problem: find a stochastic function $u : \overline{D} \times \Omega \to \mathbb{R}$ such that the following equation holds for almost every $\omega \in \Omega$ (or almost surely (a.s.)):

$$\begin{aligned} -\mathrm{div}\,[a(x,\omega)\nabla u(x,\omega)] &= f(x) \quad \text{in } D, \\ u(x,\omega) &= 0 \quad \text{on } \partial D, \end{aligned} \tag{3.1.1}$$

where $D \subset \mathbb{R}^d$ is a convex bounded polygonal domain, $a : D \times \Omega \to \mathbb{R}$ is a stochastic function with a continuous and bounded covariance function, and $f \in L^2(D)$ is a distributed deterministic control. Note that in this thesis, ∇ means differentiation with respect to $x \in D$ only.

To ensure the existence and uniqueness of the solution of the linear stochastic elliptic PDE, we assume that there are $m, M \in (0, \infty)$ such that

$$m \le a(x, \omega) \le M \quad \text{a.e. } (x, \omega) \in D \times \Omega. \tag{3.1.2}$$

We use random variables to express uncertainty in our input data $a(x, \omega)$ and reformulate traditional elliptic deterministic PDEs as stochastic elliptic PDEs. It is well known that SPDEs are used to model physical phenomenon; e.g., fluid flows in porous media, random vibration, seismic activity, oil reservoirs, and composite materials.

3.1.2 Stochastic function spaces and notations

Throughout this thesis, we use standard notations (e.g., see Adams, R. (1975)) for the Sobolev spaces $H^r(D)$ for each real number r with norms $\|\cdot\|_{H^r(D)}$. We use $H^r_0(D)$ as the subspace of $H^r(D)$ whose function value is zero on the boundary of D; e.g., $H^1_0(D)$ is the subspace of $H^1(D)$ with the zero boundary condition and the norm $\|u\|^2_{H^1_0(D)} = \int_D |\nabla u|^2 \, dx$. For a Hilbert space H^r, we denote the inner product on H^r by $(\cdot, \cdot)_{H^r}$. Also, C denotes a generic constant whose value may change with context.

Let X be an $\mathbb{R}^n$-valued random variable in a probability space $(\Omega, \mathcal{F}, P)$. If $X \in L^1_P(\Omega)$, then we define $EX = \int_\Omega X(\omega) \, dP(\omega)$ as its expected value.

Remark 3.1.1 *If the distribution of an $\mathbb{R}^n$-valued random variable X admits a density function $\rho(x)$, then the expected value of X can be computed as $EX = \int_{\mathbb{R}^n} x\rho(x) \, dx$.*

It is natural to think of the following theorem because we defined EX as an integral.

Theorem 3.1.2 *Let $a, b \in \mathbb{R}$. Suppose that random variables $X, Y \geq 0$ or $E|X|, E|Y| < \infty$. Then we have*

a) $E(X+Y) = EX + EY$

b) $E(aX+b) = aEX + b$

c) $EX \leq EY$ *if* $X \leq Y$

PROOF: The proof follows from the fact that $EX = \int_\Omega X \; dP$ is an integral.

Now we are ready to define the stochastic Sobolev spaces

$$L^2(\Omega; H^r(D)) = \{v : D \times \Omega \to \mathbb{R} \mid v \text{ is strongly measurable and } \|v\|_{L^2(\Omega;H^r(D))} < \infty\},$$

where

$$\|v\|^2_{L^2(\Omega;H^r(D))} = \int_\Omega \|v\|^2_{H^r(D)} \; dP = E\|v\|^2_{H^r(D)}.$$

For example,

$$L^2(\Omega; H^1_0(D)) = \{v : D \times \Omega \to \mathbb{R} \mid v \text{ is strongly measurable and } E\int_D |\nabla v|^2 \; dx < \infty\}.$$

Note that the stochastic Sobolev space $L^2(\Omega; H^r(D))$ is a Hilbert space and is isomorphic to the tensor product Hilbert space $H^r(D) \otimes L^2(\Omega)$; see Babuska, I., Tempone, R., and Zouraris, G. E. (2004). For instance, $L^2(\Omega; H^1_0(D))$ is a Hilbert space endowed with the inner product

$$(u, v)_{L^2(\Omega;H^1_0(D))} = E\int_D \nabla u \cdot \nabla v \; dx$$

and is isomorphic to $H^1_0(D) \otimes L^2(\Omega)$. In fact, we shall use a tensor product Hilbert space to analyze finite element approximations of the solutions of our elliptic problems because the basic nature of a stochastic function with respect to x and with respect to ω is different.

For simplicity, we put

$$\mathcal{H}^r(D) = L^2(\Omega; H^r(D)).$$

For instance,

$$\mathcal{L}^2(D) = L^2(\Omega; L^2(D)),$$

$$\mathcal{H}^1(D) = \{v \in \mathcal{L}^2(D) \mid E\|v\|^2_{H^1(D)} < \infty\},$$

and

$$\mathcal{H}^1_0(D) = \{v \in \mathcal{H}^1(D) \mid v = 0 \quad \text{on } \partial D \ a.s.\}.$$

We introduce the notations

$$b[u, v] = E \int_D a \nabla u \cdot \nabla v \, dx$$

and

$$[u, v] = E \int_D uv \, dx.$$

3.1.3 The existence and uniqueness of the solution of the stochastic elliptic partial differential equation

We have a weak formulation of (3.1.1) as follows: seek $u \in \mathcal{H}^1_0(D)$ such that

$$b[u, v] = [f, v] \quad \forall v \in \mathcal{H}^1_0(D). \tag{3.1.3}$$

We now state the existence and uniqueness for the solution to (3.1.3).

Theorem 3.1.3 *Let $f \in L^2(D)$. Then there exists a unique weak solution for* (3.1.1) *in $\mathcal{H}^1_0(D)$.*

Proof: From (3.1.2), we have

$$|b[u, v]| \leq M \|u\|_{\mathcal{H}^1_0(D)} \|v\|_{\mathcal{H}^1_0(D)} \quad \forall u, v \in \mathcal{H}^1_0(D)$$

and

$$m \|v\|^2_{\mathcal{H}^1_0(D)} \leq b[v, v] \quad \forall v \in \mathcal{H}^1_0(D).$$

On the other hand, we can easily see that

$$[f, v] \leq \|f\|_{L^2(D)} \|v\|_{\mathcal{L}^2(D)} < \infty.$$

for any $v \in \mathcal{H}_0^1(D)$. Hence, by the Lax-Milgram lemma (cf. Brenner, S. C. and Scott, L. R. (2002)), (3.1.3) has a unique solution. □

3.2 The high dimensional elliptic partial differential equation

3.2.1 The model problem with finite dimensional information

In this section, we consider (3.1.1) and the following.

$$\begin{aligned} -\text{div}\,[a(x,\omega)\nabla u(x,\omega)] = g(x,\omega) \quad \text{in } D, \\ u(x,\omega) = 0 \quad \text{on } \partial D, \end{aligned} \tag{3.2.4}$$

where $g : D \times \Omega \to \mathbb{R}$ is a stochastic function. Note that we have a unique weak solution for (3.2.4) by the Lax-Milgram theorem and that the error estimates for the solutions for (3.1.1) and (3.2.4) will be similar and both will be used later for the error estimates for the optimality system of equations.

In realistic models, the source of randomness can be expressed by a finite number of random variables that are mutually uncorrelated or mutually independent. For that reason, we assume that

$$a(x,\omega) = \sum_{n=1}^{N} c_n(x) H_n(\omega) \tag{3.2.5}$$

and

$$g(x,\omega) = g(x, H_1(\omega), H_2(\omega), \cdots, H_N(\omega)). \tag{3.2.6}$$

For the existence of the solution for stochastic elliptic PDEs with finite dimensional information, it is necessary that $\sum_{n=1}^{N} c_n(x) H_n(\omega)$ satisfy a similar condition to (3.1.2); i.e., there exist $m, M > 0$ such that

$$m \leq \sum_{n=1}^{N} c_n(x) H_n(\omega) \leq M \qquad \text{a.e. } (x,\omega) \in D \times \Omega. \tag{3.2.7}$$

We also assume that each $H_n(\Omega) \equiv \Gamma_n \subset \mathbb{R}$ is a bounded interval for $n = 1, 2, \cdots, N$ and that each H_n has a density function $\rho_n : \Gamma_n \to \mathbb{R}^+$. We use the joint density $\rho(y)$ of $(H_1, H_2, \cdots, H_N)$ for any $y \in \Gamma \equiv \prod_{n=1}^{N} \Gamma_n \subset \mathbb{R}^N$. Under these assumptions, the solution of (3.1.3) can be expressed by the finite number of random variables; i.e., $u(x, \omega) = u(x, H_1(\omega), H_2(\omega), \cdots, H_N(\omega))$; see e.g., Deb, M. K., Babuska, I., and Oden, J. T. (2001), Babuska, I. and Chatzipantelidis, P. (2002), and Babuska, I., Tempone, R., and Zouraris, G. E. (2004).

Remark 3.2.1 *Notice that* $H := (H_1, H_2, \cdots, H_N)$ *is a* Γ*-valued random variable. Thus, for* $u \in \mathcal{H}_0^1(D)$*, we have*

$$E \int_D u \; dx = \int_\Omega \int_D u(x, \omega) \; dx dP(\omega) = \int_\Gamma \rho(y) \int_D u(x, y) \; dx dy.$$

Under the above assumptions, we also have the following high-dimensional deterministic equivalent variational formulation of (3.1.3) with finite dimensional information:

$$\int_\Gamma \rho(y) \int_D a(x, y) \nabla u(x, y) \cdot \nabla v(x, y) \; dx dy = \int_\Gamma \rho(y) \int_D f(x) v(x, y) \; dx dy. \tag{3.2.8}$$

The strong formulation of this is

$$\begin{aligned} -\text{div}\,[a(x, y)\nabla u(x, y)] &= f(x) \quad \forall (x, y) \in D \times \Gamma, \\ u(x, y) &= 0 \quad \forall (x, y) \in \partial D \times \Gamma. \end{aligned} \tag{3.2.9}$$

Remark 3.2.2 *First, from* (3.2.7)*, we have well-posedness of* (3.2.9)*. Second, the solution of SPDEs can be found by solving deterministic PDEs. Third, the finite element method can be used for stochastic problems. Finally, with* $g(x, \omega)$ *and* $g(x, y)$ *instead of* $f(x)$*, we obtain the same result.*

3.2.2 Function spaces

We introduce Sobolev spaces for the high-dimensional elliptic PDE as follows:

$$L^2(\Gamma; H^r(D)) = \{v : D \times \Gamma \to \mathbb{R} \mid v \text{ is strongly measurable and } \|v\|_{L^2(\Gamma; H^r(D))} < \infty\},$$

where

$$\|v\|^2_{L^2(\Gamma;H^r(D))} = \int_\Gamma \rho \; \|v\|^2_{H^r(D)} \; dy = E\|v\|^2_{H^r(D)}.$$

For example, we have

$$L^2(\Gamma; H^1_0(D)) = \{v : D{\times}\Gamma \to \mathbb{R} \mid v \text{ is strongly measurable and } \int_\Gamma \rho \int_D |\nabla v|^2 \; dxdy < \infty\}.$$

Note that the above Sobolev space is a Hilbert space endowed with the inner product

$$(u, v)_{L^2(\Gamma;H^1_0(D))} = \int_\Gamma \rho \int_D \nabla u \cdot \nabla v \; dxdy$$

and is equivalent to $L^2(\Omega; H^1_0(D))$; see Babuska, I. and Chatzipantelidis, P. (2002).

We now give the Banach spaces that will be used as solution spaces for the solution of the optimality system of equations.

For $r = -1, 0, 1$, define

$$S^{p,r}(D) = C^p(\Gamma; H^r(D)),$$

with the norm $\|u\|_{S^{p,r}(D)} = \|u\|_{S^{0,r}(D)} + \sum_{j=1}^N \sum_{k=1}^{p_j} \|\partial^k_{y_j} u\|_{S^{0,r}(D)}$, where

$$\|u\|_{S^{0,r}(D)} = \sup_{y\in\Gamma} \|u(\cdot, y)\|_{H^r(D)}.$$

Also define

$$S^{p,1}_0(D) = C^p(\Gamma; H^1_0(D)).$$

3.2.3 Finite element spaces

Let us first consider finite element spaces on $D \subset \mathbb{R}^d$. Let X^h and G^h be families of finite element approximation subspaces of $H^1_0(D)$ and $L^2(D)$ that consist of piecewise linear continuous functions defined over a family of regular triangulations of D with a maximum grid size parameter $h > 0$. We assume that X^h and G^h satisfy the following approximation properties:

(i) for all $\phi \in H^{\alpha+1}(D) \cap H^1_0(D)$, there exists $C > 0$ and an integer l such that

$$\inf_{\phi^h \in X^h} \|\phi - \phi^h\|_{H^1_0(D)} \le Ch^\alpha \|\phi\|_{H^{\alpha+1}(D)}, \; 0 \le \alpha \le l, \tag{3.2.10}$$

where $l \geq 1$ is usually determined by the order of the piecewise polynomials used to define X^h;

(ii) for all $\phi \in H_0^1(D)$, there exists $C > 0$ such that

$$\inf_{\phi^h \in G^h} \|\phi - \phi^h\|_{L^2(D)} \leq Ch\|\phi\|_{H_0^1(D)}. \tag{3.2.11}$$

Next, we consider finite element spaces on $\Gamma \subset \mathbb{R}^N$. We partition Γ into a finite number of disjoint $\mathbb{R}^N$ boxes B_i^N, that is, for a finite index set I, we have

$$\Gamma = \bigcup_{i \in I} B_i^N = \bigcup_{i \in I} \prod_{j=1}^{N} (a_i^j,\ b_i^j),$$

where $B_k^N \cap B_l^N = \emptyset$ for $k \neq l \in I$ and $(a_i^j,\ b_i^j) \subset \Gamma_j$.

A maximum grid size parameter $\delta > 0$ is denoted by

$$\delta = \max\{|b_i^j - a_i^j|/2 : 1 \leq j \leq N \text{ and } i \in I\}.$$

Let $Y^\delta \subset L^2(\Gamma)$ be the finite element approximation space of piecewise polynomials with degree at most p_j on each direction y_j. Thus if $\psi^\delta \in Y^\delta$, then $\psi^\delta|_{B_i^N} \in \text{span}(\prod_{j=1}^N y_j^{n_j} : n_j \in \mathbb{R} \text{ and } n_j \leq p_j)$. Letting $p = (p_1, p_2, \cdots, p_N)$, we have (cf. see Brenner, S. C. and Scott, L. R. (2002)) the following property: for all $\psi \in C^{p+1}(\Gamma)$,

$$\inf_{\psi^\delta \in Y^\delta} \|\psi - \psi^\delta\|_{C^0(\Gamma)} \leq \delta^\gamma \sum_{j=1}^{N} \frac{\|\partial_{y_j}^{p_j+1}\psi\|_{C^0(\Gamma)}}{(p_j+1)!}, \tag{3.2.12}$$

where $\gamma = \min_{1 \leq j \leq N}\{p_j + 1\}$.

We now are ready to think of tensor product finite element spaces on $D \times \Gamma$. If $V^{h\delta} \equiv X^h \otimes Y^\delta$, then if $v^{h\delta} \in V^{h\delta}$, $v^{h\delta} \in \text{span}(\phi^h\psi^\delta : \phi^h(x) \in X^h \text{ and } \psi^\delta(y) \in Y^\delta)$.

We denote by R^h the $H^1(D)$-projection from $H_0^1(D)$ onto X^h and P^δ the $L^2(\Gamma)$-projection from $L^2(\Gamma)$ onto Y^δ. Namely for each $\phi \in H_0^1(D)$,

$$(R^h\phi, \phi^h)_{H_0^1(D)} = (\phi, \phi^h)_{H_0^1(D)} \quad \forall \phi^h \in X^h;$$

for each $\psi \in L^2(\Gamma)$,

$$(P^\delta\psi, \psi^\delta)_{L^2(\Gamma)} = (\psi, \psi^\delta)_{L^2(\Gamma)} \quad \forall \psi^\delta \in Y^\delta.$$

Remark 3.2.3 *In this thesis, for all ψ_1 and $\psi_2 \in C^0(\Gamma)$, the inner product of ψ_1 and ψ_2 is defined by the $L^2(\Gamma)$-inner product.*

It follows from (3.2.10) that for all $\phi \in H_0^1(D) \cap H^{\alpha+1}$ and for some $C > 0$ we have

$$\|\phi - R^h\phi\|_{H_0^1(D)} \leq Ch^\alpha \|\phi\|_{H^{\alpha+1}(D)},\ 0 \leq \alpha \leq l \tag{3.2.13}$$

and from (3.2.12) that for all $\psi \in C^{p+1}(\Gamma)$ we obtain

$$\|\psi - P^\delta\psi\|_{C^0(\Gamma)} \leq \delta^\gamma \sum_{j=1}^N \frac{\|\partial_{y_j}^{p_j+1}\psi\|_{C^0(\Gamma)}}{(p_j+1)!}. \tag{3.2.14}$$

By using the last two inequalities, we have (cf. see Babuska, I., Tempone, R., and Zouraris, G. E. (2004)) the following property: for all $u \in C^{p+1}(\Gamma; H^{\alpha+1}(D) \cap H_0^1(D))$, there exists $C > 0$, which is independent of h, δ, N, and p, such that

$$\inf_{u^{h\delta} \in V^{h\delta}} \|u - u^{h\delta}\|_{S_0^{0,1}(D)} \leq C \left(h^\alpha \|u\|_{S^{0,\alpha+1}(D)} + \delta^\gamma \sum_{j=1}^N \frac{\|\partial_{y_j}^{p_j+1} u\|_{S_0^{0,1}(D)}}{(p_j+1)!} \right). \tag{3.2.15}$$

Note that from the definition of the space $S^{p,r}(D)$ and its norm in Section 3.2.2, we may assume that $0 < \|\partial_{y_j}^k\|_{op} < 1$ for all $1 \leq j \leq N$ and $1 \leq k \leq p_j + 1$. Also we know that for the solution u of (3.2.9), each $\partial_{y_j}^k u$ is continuous on the bounded domain Γ_j and, hence, $\partial_{y_j}^k u$ is bounded. We now assume further that for each $\partial_{y_j}^k$ there is a constant $\tilde{c} > 0$ such that $0 < \|\partial_{y_j}^k\|_{op} \leq \tilde{c} < 1$. With the help of (3.2.15), we state the approximation property for the solution:

$$\inf_{u^{h\delta} \in V^{h\delta}} \|u - u^{h\delta}\|_{S_0^{p+1,1}(D)} \leq C \left(h^\alpha \|u\|_{S^{0,\alpha+1}(D)} + \delta^\gamma \sum_{j=1}^N \frac{\|\partial_{y_j}^{p_j+1} u\|_{S_0^{0,1}(D)}}{(p_j+1)!} \right). \tag{3.2.16}$$

Remark 3.2.4 *Because $a(x,y) = \sum_{j=1}^N c_j(x) y_j \in C^{p+1}(\overline{D \times \Gamma})$, it is well known that the solution u of (3.2.9) satisfies $u \in C^{p+1}(\Gamma; H^2(D) \cap H_0^1(D))$; see e.g., Lemma 4.1 in Lagness, J. E. (1972) and Remark 5.1 in Babuska, I., Tempone, R., and Zouraris, G. E. (2005). Also, if we assume that $g(x,y) \in S^{p+1,0}(D)$, then the solution u of the* *following problem also satisfies* $u \in C^{p+1}(\Gamma; H^2(D) \cap H_0^1(D))$:

$$-\text{div}\,[a(x,y)\nabla u(x,y)] = g(x,y) \quad \forall (x,y) \in D \times \Gamma,$$
$$u(x,y) = 0 \quad \forall (x,y) \in \partial D \times \Gamma. \tag{3.2.17}$$

Hence, under our assumptions, (3.2.16) *makes sense for some* α.

3.2.4 Error estimates on the solution of the high dimensional elliptic problem

Recall that our goal is to solve the high-dimensional deterministic problem (3.2.9). The weak formulation of (3.2.9) is as follows: seek $u \in S_0^{p+1,1}(D)$ such that for all $v \in S_0^{p+1,1}(D)$,

$$b[u,v] = [f,v]. \tag{3.2.18}$$

Then we have the finite element weak formulation: find $u^{h\delta} \in V^{h\delta}$ such that

$$b[u^{h\delta}, v^{h\delta}] = [f, v^{h\delta}] \tag{3.2.19}$$

for all $v^{h\delta} \in V^{h\delta}$.

Our goal in this section is to estimate the error between solutions for (3.2.18) and (3.2.19) in $S_0^{p+1,1}(D)$. We note that our discrete error estimates on the solution will be obtained under weaker regularity requirements in the spatial domain than in the literature. Also, we do the same thing with a finite data $g(x,y)$ instead of $f(x)$. For these, we need the following lemmas.

Lemma 3.2.5 *Let* $\epsilon > 0$ *and* $f(x) \in H^{-1+\epsilon}(D)$. *Then for any* $y \in \Gamma$, $u(\cdot, y) \in H^{1+\epsilon}(D)$ *and there exists* $C > 0$ *such that*

$$\|u(\cdot,y)\|_{H^{1+\epsilon}(D)} \leq C\|f\|_{H^{-1+\epsilon}(D)}.$$

Proof: This follows from a regularity theorem (see Evans, L. C. (1998)) for the solution of the elliptic equation and interpolation theorem (see Lions, J. L. and Magenes, E. (1972)).

Remark 3.2.6 *For problems with $g(\cdot, y) \in H^{-1+\epsilon}(D)$, we have*

$$\|u(\cdot, y)\|_{H^{1+\epsilon}(D)} \leq C\|g(\cdot, y)\|_{H^{-1+\epsilon}(D)}.$$

Lemma 3.2.7 *Let $\epsilon > 0$, $f(x) \in H^{-1+\epsilon}(D)$, $u \in S_0^{p+1,1}(D)$, and $c_j(x) \in L^\infty(D)$. Then for all $j = 1, 2, \cdots, N$ and for any $y \in \Gamma$, there exists $C > 0$ such that*

$$\frac{\|\partial_{y_j}^{p_j+1} u(\cdot, y)\|_{H_0^1(D)}}{(p_j+1)!} \leq C\|c_j\|_{L^\infty(D)}^{p_j+1}\|f\|_{H^{-1+\epsilon}(D)}.$$

Proof: Without loss of generality, we show this for only $j = 1$. Recall that $a(x, y) = \sum_{j=1}^N c_j(x) y_j$. If we take derivatives with respect to y_1 in (3.2.9) we find

$$-\text{div}\,[c_1(x)\nabla u(x, y) + a(x, y)\nabla\partial_{y_1} u(x, y)] = 0.$$

Note that because $u(x, y) = 0$ for any $(x, y) \in \partial D \times \Gamma$, $\partial_{y_1} u(x, y) = 0$ for any $(x, y) \in \partial D \times \Gamma$. Thus, by integrating over D after multiplying by $\partial_{y_1} u$, we see that

$$\int_D c_1(x)\nabla u(x, y) \cdot \nabla\partial_{y_1} u(x, y)\, dx + \int_D a(x, y)|\nabla\partial_{y_1} u(x, y)|^2\, dx = 0.$$

This by the coercivity, implies

$$\|\partial_{y_1} u(\cdot, y)\|_{H_0^1(D)}^2 \leq C\|c_1\|_{L^\infty(D)}\|u(\cdot, y)\|_{H^1(D)}\|\partial_{y_1} u(\cdot, y)\|_{H^1(D)}.$$

From Lemma 3.2.5, we have

$$\|\partial_{y_1} u(\cdot, y)\|_{H_0^1(D)} \leq C\|c_1\|_{L^\infty(D)}\|f\|_{H^{-1+\epsilon}(D)}.$$

We now assume that the following is true:

$$\|\partial_{y_1}^{p_1} u(\cdot, y)\|_{H_0^1(D)} \leq Cp_1!\|c_1\|_{L^\infty(D)}^{p_1}\|f\|_{H^{-1+\epsilon}(D)}.$$

Taking derivatives p_1+1 times with respect to y_1 in (3.2.9), we obtain

$$-\mathrm{div}\,[(p_1+1)c_1(x)\nabla\partial_{y_1}^{p_1}u(x,y)+a(x,y)\nabla\partial_{y_1}^{p_1+1}u(x,y)]=0.$$

After multiplying by $\partial_{y_1}^{p_1+1}u$ integrating over D yields

$$(p_1+1)\int_D c_1(x)\nabla\partial_{y_1}^{p_1}u(\cdot,y)\cdot\nabla\partial_{y_1}^{p_1+1}u(\cdot,y)\;dx+\int_D a(x,y)|\nabla\partial_{y_1}^{p_1+1}u(\cdot,y)|^2\;dx=0.$$

By the coercivity, Lemma 3.2.5, and our induction hypothesis, we find

$$\|\partial_{y_1}^{p_1+1}u(\cdot,y)\|_{H_0^1(D)}\le C(p_1+1)\|c_1\|_{L^\infty(D)}(p_1!\|c_1\|_{L^\infty(D)}^{p_1}\|f\|_{H^{-1+\epsilon}(D)}).$$

Thus, the assertion for $j=1$ in Lemma 3.2.7 follows from the last inequality by induction. ∎

Remark 3.2.8 *For problems with $g(x,y)\in C^{p+1}(\Gamma;H^{-1+\epsilon}(D))$, we have*

$$\frac{\|\partial_{y_j}^{p_j+1}u(\cdot,y)\|_{H_0^1(D)}}{(p_j+1)!}\le C\sum_{k=0}^{p_j+1}\frac{1}{k!}\;\|c_j\|_{L^\infty(D)}^{p_j+1-k}\|\partial_{y_j}^k g(\cdot,y)\|_{H^{-1+\epsilon}(D)}.$$

As a consequence of (3.2.16), Lemma 3.2.5, and Lemma 3.2.7, we have the following theorem.

Theorem 3.2.9 *Let $f(x)\in H^{-1+\epsilon}(D)$, let u be the solution of* (3.2.18), *and let $u^{h\delta}$ be the finite element solution of* (3.2.19). *Then there exists $C>0$ such that*

$$\|u-u^{h\delta}\|_{S_0^{p+1,1}(D)}\le C(h^\epsilon+\delta^\gamma)\sum_{j=1}^N\max\{1,\|c_j\|_{L^\infty(D)}^{p_j+1}\}\|f\|_{H^{-1+\epsilon}(D)}.$$

Similarly, (3.2.16), Remark 3.2.6, and Remark 3.2.8 give the following remark.

Remark 3.2.10 *For problems with $g(x,y)\in S^{p+1,-1+\epsilon}(D)$, we have*

$$\begin{aligned}&\|u-u^{h\delta}\|_{S_0^{p+1,1}(D)}\\&\quad\le C(h^\epsilon+\delta^\gamma)\sum_{j=1}^N\max_{0\le k\le p_j+1}\{1,\frac{1}{k!}\;\|c_j\|_{L^\infty(D)}^{p_j+1-k}\}(\sum_{k=0}^{p_j+1}\|\partial_{y_j}^k g\|_{S^{0,-1+\epsilon}(D)}).\end{aligned}$$

CHAPTER 4. STOCHASTIC OPTIMAL CONTROL PROBLEMS

In this chapter, we shall solve stochastic optimal control problems; that is, we try to find the solutions of our constraint equations, stochastic elliptic PDEs, with flexible input data to minimize an objective functional of the solution. We first prove the existence of an optimal solution of our stochastic optimal control problem. Next, after checking the existence of a Lagrange multiplier (we use the method of Lagrange multipliers), we derive the stochastic optimality system of equations. Finally, we establish the error estimate for the solution of our optimality system of equations.

4.1 Constrained minimization problems

4.1.1 The stochastic functional

We introduce a stochastic functional that we want to minimize. Let $U : \overline{D} \times \Omega \to \mathbb{R}$, a target solution, be given. Consider

$$\mathcal{J}_\beta(u, f) = E\left(\frac{1}{2}\int_D |u - U|^2 \, dx + \frac{\beta}{2}\int_D |f|^2 \, dx\right), \tag{4.1.1}$$

where β is a positive constant (β can be regarded as measuring the importance of the two terms in (4.1.1)) and f is a deterministic control function. Notice that another expression for (4.1.1) is as follows:

$$\mathcal{J}_\beta(u, f) = \frac{1}{2}\int_\Omega\int_D |u - U|^2 \, dxdP + \frac{\beta}{2}\int_\Omega\int_D |f|^2 \, dxdP. \tag{4.1.2}$$

We used a cost functional and adopted the same notation as that in Hou, L. S. and Lee, J. (2008) or in most papers on optimal control problems. However, we point out that the cost functional in this thesis is the expectation.

We are going to minimize the stochastic functional (4.1.2) with suitable deterministic function f subject to the following stochastic elliptic PDE:

$$
\begin{aligned}
-\mathrm{div}\,[a(x,\omega)\nabla u(x,\omega)] &= f(x) \quad \text{in } D,\\
u(x,\omega) &= 0 \quad \text{on } \partial D.
\end{aligned}
\tag{4.1.3}
$$

4.1.2 The existence of an optimal solution

In Section 3.1.3, we proved that there is a unique solution of our constraint equation. We now examine the existence of an optimal solution that minimizes $\mathcal{J}_\beta(\cdot,\cdot)$. Let the admissibility set be defined by

$$\mathcal{U}_{ad} = \{(u,f) \in \mathcal{H}_0^1(D) \times L^2(D) \text{ such that (3.1.3) satisfied and } \mathcal{J}_\beta(u,f) < \infty\}. \tag{4.1.4}$$

We say that $(\hat{u},\hat{f}) \in \mathcal{U}_{ad}$ is an *optimal solution* of $\mathcal{J}_\beta(u,f)$ if for all $(u,f) \in \mathcal{U}_{ad}$ satisfying that $\|u-\hat{u}\|_{\mathcal{H}_0^1(D)} + \|f-\hat{f}\|_{L^2(D)} \le \epsilon$ for some $\epsilon > 0$,

$$\mathcal{J}_\beta(\hat{u},\hat{f}) \le \mathcal{J}_\beta(u,f). \tag{4.1.5}$$

Lemma 4.1.1 *Let $(u,f) \in \mathcal{U}_{ad}$. Then we have*

$$\|u\|_{\mathcal{H}_0^1(D)} \le C\|f\|_{L^2(D)} \tag{4.1.6}$$

for some positive constant C.

Proof: From (3.1.2) and (3.1.3), we see that

$$m\|u\|^2_{\mathcal{H}_0^1(D)} \le b[u,u] = [f,u]. \tag{4.1.7}$$

We know from Theorem 3.1.3 that,

$$[f,u]^2 \le \|f\|^2_{L^2(D)}\|u\|^2_{\mathcal{L}^2(D)}. \tag{4.1.8}$$

Now recall that by the Poincare inequality there is a positive constant C_p such that

$$\int_D |u|^2 \, dx \leq C_p \int_D |\nabla u|^2 \, dx \text{ a.s.} \tag{4.1.9}$$

Note that $E[\int_D |u|^2 \, dx]$ and $E[\int_D |\nabla u|^2 \, dx]$ are finite.

Thus, by Theorem 3.1.2, we have

$$\|u\|^2_{\mathcal{L}^2(D)} = E \int_D |u|^2 \, dx \leq C_p E \int_D |\nabla u|^2 \, dx = C_p \|u\|^2_{\mathcal{H}^1_0(D)}. \tag{4.1.10}$$

Combining (4.1.7), (4.1.8), and (4.1.10), we arrive at

$$m\|u\|^2_{\mathcal{H}^1_0(D)} \leq C_p \|f\|_{L^2(D)} \|u\|_{\mathcal{H}^1_0(D)}.$$

Finally, (4.1.6) follows from the last inequality. □

Theorem 4.1.2 *There is a unique optimal solution* $(\hat{u}, \hat{p}) \in \mathcal{U}_{ad}$ *of* $\mathcal{J}_\beta(u, f)$.

Proof: By Theorem 3.1.3, $\mathcal{U}_{ad}$ is not empty. Let $\{(u^{(n)}, f^{(n)})\}$ be a minimizing sequence in $\mathcal{U}_{ad}$, that is,

$$\lim_{n\to\infty} \mathcal{J}_\beta(u^{(n)}, f^{(n)}) = \inf_{(u,f)\in\mathcal{U}_{ad}} \mathcal{J}_\beta(u, f). \tag{4.1.11}$$

Because a convergent sequence is bounded, we have that $\|f^{(n)}\|_{L^2(D)} \leq C$ for some $C > 0$. That is, the sequence $\{f^{(n)}\}$ is uniformly bounded in $L^2(D)$. Thus, by Lemma 4.1.1, $\{u^{(n)}\}$ is a uniformly bounded sequence in $\mathcal{H}^1_0(D)$.

As a result, there is a convergent subsequence $\{(u^{(n_i)}, f^{(n_i)})\}$ such that

$$u^{(n_i)} \rightharpoonup \hat{u} \text{ weakly in } \mathcal{H}^1_0(D) \text{ and } f^{(n_i)} \rightharpoonup \hat{f} \text{ weakly in } L^2(D) \tag{4.1.12}$$

for some $(\hat{u}, \hat{f}) \in \mathcal{H}^1_0(D) \times L^2(D)$.

Note that $\|f^{(n)}\|_{\mathcal{L}^2(D)} = \|f^{(n)}\|_{L^2(D)} \leq C$. Hence, we also see that

$$f^{(n_i)} \rightharpoonup \hat{f} \text{ weakly in } \mathcal{L}^2(D). \tag{4.1.13}$$

This implies that

$$[f^{(n_i)}, v] \to [\hat{f}, v] \quad \forall v \in \mathcal{L}^2(D). \tag{4.1.14}$$

Because $\|\nabla u^{(n)}\|_{\mathcal{L}^2(D)} \le \|u^{(n)}\|_{\mathcal{H}_0^1(D)} \le C$, we have

$$\nabla u^{(n_i)} \rightharpoonup \nabla \hat{u} \;\; \text{weakly} \quad \text{in} \quad \mathcal{L}^2(D).$$

This yields

$$[\nabla u^{(n_i)}, w] \to [\nabla \hat{u}, w] \quad \forall w \in \mathcal{L}^2(D).$$

The fact that $\nabla v \in \mathcal{L}^2(D)$ for $v \in \mathcal{H}_0^1(D)$ lead us to

$$[\nabla u^{(n_i)}, \nabla v] \to [\nabla \hat{u}, \nabla v] \quad \forall v \in \mathcal{H}_0^1(D).$$

Thus, we obtain

$$b[u^{(n_i)}, v] \to b[\hat{u}, v] \quad \forall v \in \mathcal{H}_0^1(D) \tag{4.1.15}$$

because $a\nabla v \in \mathcal{L}^2(D)$ for $v \in \mathcal{H}_0^1(D)$.

With the help of (4.1.14) and (4.1.15), we can show that

$$b[\hat{u}, v] = \lim_{n_i \to \infty} b[u^{(n_i)}, v] = \lim_{n_i \to \infty} [f^{(n_i)}, v] = [\hat{f}, v] \quad \forall v \in \mathcal{H}_0^1(D). \tag{4.1.16}$$

That is, $(\hat{u}, \hat{f})$ satisfies (3.1.3) and hence $(\hat{u}, \hat{f}) \in \mathcal{U}_{ad}$. Using the weak convergence (4.1.12) and the weak lower continuity of the functional $\mathcal{J}_\beta(\cdot,\cdot)$ we arrive at

$$\mathcal{J}_\beta(\hat{u}, \hat{f}) \le \lim_{n_i \to \infty} \inf \mathcal{J}_\beta(u^{(n_i)}, f^{(n_i)}) = \inf_{(u,f) \in \mathcal{U}_{ad}} \mathcal{J}_\beta(u, f). \tag{4.1.17}$$

Therefore, $(\hat{u}, \hat{f})$ is an optimal solution.

The uniqueness of $(\hat{u}, \hat{f})$ follows from the strict convexity of the functional, the convexity of $\mathcal{U}_{ad}$, and the linearity of the constraints. □

4.2 The stochastic optimality system of equations

We will derive a stochastic optimality system of equations by using the Lagrange multiplier rule for the constrained minimization problem:

$$\min_{(u,f)\in\mathcal{U}_{ad}} \mathcal{J}_\beta(u,f) \quad \text{subject to} \quad (3.1.3). \tag{4.2.18}$$

For the deterministic PDE constraint case, we know that there exists a Lagrange multiplier; see e.g., Evans, L. C. (1998). Thus, without proving the existence of a Lagrange multiplier, we could derive an optimality system of equations for a deterministic minimization problem with a deterministic PDE constraint; e.g., Hou, L. S. and Lee, J. (2008). In the SPDE constraint case, however, we should show that there is a Lagrange multiplier before using the method of Lagrange multipliers to derive a stochastic optimality system of equations. To show the existence of a Lagrange multiplier, we follow the method given in Gunzburger, M. D. and Hou, L. S. (1996).

4.2.1 The abstract minimization problem

We begin with the definition of the abstract class of minimization problems. Let G, X, and Y be reflexive Banach Spaces whose norm are denoted by $\|\cdot\|_G, \|\cdot\|_X$, and $\|\cdot\|_Y$ and whose dual spaces are denoted by G^*, X^*, and Y^*, respectively. Let Θ be the control set that is a closed convex subset of G.

We assume that the functional to be minimized takes the form

$$\mathcal{J}(v,z) = \lambda\mathcal{F}(v) + \lambda\mathcal{E}(z) \quad \forall (v,z)\in X\times\Theta, \tag{4.2.19}$$

where $\mathcal{F}$ is a functional on X, $\mathcal{E}$ is a functional on Θ, and λ is a given parameter that is assumed to belong to a compact interval $\Lambda \subset \mathbb{R}_+$.

We define the function $M : X\times\Theta \to X$ for the constraint equation $M(v,z) = 0$ as follows:

$$M(v,z) = v + \lambda TN(v) + \lambda TK(z) \quad \forall (v,z)\in X\times\Theta, \tag{4.2.20}$$

where $N : X \to Y$ is a differentiable map, $K : \Theta \to Y$ is a bounded linear operator, $T : Y \to X$ is a bounded linear operator, and $\lambda \in \Lambda$.

With these definitions, we now consider the following constrained minimization problem:

$$\min_{(v,z)\in X\times\Theta} \mathcal{J}(v,z) \quad \text{subject to} \quad M(v,z) = 0. \tag{4.2.21}$$

Note that in our minimization problem (4.2.18) we had a local minima from the definition of the optimal solution. However, because we showed the unique existence of the optimal solution, our minima is actually the global minima.

4.2.2 Hypotheses concerning the abstract minimization problem

The set of hypotheses needed to justify the use of the Lagrange multiplier rule and to derive an optimality system from which optimal states and controls can be determined is given by

(HE1) for each $z \in \Theta$, $v \mapsto \mathcal{J}(v,z)$ and $v \mapsto M(v,z)$ are Fréchet differentiable;

(HE2) $z \mapsto \mathcal{E}(z)$ is convex;

(HE3) for $v \in X$, $N'(v)$ maps from X into $Z \hookrightarrow\hookrightarrow Y$, where N' denotes the Fréchet derivative of N.

4.2.3 Results concerning the existence of Lagrange multipliers

In this section we give some useful theorems about the abstract Lagrange multiplier rule.

Theorem 4.2.1 *Let X_1 and X_2 be two Banach spaces and Θ an arbitrary set. Suppose that $\mathcal{J}$ is a functional on $X_1 \times \Theta$ and M a mapping from $X_1 \times \Theta$ to X_2. Assume that $(u,g) \in X_1 \times \Theta$ is a solution to the following constrained minimization problem:*

$$M(u,g) = 0 \text{ and there exists an } \epsilon > 0 \text{ such that } \mathcal{J}(u,g) \le \mathcal{J}(v,z)$$
$$\text{for all } (v,z) \in X_1 \times \Theta \text{ such that } \|u - v\|_{X_1} \le \epsilon \text{ and } M(v,z) = 0. \tag{4.2.22}$$

Let U be an open neighborhood of $u \in X_1$. Assume further that the following conditions are satisfied:

for each $z \in \Theta, v \mapsto \mathcal{J}(v,z)$ and $v \mapsto M(v,z)$ are Frechet-differentiable at $v = u$;

for any $v \in U, z_1, z_2 \in \Theta$, and $\gamma \in [0,1]$, there exists a $z_\gamma = z_\gamma(v, z_1, z_2)$ such that

$$M(v, z_\gamma) = \gamma M(v, z_1) + (1-\gamma) M(v, z_2)$$

and

$$\mathcal{J}(v, z_\gamma) \leq \gamma \mathcal{J}(v, z_1) + (1-\gamma)\mathcal{J}(v, z_2);$$

The range of $M_u(u,g)$ is closed with a finite codimension,

where $M_u(u,g)$ is denotes the Frechet derivative of M with respect to u. Then, there exists a $k \in \mathbb{R}$ and a $\mu \in X_2^*$ that are not both equal to zero such that

$$k\langle \mathcal{J}_u(u,g), v\rangle - \langle \mu, M_u(u,g)v\rangle = 0 \quad \forall v \in X_1$$

and

$$\min_{z \in \Theta} \mathcal{L}(u, z, \mu, k) = \mathcal{L}(u, g, \mu, k),$$

where $\mathcal{L}(u,z,\mu,k) = k(J)(u,g) - \langle \mu, M(u,g)\rangle$ is the Lagrangian for the constrained minimization problem (4.2.22) *and where $\mathcal{J}_u(u,g)$ denotes the Frechet derivative of $\mathcal{J}$ with respect to u.*

Proof: see Tikhomirov, V. (1982).

By using Theorem 4.2.1, we have the following result.

Theorem 4.2.2 *Let $\lambda \in \Lambda$ be given. Assume that there exists an optimal solution (u, f) of* (4.2.21) *in $X \times \Theta$ and that $(HE1) - (HE3)$ hold. Then there exists a $k \in \mathbb{R}$ and a $\mu \in X^*$, not both equal to zero, such that*

$$k\langle \mathcal{J}_u(u,f), w\rangle - \langle \mu, M_u(u,f)\cdot w\rangle = 0 \quad \forall w \in X \tag{4.2.23}$$

and

$$\min_{z\in\Theta} \mathcal{L}(u,z,\mu,k) = \mathcal{L}(u,f,\mu,k). \tag{4.2.24}$$

PROOF: see Gunzburger, M. D. and Hou, L. S. (1996).

Θ has only been assumed to be a closed and convex subset of G. Now we assume that $\Theta = G$ to have a more concrete structure.

Theorem 4.2.3 *Let $\lambda \in \Lambda$ be given. Assume that there exists an optimal solution (u, f) of (4.2.21) in $X \times G$, that $(HE1) - (HE3)$ hold, and that the mapping $z \mapsto \mathcal{E}(z)$ is Frechet differentiable on G. Then there exists a $k \in \mathbb{R}$ and a $\mu \in X^*$, not both equal to zero, such that*

$$k\langle \mathcal{J}_u(u,f), w\rangle - \langle \mu, (I + \lambda T N'(u)) \cdot w\rangle = 0 \quad \forall w \in X \tag{4.2.25}$$

and

$$k\langle \mathcal{E}'(f), z\rangle - \langle \mu, TKz\rangle = 0 \quad \forall z \in G. \tag{4.2.26}$$

PROOF: see Gunzburger, M. D. and Hou, L. S. (1996).

Remark 4.2.4 *For two Theorems 4.2.2 and 4.2.3, if $1/\lambda$ is not in $\sigma(-TN'(u))$, we may choose $k = 1$; see Gunzburger, M. D. and Hou, L. S. (1996).*

4.2.4 The existence of a Lagrange multiplier and the stochastic optimality system of equations

We are now ready to prove the existence of a Lagrange multiplier for our minimization problem (4.2.18). The Lagrange multiplier rule may be used to convert the constrained minimization problem into an unconstrained one.

Note that since our stochastic elliptic PDE has a unique solution regardless of the choice of λ, a parameter in the abstract setting, we take $\lambda = 1$.

Recall the stochastic optimal control problem:

$$\min\{\mathcal{J}_\beta(u,f) \mid (u,f) \in \mathcal{H}_0^1(D) \times L^2(D)\} \text{ subject to } M(u,f) = 0 \quad \forall v \in \mathcal{H}_0^1(D), \tag{4.2.27}$$

where $M(u,f) = b[u,v] - [f,v]$.

We define $X = \mathcal{H}_0^1(D)$, $Y = \mathcal{H}^{-1}(D)$, $G = L^2(D)$, and $Z = \{0\}$. Then clearly we have $Z \hookrightarrow\hookrightarrow Y$. For the time being, we assume that the admissible set Θ for the control f is a closed, convex subset of G. We define the continuous linear operator $T \in \mathcal{L}(Y;X)$ as follows: for $g \in Y$, $Tg = u \in X$ is the unique solution of

$$b[u,v] = [g,v] \quad \forall v \in X. \tag{4.2.28}$$

We define the (differentiable) mapping $N : X \to Y$ by

$$N(u) = 0 \quad \forall u \in X \tag{4.2.29}$$

or, equivalently,

$$\langle N(u), v\rangle = 0 \quad \forall v \in X \tag{4.2.30}$$

and define $K : G \to Y$ by

$$K(f) = -f \tag{4.2.31}$$

or, equivalently,

$$\langle Kf, \eta\rangle = -\langle f, \eta\rangle \quad \forall \eta \in X. \tag{4.2.32}$$

Then it is clear that $u + TKf = 0$. In (4.1.1), we see that

$$\mathcal{F}(u) = E\left(\frac{1}{2}\int_D |u-U|^2\, dx\right) \text{ and } \mathcal{E}(f) = E\left(\frac{\beta}{2}\int_D |f|^2\, dx\right). \tag{4.2.33}$$

Next, we verify the hypotheses for the existence of Lagrange multipliers. First, notice that $(HE1)$ is obvious. Second, $(HE2)$ holds because $f \mapsto \mathcal{E}(f) = \frac{\beta}{2}\|f\|^2_{\mathcal{L}^2(D)}$ is convex. Third, because $N'(u)\cdot v = 0 \in Z \hookrightarrow\hookrightarrow Y$ for $\forall u, v \in X$, $(HE3)$ holds.

The Lagrangian is given by

$$\mathcal{L}(u, f, \xi, k) = k\mathcal{J}(u, f) - b[u, \xi] + [f, \xi]$$

for $\forall(u, f, \xi, k) \in X \times G \times X \times \mathbb{R}$.

By Theorem 4.2.2, there exists $\xi = T^*\mu \in X$ such that

$$\xi - kT^*\mathcal{F}'(u) = 0 \tag{4.2.34}$$

and

$$\mathcal{L}(u, f, \xi, k) \leq \mathcal{L}(u, z, \xi, k) \quad \forall z \in \Theta \tag{4.2.35}$$

We note that we may choose $k = 1$ in (4.2.34) and (4.2.35).

With $k = 1$, (4.2.34) becomes

$$b[\xi, \zeta] = [u - U, \zeta] \quad \forall \zeta \in X \tag{4.2.36}$$

and (4.2.35) implies that

$$\frac{\beta}{2}[z, z] + [z, \xi] - \frac{\beta}{2}[f, f] + [f, \xi] \geq 0 \quad \forall z \in \Theta \subseteq G. \tag{4.2.37}$$

For each $\epsilon \in (0, 1)$ and each $t \in \Theta$, set $z = \epsilon t + (1 - \epsilon)f \in \Theta$. Then from (4.2.37), we have

$$\frac{\beta\epsilon}{2}[t - f, t - f] + \beta[t - f, f] + [t - f, \xi] \geq 0 \quad \forall t \in \Theta. \tag{4.2.38}$$

By letting $\epsilon \to 0^+$ in the above inequality, we have

$$[t - f, \beta f + \xi] \geq 0 \quad \forall t \in \Theta. \tag{4.2.39}$$

We now consider the case $\Theta = G$. Note that the mapping $z \mapsto \mathcal{E}(z)$ is Fréchet differentiable on G. Hence, by Theorem 4.2.3, (4.2.39) becomes an equality and by letting $z = t - f$ we obtain

$$[\beta f + \xi, z] = 0 \quad \forall z \in G. \tag{4.2.40}$$

The system formed by equations (3.1.3), (4.2.36), and (4.2.40), which are necessary conditions for an optimum, is called a stochastic optimality system. We now conclude this section with the following theorem.

Theorem 4.2.5 *Let* $(u, f) \in \mathcal{H}_0^1(D) \times L^2(D)$ *be an optimal solution of* (4.2.18)*. Then there exists* $\xi \in \mathcal{H}_0^1(D)$ *such that* (4.2.36) *and* (4.2.40) *hold.*

4.3 Discrete approximations of the optimality system

4.3.1 Description of the Brezzi-Rappaz-Raviart theory

The Brezzi-Rappaz-Raviart (B-R-R) theory implies that the error of approximation of solutions of certain nonlinear problems under certain hypotheses is basically the same as the error of approximation of solutions of related linear problems; see Brezzi, F., Rappaz, J., and Raviart, P. (1980), Crouzeix, M. and Rappaz, J. (1990), and Girault, V. and Raviart, P. (1986). Here for the sake of completeness, we will state the relevant results, specialized to our needs.

Consider the following type of nonlinear problems: seek $\psi \in \mathcal{X}$ such that

$$\psi + \mathcal{T}\mathcal{G}(\psi) = 0, \tag{4.3.41}$$

where $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, $\mathcal{G}$ is a C^2 mapping from $\mathcal{X}$ into $\mathcal{Y}$, and $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. We say that ψ is a regular solution of (4.3.41) if (4.3.41) holds and $\psi + \mathcal{T}\mathcal{G}_\psi(\psi)$ is an isomorphism from $\mathcal{X}$ into $\mathcal{X}$. Here $\mathcal{G}_\psi$ denotes the Frechet derivative of $\mathcal{G}$ with respect to ψ. We assume that there exists another Banach space $\mathcal{Z}$, contained in $\mathcal{Y}$, with continuous imbedding, such that

$$\mathcal{G}_\psi(\psi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z}) \quad \forall \psi \in \mathcal{X}. \tag{4.3.42}$$

Approximations are defined by introducing a subspace $\mathcal{X}^h \subset \mathcal{X}$ and an approximating operator $\mathcal{T}^h \in \mathcal{L}(\mathcal{Y};\mathcal{X}^h)$. We seek $\psi^h \in \mathcal{X}^h$ such that

$$\psi^h + \mathcal{T}^h \mathcal{G}(\psi^h) = 0. \tag{4.3.43}$$

Concerning the operator $\mathcal{T}^h$, we assume the approximation properties

$$\lim_{h\to 0} \|(\mathcal{T}^h - \mathcal{T})\omega\|_{\mathcal{X}} = 0 \quad \forall \omega \in \mathcal{Y} \tag{4.3.44}$$

and

$$\lim_{h\to 0} \|\mathcal{T}^h - \mathcal{T}\|_{\mathcal{L}(\mathcal{Z};\mathcal{X})} = 0 \tag{4.3.45}$$

Note that whenever the imbedding $\mathcal{Z} \subset \mathcal{Y}$ is compact, (4.3.45) follows from (4.3.44) and, moreover, (4.3.42) implies that the operator $\mathcal{T}\mathcal{G}_\psi(\psi) \in \mathcal{L}(\mathcal{X};\mathcal{X})$ is compact.

We now state the result of Brezzi, F., Rappaz, J., and Raviart, P. (1980) that will be used in the sequel. In the statement of the theorem, $D^2\mathcal{G}$ represents any and all second Frechet derivatives of $\mathcal{G}$.

Theorem 4.3.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Assume that $\mathcal{G}$ is a C^2 mapping from $\mathcal{X}$ to $\mathcal{Y}$ and that $D^2\mathcal{G}$ is bounded on all bounded sets of $\mathcal{X}$. Assume that* (4.3.42), (4.3.44), *and* (4.3.45) *hold and that ψ is a regular solution of* (4.3.41). *Then there exists a neighborhood $\mathcal{O}$ of the origin in $\mathcal{X}$ and, for $h \le h_0$ small enough, a unique $\psi^h \in \mathcal{X}^h$ such that ψ^h is a regular solution of* (4.3.43). *Moreover, there exists a constant $C > 0$, independent of h, such that*

$$\|\psi^h - \psi\|_{\mathcal{X}} \le C\|(\mathcal{T}^h - \mathcal{T})\mathcal{G}(\psi)\|_{\mathcal{X}}. \tag{4.3.46}$$

4.3.2 Recasting the optimality system and its discrete approximation into the B-R-R framework

We first fit our optimality system and its discrete approximation into the B-R-R framework. Then we obtain the desired error estimates on the solution of the optimality system of equations by verifying each assumption of the B-R-R theory.

We set $\mathcal{X} = S_0^{p+1,1}(D) \times L^2(D) \times S_0^{p+1,1}(D)$ and $\mathcal{Y} = H^{-1}(D) \times S^{p+1,-1}(D)$. We define the linear operator $\mathcal{T} \in \mathcal{L}(\mathcal{Y};\mathcal{X})$ as follows:

$$(\tilde{u}, \tilde{f}, \tilde{\xi}) = \mathcal{T}(\tilde{r}, \tilde{\tau})$$

if and only if

$$b[\tilde{u}, v] = [\tilde{r}, v] \quad \forall v \in S_0^{p+1,1}(D), \tag{4.3.47}$$

$$b[\tilde{\xi}, \zeta] = [\tilde{\tau}, \zeta] \quad \forall \zeta \in S_0^{p+1,1}(D), \tag{4.3.48}$$

and

$$[\beta\tilde{f} + \tilde{\xi}, z] = 0 \quad \forall z \in L^2(D). \tag{4.3.49}$$

We define $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{G}(\tilde{u}, \tilde{f}, \tilde{\xi}) = (-\tilde{f}, -\tilde{u} + U).$$

It is clear that the optimality system (3.1.3), (4.2.36), and (4.2.40) can be written as

$$(u, f, \xi) + \mathcal{T}(\mathcal{G}(u, f, \xi)) = 0. \tag{4.3.50}$$

Hence, the optimality system is recast into the form of (4.3.41).

We now set $\mathcal{X}^{h\delta} = V^{h\delta} \times G^h \times V^{h\delta}$, where $V^{h\delta}$ and G^h are from Section 3.2.3.

We define the discrete operator $\mathcal{T}^{h\delta} \in \mathcal{L}(\mathcal{Y};\mathcal{X}^{h\delta})$ as follows:

$$(\tilde{u}^{h\delta}, \tilde{f}^h, \tilde{\xi}^{h\delta}) = \mathcal{T}^{h\delta}(\tilde{r}, \tilde{\tau})$$

if and only if

$$b[\tilde{u}^{h\delta}, v^{h\delta}] = [\tilde{r}, v^{h\delta}] \quad \forall v^{h\delta} \in V^{h\delta}, \tag{4.3.51}$$

$$b[\tilde{\xi}^{h\delta}, \zeta^{h\delta}] = [\tilde{\tau}, \zeta^{h\delta}] \quad \forall \zeta^{h\delta} \in V^{h\delta}, \tag{4.3.52}$$

and

$$[\beta\tilde{f}^h + \tilde{\xi}^{h\delta}, z^h] = 0 \quad \forall z^h \in G^h. \tag{4.3.53}$$

Then it is clear that the discrete optimality system,

$$b[u^{h\delta}, v^{h\delta}] = [f^h, v^{h\delta}] \quad \forall v^{h\delta} \in V^{h\delta}, \tag{4.3.54}$$

$$b[\xi^{h\delta}, \zeta^{h\delta}] = [u^{h\delta} - U, \zeta^{h\delta}] \quad \forall \zeta^{h\delta} \in V^{h\delta}, \tag{4.3.55}$$

and

$$[\beta f^h + \xi^{h\delta}, z^h] = 0 \quad \forall z^h \in G^h, \tag{4.3.56}$$

can be written as

$$(u^{h\delta}, f^h, \xi^{h\delta}) + \mathcal{T}^{h\delta}(\mathcal{G}(u^{h\delta}, f, \xi^{h\delta})) = 0.$$

Hence, the discrete optimality system is recast into the form of (4.3.43).

4.3.3 Error estimates for discrete finite element approximation of the optimality system

In this section, we proceed to verify all assumptions in Theorem 4.3.1. We define first a space $\mathcal{Z} = L^2(D) \times S^{p+1,0}(D)$. Then clearly this space is continuously embedded into $\mathcal{Y} = H^{-1}(D) \times S^{p+1,-1}(D)$.

Denote the Fréchet derivative of $\mathcal{G}(u, f, \xi)$ with respect to (u, f, ξ) by $D\mathcal{G}(u, f, \xi)$ or $\mathcal{G}_{(u,f,\xi)}(u, f, \xi)$. Then for $(u, f, \xi) \in \mathcal{X}$, we obtain

$$D\mathcal{G}(u, f, \xi) \cdot (\tilde{u}, \tilde{f}, \tilde{\xi}) = (-\tilde{f}, -\tilde{u}) \quad \forall (\tilde{u}, \tilde{f}, \tilde{\xi}) \in \mathcal{X}.$$

Proposition 4.3.2 *$D\mathcal{G}(u, f, \xi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z})$ for all $(u, f, \xi) \in \mathcal{X}$.*

Proof: It is clear that

$$\|D\mathcal{G}(u, f, \xi) \cdot (\tilde{u}, \tilde{f}, \tilde{\xi})\|_{\mathcal{Z}} = \|\tilde{f}\|_{L^2(D)} + \|\tilde{u}\|_{S^{p+1,0}(D)} < \infty.$$

Therefore, $D\mathcal{G}(u, f, \xi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z})$. □

Proposition 4.3.3 *$\mathcal{G}$ is twice continuously differentiable and $D^2\mathcal{G}$ is bounded on all bounded sets of $\mathcal{X}$.*

PROOF: For any $(u, f, \xi) \in \mathcal{X}$,

$$D^2\mathcal{G}(u, f, \xi) \cdot (\tilde{u}, \tilde{f}, \tilde{\xi}) = (0, 0) \quad \forall(\tilde{u}, \tilde{f}, \tilde{\xi}) \in \mathcal{X}.$$

Thus, it is easy to show that $D^2\mathcal{G}$ is well defined, continuous, and bounded on all bounded sets of $\mathcal{X}$. □

Before we show (4.3.44) and (4.3.45), we consider the following lemmas.

Lemma 4.3.4 *Let $\tilde{f} \in L^2(D)$ and $\tilde{\xi} \in S_0^{p+1,1}(D)$ in (4.3.49). Let $\tilde{f}^h \in G^h$ and $\tilde{\xi} \in V^{h\delta}$ in (4.3.53). Then there exists $C > 0$ such that*

$$\|\tilde{f} - \tilde{f}^h\|^2_{\mathcal{L}^2(D)} \leq C(\|\tilde{\xi} - \tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \|\tilde{f} - g^h\|^2_{\mathcal{L}^2(D)}) \tag{4.3.57}$$

for all $g^h \in G^h$.

PROOF: From (4.3.49) and from (4.3.53), we see that

$$[\beta\tilde{f}, z^h] = -[\tilde{\xi}, z^h] \quad \forall z^h \in G^h \tag{4.3.58}$$

and

$$[\beta\tilde{f}^h, z^h] = -[\tilde{\xi}^{h\delta}, z^h] \quad \forall z^h \in G^h. \tag{4.3.59}$$

Subtracting (4.3.59) from (4.3.58) leads us to

$$[\beta(\tilde{f} - \tilde{f}^h), z^h] = -[\tilde{\xi} - \tilde{\xi}^{h\delta}, z^h] \quad \forall z^h \in G^h. \tag{4.3.60}$$

Thus, for any $g^h \in G^h$, we find

$$\begin{aligned}[\tilde{f} - \tilde{f}^h, \tilde{f} - \tilde{f}^h] &= [\tilde{f} - \tilde{f}^h, \tilde{f} - g^h] + [\tilde{f} - \tilde{f}^h, g^h - \tilde{f}^h] \\ &= [\tilde{f} - \tilde{f}^h, \tilde{f} - g^h] + \frac{1}{\beta}[\tilde{\xi} - \tilde{\xi}^{h\delta}, \tilde{f}^h - g^h].\end{aligned} \tag{4.3.61}$$

The Hölder inequality implies

$$\|\tilde{f}-\tilde{f}^h\|^2_{\mathcal{L}^2(D)} \le \|\tilde{f}-\tilde{f}^h\|_{\mathcal{L}^2(D)}\|\tilde{f}-g^h\|_{\mathcal{L}^2(D)} + \frac{1}{\beta}\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|_{\mathcal{L}^2(D)}\|\tilde{f}^h-\tilde{f}\|_{\mathcal{L}^2(D)} + \frac{1}{\beta}\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|_{\mathcal{L}^2(D)}\|\tilde{f}-g^h\|_{\mathcal{L}^2(D)}. \tag{4.3.62}$$

By Cauchy's inequality with $\epsilon > 0$, for some C_ϵ that depends on ϵ, we have

$$\begin{aligned}\|\tilde{f}-\tilde{f}^h\|^2_{\mathcal{L}^2(D)} \le \epsilon\|\tilde{f}-\tilde{f}^h\|^2_{\mathcal{L}^2(D)} + C_\epsilon\|\tilde{f}-g^h\|^2_{\mathcal{L}^2(D)} \\ + \frac{C_\epsilon}{\beta}\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \epsilon\|\tilde{f}^h-\tilde{f}\|^2_{\mathcal{L}^2(D)} \\ + \frac{1}{2\beta}\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \frac{1}{2}\|\tilde{f}-g^h\|^2_{\mathcal{L}^2(D)}.\end{aligned} \tag{4.3.63}$$

Now choose $\epsilon = \frac{1}{4}$. Then there exists $C > 0$ such that

$$\|\tilde{f}-\tilde{f}^h\|^2_{\mathcal{L}^2(D)} \le C(\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|^2_{\mathcal{L}^2(D)} + \|\tilde{f}-g^h\|^2_{\mathcal{L}^2(D)}) \tag{4.3.64}$$

for any $g^h \in G^h$. □

Lemma 4.3.5 *Let $\tilde{r} \in H^{-1}(D)$, $\tilde{u}$ be the solution of*

$$b[\tilde{u}, v] = [\tilde{r}, v] \quad \forall v \in S_0^{p+1,1}(D), \tag{4.3.65}$$

and $\tilde{u}^{h\delta}$ be the solution of

$$b[\tilde{u}^{h\delta}, v^{h\delta}] = [\tilde{r}, v^{h\delta}] \quad \forall v^{h\delta} \in V^{h\delta}. \tag{4.3.66}$$

Then we have

$$\|\tilde{u}-\tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} \to 0 \text{ as } h, \delta \to 0.$$

Proof: Let $\epsilon > 0$ be given. Let $\tilde{r} \in H^{-1}(D)$. Then there is a sequence of C^∞-functions $\{\tilde{r}_k\} \subset L^2(D)$ such that $\tilde{r}_k \to \tilde{r}$ in $H^{-1}(D)$; i.e., there exists k_0 such that

$$\|\tilde{r}-\tilde{r}_{k_0}\|_{H^{-1}(D)} < \epsilon. \tag{4.3.67}$$

Now consider the following problems:

$$b[\tilde{u}_{k_0}, v] = [\tilde{r}_{k_0}, v] \quad \forall v \in S_0^{p+1,1}(D) \tag{4.3.68}$$

and

$$b[\tilde{u}_{k_0}^{h\delta}, v^{h\delta}] = [\tilde{r}_{k_0}, v^{h\delta}] \quad \forall v^{h\delta} \in V^{h\delta}. \tag{4.3.69}$$

Then from (4.3.65) and (4.3.68) and from (4.3.66) and (4.3.69), there exists $C > 0$ such that

$$\|\tilde{u} - \tilde{u}_{k_0}\|_{S_0^{p+1,1}(D)} \leq C\|\tilde{r} - \tilde{r}_{k_0}\|_{H^{-1}(D)} \tag{4.3.70}$$

and

$$\|\tilde{u}^{h\delta} - \tilde{u}_{k_0}^{h\delta}\|_{S_0^{p+1,1}(D)} \leq C\|\tilde{r} - \tilde{r}_{k_0}\|_{H^{-1}(D)}, \tag{4.3.71}$$

respectively.

Hence, obviously,

$$\begin{aligned}
&\|\tilde{u}-\tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} \\
&\quad \leq \|\tilde{u} - \tilde{u}_{k_0}\|_{S_0^{p+1,1}(D)} + \|\tilde{u}_{k_0} - \tilde{u}_{k_0}^{h\delta}\|_{S_0^{p+1,1}(D)} + \|\tilde{u}_{k_0}^{h\delta} - \tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} \\
&\quad \leq C\|\tilde{r} - \tilde{r}_{k_0}\|_{H^{-1}(D)} + \|\tilde{u}_{k_0} - \tilde{u}_{k_0}^{h\delta}\|_{S_0^{p+1,1}(D)} + C\|\tilde{r} - \tilde{r}_{k_0}\|_{H^{-1}(D)}
\end{aligned} \tag{4.3.72}$$

On the other hand, because $\tilde{r}_{k_0} \in L^2(D)$, Theorem 3.2.9 yields

$$\|\tilde{u}_{k_0} - \tilde{u}_{k_0}^{h\delta}\|_{S_0^{p+1,1}(D)} \leq C(h + \delta^\gamma) \sum_{j=1}^{N} \max\{1, \|c_j\|_{L^\infty(D)}^{p_j+1}\} \|\tilde{r}_{k_0}\|_{L^2(D)}. \tag{4.3.73}$$

Thus, from the last inequality, by letting $h, \delta \to 0$, we obtain

$$\|\tilde{u}_{k_0} - \tilde{u}_{k_0}^{h\delta}\|_{S_0^{p+1,1}(D)} < \epsilon. \tag{4.3.74}$$

Combining (4.3.67), (4.3.72) and (4.3.74)

$$\|\tilde{u} - \tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} < (2C + 1)\epsilon. \tag{4.3.75}$$

Because ϵ is arbitrary, this complete the proof of Lemma 4.3.5. $\square$

Remark 4.3.6 *Likewise, for $\tilde{\xi}$ in (4.3.48), $\tilde{\xi}^{h\delta}$ in (4.3.52), and $\tilde{\tau} \in S^{p+1,-1}(D)$, we have*

$$\|\tilde{\xi} - \tilde{\xi}^{h\delta}\|_{S_0^{p+1,1}(D)} \to 0 \text{ as } h, \delta \to 0.$$

Lemma 4.3.7 *Let $\tilde{f} \in L^2(D)$, $g^h \in G^h$, and $\tilde{\tau} \in S^{p+1,-1}(D)$ in (4.3.48). Then there exists $C > 0$ such that*

$$\|\tilde{f} - g^h\|_{L^2(D)} \leq Ch\|\tilde{\tau}\|_{\mathcal{H}^{-1}(D)}. \tag{4.3.76}$$

PROOF: From (4.3.49), we see that

$$\int_D \beta \tilde{f} z \; dx = -\int_D (E\tilde{\xi}) z \; dx.$$

The Hölder inequality implies that

$$\begin{aligned} \int_D |\nabla \tilde{f}|^2 \; dx &= \frac{1}{\beta^2} \int_D (E\nabla\tilde{\xi})^2 \; dx \\ &\leq \frac{1}{\beta^2} \int_D E|\nabla\tilde{\xi}|^2 \; dx = \frac{1}{\beta^2} E \int_D |\nabla\tilde{\xi}|^2 \; dx < \infty. \end{aligned} \tag{4.3.77}$$

On the other hand, choose $g^h = \tilde{P}^h \tilde{f}$, where $\tilde{P}^h$ is a $L^2(D)$-projection from $L^2(D)$ onto G^h. Because $\tilde{f} \in H_0^1(D)$, by the approximation property (3.2.11), there exists $C > 0$ such that

$$\|\tilde{f} - g^h\|_{L^2(D)} = \|\tilde{f} - \tilde{P}^h \tilde{f}\|_{L^2(D)} \leq Ch\|\tilde{f}\|_{H_0^1(D)}. \tag{4.3.78}$$

Thus, (4.3.76) follows by combining the last two inequalities and (4.3.48) because $\tilde{\tau} \in S^{p+1,-1}(D) \subset \mathcal{H}^{-1}(D)$. □

Proposition 4.3.8 *For any $(\tilde{r}, \tilde{\tau}) \in \mathcal{Y}$, $\|(\mathcal{T} - \mathcal{T}^{h\delta})(\tilde{r}, \tilde{\tau})\|_{\mathcal{X}} \to 0$ as $h, \delta \to 0$.*

PROOF: By Lemma 4.3.4, we see that for any $g^h \in G^h$, there exists $C > 0$ such that

$$\begin{aligned} \|(\mathcal{T} - \mathcal{T}^{h\delta})(\tilde{r}, \tilde{\tau})\|_{\mathcal{X}} &= \|(\tilde{u} - \tilde{u}^{h\delta}, \tilde{f} - \tilde{f}^h, \tilde{\xi} - \tilde{\xi}^{h\delta})\|_{\mathcal{X}} \\ &= \|\tilde{u} - \tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} + \|\tilde{f} - \tilde{f}^h\|_{L^2(D)} + \|\tilde{\xi} - \tilde{\xi}^{h\delta}\|_{S_0^{p+1,1}(D)} \quad (4.3.79) \\ &\leq \|\tilde{u} - \tilde{u}^{h\delta}\|_{S_0^{p+1,1}(D)} + C(\|\tilde{\xi} - \tilde{\xi}^{h\delta}\|_{S_0^{p+1,1}(D)} + \|\tilde{f} - g^h\|_{L^2(D)}). \end{aligned}$$

Thus, by Lemma 4.3.5, Remark 4.3.6, and Lemma 4.3.7, we have

$$\|(\mathcal{T}-\mathcal{T}^{h\delta})(\tilde{r},\tilde{\tau})\|_{\mathcal{X}} \to 0 \text{ as } h,\delta \to 0. \qquad \square$$

Proposition 4.3.9 $\|\mathcal{T}-\mathcal{T}^{h\delta}\|_{\mathcal{L}(\mathcal{Z},\mathcal{X})} \to 0$ *as* $h,\delta \to 0$.

PROOF: Let $\tilde{\tau} \in S^{p+1,0}(D)$. From Remark 3.2.10 there exists $C > 0$ such that

$$\|\tilde{\xi}-\tilde{\xi}^{h\delta}\|^2_{S_0^{p+1,1}(D)} \le C(h^2+\delta^{2\gamma})K\|\tilde{\tau}\|^2_{S^{p+1,0}(D)}, \tag{4.3.80}$$

where $K = \max\{1, \dfrac{1}{(k!)^2}\ \|c_j\|^{2(p_j+1-k)}_{L^\infty(D)}\ :\ 1 \le j \le N,\ 0 \le k \le p_j+1\}$.

Theorem 3.2.9, Lemma 4.3.4, Lemma 4.3.7, and (4.3.80) yield

$$\begin{aligned}\|(\mathcal{T}-\mathcal{T}^{h\delta})(\tilde{r},\tilde{\tau})\|^2_{\mathcal{X}} &\le C(h^2+\delta^{2\gamma})K(\|\tilde{r}\|^2_{L^2(D)}+\|\tilde{\tau}\|^2_{S^{p+1,0}(D)})\\ &\le C(h^2+\delta^{2\gamma})K\|(\tilde{r},\tilde{\tau})\|^2_{\mathcal{Z}}\end{aligned} \tag{4.3.81}$$

for some $C > 0$.

Hence, we see that

$$\begin{aligned}\|\mathcal{T}-\mathcal{T}^{h\delta}\|^2_{\mathcal{L}(\mathcal{Z},\mathcal{X})} &= \sup_{\|(\tilde{r},\tilde{\tau})\|_{\mathcal{Z}}\neq 0} \frac{\|(\mathcal{T}-\mathcal{T}^{h\delta})(\tilde{r},\tilde{\tau})\|^2_{\mathcal{X}}}{\|(\tilde{r},\tilde{\tau})\|^2_{\mathcal{Z}}}\\ &\le \sup_{\|(\tilde{r},\tilde{\tau})\|_{\mathcal{Z}}\neq 0} C(h^2+\delta^{2\gamma})K \to 0 \text{ as } h,\delta\to 0. \quad \square\end{aligned} \tag{4.3.82}$$

Proposition 4.3.10 *A solution of* (4.3.50) *is regular.*

PROOF: A proof follows from the linearity and well-posedness of (4.3.47), (4.3.48), and (4.3.49). $\square$

Through Propositions 4.3.2 - 4.3.10 we have verified all of the assumptions of Theorem 4.3.1. Thus, by that theorem, we obtain the following results.

Theorem 4.3.11 *Assume that* $U \in S_0^{p+1,1}(D)$. *Let* $(u,f,\xi) \in S_0^{p+1,1}(D) \times L^2(D) \times S_0^{p+1,1}(D)$ *be the solution of the optimality system* (3.1.3), (4.2.36), *and* (4.2.40). *Let*

$(u^{h\delta}, f^h, \xi^{h\delta}) \in V^{h\delta} \times G^h \times V^{h\delta}$ *be the solution of the discrete optimality system* (4.3.54), (4.3.55), *and* (4.3.56). *Then we have*

$$\|u - u^{h\delta}\|_{S_0^{p+1,1}(D)} + \|f - f^h\|_{L^2(D)} + \|\xi - \xi^{h\delta}\|_{S_0^{p+1,1}(D)} \to 0 \text{ as } h, \delta \to 0.$$

Moreover, there exists $C > 0$ *such that*

$$\|u - u^{h\delta}\|^2_{S_0^{p+1,1}(D)} + \|f - f^h\|^2_{L^2(D)} + \|\xi - \xi^{h\delta}\|^2_{S_0^{p+1,1}(D)} \tag{4.3.83}$$

$$\leq C(h^2 + \delta^{2\gamma})K(\|f\|^2_{L^2(D)} + \|u - U\|^2_{S^{p+1,0}(D)}), \tag{4.3.84}$$

where $K = \max\{1, \dfrac{1}{(k!)^2}\ \|c_j\|^{2(p_j+1-k)}_{L^\infty(D)}\ :\ 1 \leq j \leq N,\ 0 \leq k \leq p_j + 1\}$.

CHAPTER 5. NUMERICAL EXPERIMENTS

In this chapter, we consider another approximation space $Z^p \subset L^2(D)$, where $Z^p = Z_1^{p_1} \bigotimes Z_2^{p_2} \bigotimes \cdots \bigotimes Z_N^{p_N}$ and $Z_n^{p_n} = \{v : \Gamma_n \to \mathbb{R} : v \in span(1, y, \cdots, y^{p_n})\}$. This space is a particular case of the space Y^δ in Section 3.2.3 with no partition of Γ (instead, we increase only the polynomial degree). We define the tensor product finite element space $V^{hp} = X^h \bigotimes Z^p$ on $D \times \Gamma$.

Let $\{\varphi_i(x)\}$ be a basis of the space $X^h \subset H_0^1(D)$ and let $\{\psi_j(y)\}$ be a basis of the space $Z^p \subset L^2(D)$. Then the solution of the discrete optimality system of equations is given by

$$\begin{aligned} u^{hp}(x, y) &= \sum_{i,j} u_{ij} \varphi_i(x) \psi_j(y), \\ \xi^{hp}(x, y) &= \sum_{i,j} \xi_{ij} \varphi_i(x) \psi_j(y), \\ f^h(x) &= \sum_i f_i \varphi_i(x). \end{aligned} \tag{5.0.1}$$

Recall the discrete optimality system of equations:

$$\int_\Gamma \rho \int_D a \nabla u^{hp} \cdot \nabla v^{hp} dx dy \qquad - \int_\Gamma \rho \int_D f^h v^{hp} dx dy = 0 \ \forall v^{hp} \in V^{hp},$$

$$- \int_\Gamma \rho \int_D \xi^{hp} \eta^h dx dy + \beta \int_\Gamma \rho \int_D f^h \eta^h dx dy = 0 \ \forall \eta^h \in G^h,$$

$$\int_\Gamma \rho \int_D u^{hp} \lambda^{hp} dx dy + \int_\Gamma \rho \int_D a \nabla \xi^{hp} \nabla \lambda^{hp} dx dy = \int_\Gamma \rho \int_D U \lambda^{hp} dx dy \ \forall \lambda^{hp} \in V^{hp}.$$

By substituting (5.0.1) into this system of equations, for any test function $\varphi_k(x)\psi_l(y)$, we have

$$\int_\Gamma \rho(y) \int_D a(x,y)\nabla u^{hp}(x,y)\nabla v^{hp}(x,y)\ dxdy \\ = \sum_{i,j} \left(\int_\Gamma \rho(y)\psi_j(y)\psi_l(y) \int_D a(x,y)\nabla\varphi_i(x)\nabla\varphi_k(x)\ dxdy \right) u_{ij},$$

$$\int_\Gamma \rho(y) \int_D f^h(x)v^{hp}(x,y)\ dxdy = \sum_i \left(\int_\Gamma \rho(y)\psi_l(y) \int_D \varphi_i(x)\varphi_k(x)\ dxdy \right) f_i,$$

$$\int_\Gamma \rho(y) \int_D \xi^{hp}(x,y)\varphi_k(x)\ dxdy = \sum_{i,j} \left(\int_\Gamma \rho(y)\psi_j(y) \int_D \varphi_i(x)\varphi_k(x)\ dxdy \right) \xi_{ij},$$

$$\beta \int_\Gamma \rho(y) \int_D f^h(x)\varphi_k(x)\ dxdy = \sum_i \left(\beta \int_\Gamma \rho(y) \int_D \varphi_i(x)\varphi_k(x)\ dxdy \right) f_i,$$

$$\int_\Gamma \rho(y) \int_D u^{hp}(x,y)v^{hp}(x,y)\ dxdy = \sum_{i,j} \left(\int_\Gamma \rho(y)\psi_j(y)\psi_l(y) \int_D \varphi_i(x)\varphi_k(x)\ dxdy \right) u_{ij},$$

$$\int_\Gamma \rho(y) \int_D a(x,y)\nabla \xi^{hp}(x,y)\nabla v^{hp}(x,y)\ dxdy \\ = \sum_{i,j} \left(\int_\Gamma \rho(y)\psi_j(y)\psi_l(y) \int_D a(x,y)\nabla\varphi_i(x)\nabla\varphi_k(x)\ dxdy \right) \xi_{ij},$$

$$\int_\Gamma \rho(y) \int_D U(x,y)v^{hp}(x,y)\ dxdy = \int_\Gamma \rho(y) \int_D U(x,y)\varphi_k(x)\psi_l(y)\ dxdy.$$

We now look at only the right hand side of the first equation. Note that for $\psi_j(y) \in Z^p = Z^{p_1} \bigotimes Z^{p_2} \bigotimes \cdots \bigotimes Z^{p_N}$, we have $\psi_j(y) = \prod_{m=1}^N \psi_{jm}(y_m)$, where $\psi_{jm} : \Gamma_m \to \mathbb{R}$ is a basis function of Z^{p_m}. With a finite K-L expansion of $a(x,y)$, we have

$$\begin{aligned}
&\int_\Gamma \rho(y)\psi_j(y)\psi_l(y)\int_D a(x,y)\nabla\varphi_i(x)\nabla\varphi_k(x)\ dxdy\\
&\quad= \int_\Gamma \rho(y)\psi_j(y)\psi_l(y)\int_D (Ea(x)+\sum_{n=1}^N\sqrt{\lambda_n}\phi_n(x)y_n)\nabla\varphi_i(x)\nabla\varphi_k(x)\ dxdy\\
&\quad= K^0_{i,k}\int_\Gamma \rho(y)\psi_j(y)\psi_l(y)\ dy+\sum_{n=1}^N K^n_{i,k}\int_\Gamma y_n\rho(y)\psi_j(y)\psi_l(y)\ dy\\
&\quad= K^0_{i,k}\int_\Gamma \prod_{m=1}^N \rho_m(y_m)\psi_{jm}(y_m)\psi_{lm}(y_m)\ dy\\
&\qquad\qquad+\sum_{n=1}^N K^n_{i,k}\int_\Gamma y_n\prod_{m=1}^N \rho_m(y_m)\psi_{jm}(y_m)\psi_{lm}(y_m)\ dy,
\end{aligned}$$

where

$$K^0_{i,k}=\int_D Ea(x)\nabla\varphi_i(x)\nabla\varphi_k(x)\ dx$$

and

$$K^n_{i,k}=\int_D \sqrt{\lambda_n}\phi_n(x)\nabla\varphi_i(x)\nabla\varphi_k(x)\ dx.$$

On the same way, it is easy to calculate the other equations. Next, we solve the linear system to determine u_{ij},ξ_{ij}, and f_i that are coefficients of solutions of the discrete optimality system of equations.

In our numerical experiments, we assume for simplicity in calculation that our deterministic domain D is $[-1,1]$. Also we suppose that we have a constant density function. The assumptions $EX_n=0$ and $\mathrm{Var}X_n=1$ in the K-L expansion imply that $\Gamma_n=[-\sqrt{3},\sqrt{3}\ ]$ and each constant density function $\rho(X_n)$ is $\frac{1}{2\sqrt{3}}$. We thus assume that the joint probability density function ρ of $(X_1,X_2,\cdots,X_N)$ in our numerical experiments is $\frac{1}{(2\sqrt{3})^N}$.

Now let $C(x_1,x_2)=e^{-|x_1-x_2|}$ be a covariance function in our numerical experiment and solve the following eigenvalue problem:

$$\int_D e^{-|x_1-x_2|}\phi_n(x_1)\ dx_1=\lambda_n\phi_n(x_2).$$

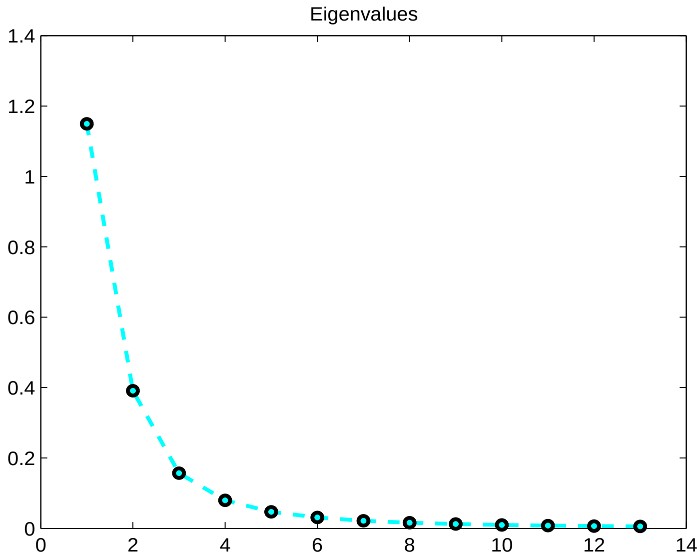

Figure 5.0.1 Eigenvalue decay

Then we have

$$\phi_n(x) = \frac{1}{\sqrt{1 + \frac{\sin(2v_n)}{2v_n}}} \cos(v_n x) \quad \text{if n is odd,}$$

$$\phi_n(x) = \frac{1}{\sqrt{1 - \frac{\sin(2w_n)}{2w_n}}} \sin(w_n x) \quad \text{if n is even,}$$

$$\lambda_n = \frac{2}{v_n^2 + 1} \quad \text{if n is odd, and}$$
$$\lambda_n = \frac{2}{w_n^2 + 1} \quad \text{if n is even,}$$

where v_n is a solution of $1 - v\tan(v) = 0$ and w_n is a solution of $w + \tan(w) = 0$; see Ghanem, R. G. and Spanos, P. D. (1991).

Note that λ_n gets smaller as v_n or w_n gets larger; see Figure 5.0.1.

We now consider our model problem with a target solution $U = 1$: find the solution of

$$-(a(x,y)u'(x,y))' = f(x) \quad \forall (x,y) \in (-1,1) \times \prod_{n=1}^{N}(-\sqrt{3},\sqrt{3}), \tag{5.0.2}$$
$$u(x,y) = 0 \quad \forall (x,y) \in \{-1,1\} \times \prod_{n=1}^{N}(-\sqrt{3},\sqrt{3})$$

with flexible input data $f(x)$ to minimize

$$\mathcal{J}_\beta(u,f) = \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{(2\sqrt{3})^N} \int_{-1}^{1} |u-1|^2 \, dxdy + \frac{\beta}{2}\int_{-1}^{1} |f|^2 \, dx. \tag{5.0.3}$$

Note that here $'$ means differentiation with respect to x only and that the finite K-L expansion of $a(x,y)$ is given by

$$a(x,y) = 10 + \sum_{n=1}^{N} \sqrt{\lambda_n}\phi_n(x)y_n,$$

where $(\lambda_n, \phi_n)_{1\le n\le N}$ are eigenpairs of

$$\int_D e^{-|x_1-x_2|}\phi_n(x_1)\, dx_1 = \lambda_n\phi_n(x_2).$$

In our model problem, because our objective functional of solution is positive, we have 0 as the possible minimum value of our functional. To have the minimum zero, we should have $u = 1$ and $f = 0$. In fact, in our constraint equation, if $u = 1$, then the left hand side of our constraint PDE is 0. This implies that our control $f = 0$ in the right hand side of the constraint equation. We thus expect that $Eu^{hp} = 1$ and $f^h = 1$ from our simulation.

Figure 5.0.2 shows numerical results for both Eu^{hp} and f^h when our target solution U is assumed to be simply 1, the expected value of our stochastic coefficient $Ea(x)$ in the K-L expansion of $a(x,y)$ is 10, a maximum grid size parameter $h = 0.25$, the maximum degrees of polynomials in y_1 and y_2 directions are 2 and 1, respectively, and the number of terms in the K-L expansion is 3. Actually as we expected, Eu^{hp} is almost 1 and f^h is almost 0; see Table 5.0.1.

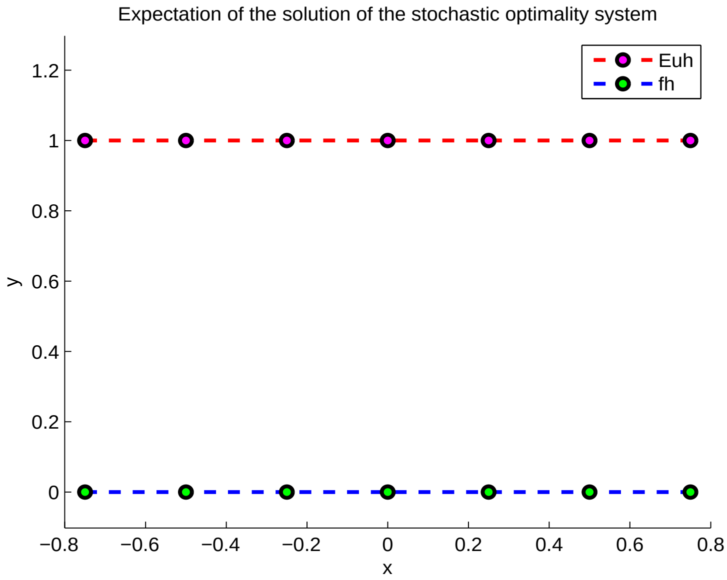

Figure 5.0.2 $N = 2,\ p = (2, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

x	Eu^{hp}	f^h
-0.75	1.00000002127853	0.00000101674666
-0.50	1.00000003724766	0.00000101596809
-0.25	1.00000004682037	0.00000101540405
0.00	1.00000005000783	0.00000101519861
0.25	1.00000004682037	0.00000101540405
0.50	1.00000003724766	0.00000101596809
0.75	1.00000002127853	0.00000101674666

Table 5.0.1 $N = 2,\ p = (2, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

Also, under the same conditions as above, except for the maximum degrees of polynomials in the y-direction and the number of terms in the K-L expansion, we have results in Figure 5.0.3 and Table 5.0.2, Figure 5.0.4 and Table 5.0.3, Figure 5.0.5 and Table 5.0.4, Figure 5.0.6 and Table 5.0.5, and Figure 5.0.7 and Table 5.0.6.

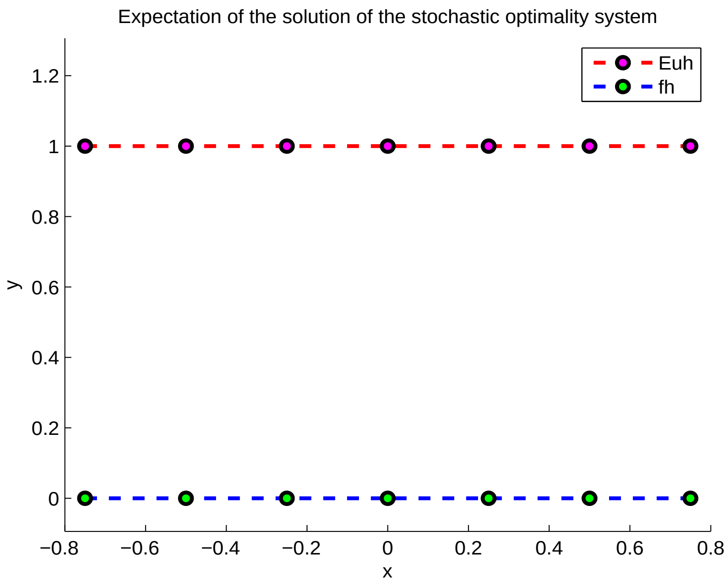

Figure 5.0.3 $N = 3,\ p = (3, 2, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

x	Eu^{hp}	f^h
-0.75	1.00000002133737	0.00000101815773
-0.50	1.00000003733506	0.00000101737070
-0.25	1.00000004692201	0.00000101680426
0.00	1.00000005011761	0.00000101659822
0.25	1.00000004692201	0.00000101680426
0.50	1.00000003733506	0.00000101737070
0.75	1.00000002133737	0.00000101815773

Table 5.0.2 $N = 3,\ p = (3, 2, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

x	Eu^{hp}	f^h
-0.75	1.00000002136009	0.00000101860770
-0.50	1.00000003736690	0.00000101783114
-0.25	1.00000004696721	0.00000101741985
0.00	1.00000005016641	0.00000101730710
0.25	1.00000004696721	0.00000101741985
0.50	1.00000003736690	0.00000101783114
0.75	1.00000002136009	0.00000101860770

Table 5.0.3 $N = 4,\ p = (4, 2, 2, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

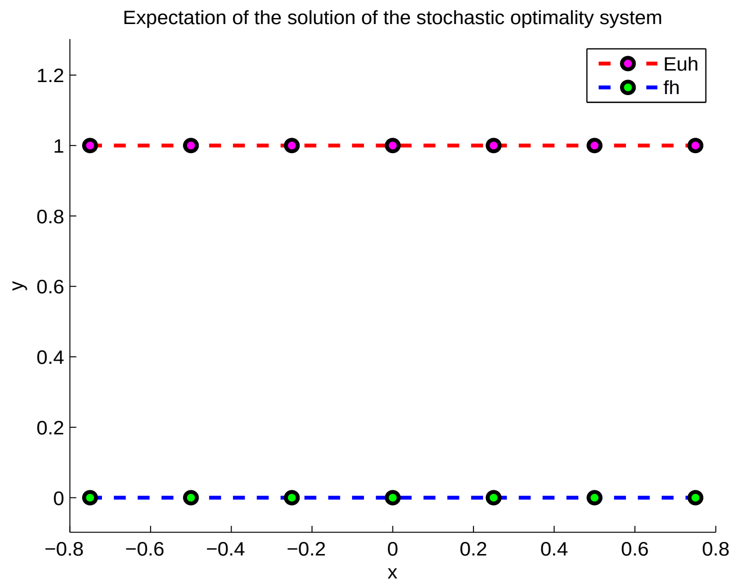

Figure 5.0.4 $N = 4,\ p = (4, 2, 2, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

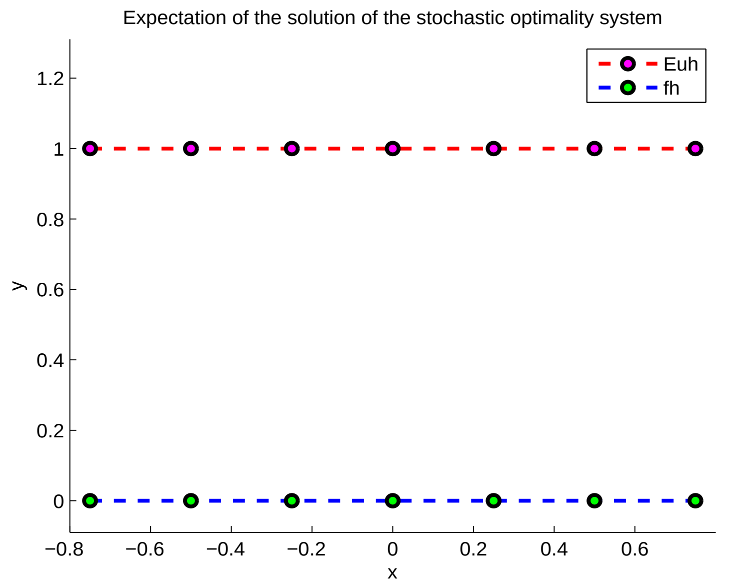

Figure 5.0.5 $N = 5,\ p = (3, 2, 1, 1, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

x	Eu^{hp}	f^h
-0.75	1.00000002137040	0.00000101885355
-0.50	1.00000003738355	0.00000101807682
-0.25	1.00000004698822	0.00000101766542
0.00	1.00000005018878	0.00000101755245
0.25	1.00000004698822	0.00000101766542
0.50	1.00000003738355	0.00000101807682
0.75	1.00000002137041	0.00000101885355

Table 5.0.4 $N = 5,\ p = (3, 2, 1, 1, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

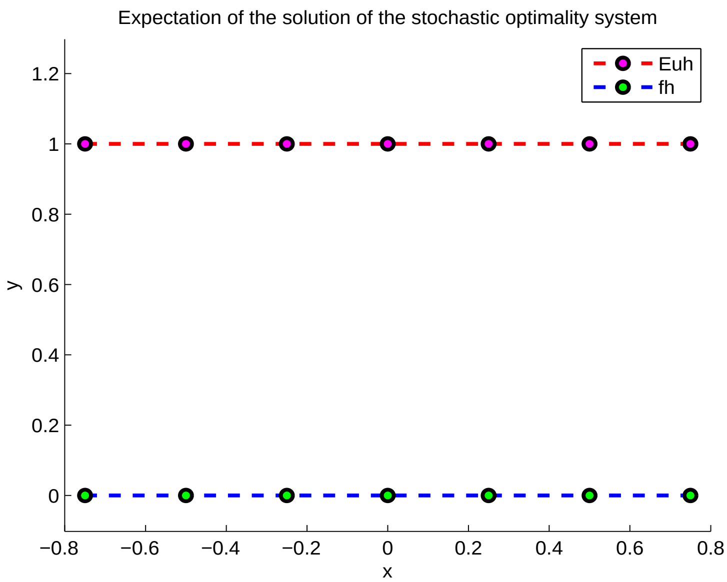

Figure 5.0.6 $N = 6,\ p = (2, 2, 1, 1, 1, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

x	Eu^{hp}	f^h
-0.75	1.00000002137419	0.00000101892467
-0.50	1.00000003739228	0.00000101818290
-0.25	1.00000004699783	0.00000101776594
0.00	1.00000005019902	0.00000101762202
0.25	1.00000004699783	0.00000101776594
0.50	1.00000003739228	0.00000101818290
0.75	1.00000002137419	0.00000101892467

Table 5.0.5 $N = 6,\ p = (2, 2, 1, 1, 1, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

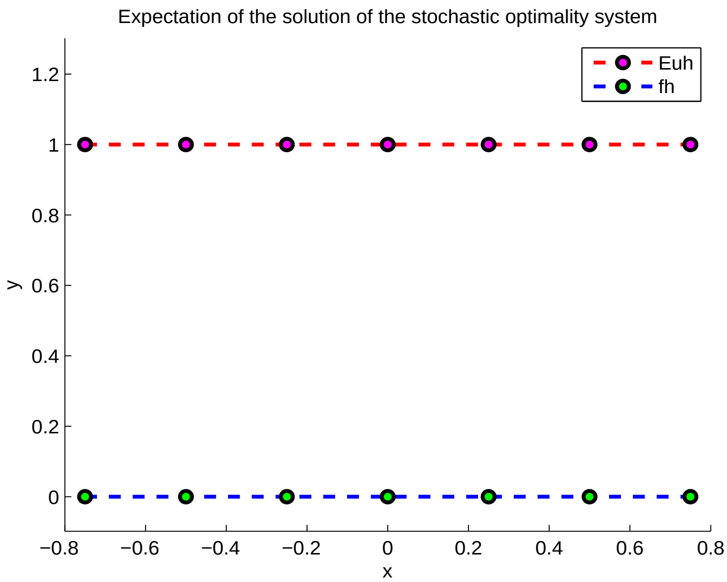

Figure 5.0.7 $N = 7,\ p = (2, 1, 1, 1, 1, 1, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

x	Eu^{hp}	f^h
-0.75	1.00000002137533	0.00000101894930
-0.50	1.00000003739580	0.00000101821011
-0.25	1.00000004700272	0.00000101779418
0.00	1.00000005020407	0.00000101765050
0.25	1.00000004700272	0.00000101779418
0.50	1.00000003739580	0.00000101821011
0.75	1.00000002137533	0.00000101894930

Table 5.0.6 $N = 7,\ p = (2, 1, 1, 1, 1, 1, 1),\ h = 0.25,\ Ea(x) = 10,\ U = 1$

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