Intermediate Micro Review

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Contents

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In these notes we briefly discuss the two core topics of Microeconomics: (1) the theory of consumer choice and (2) the theory of producer choice. Modern Macroeconomic theory uses models with explicit decision making of households and firms, and these notes will hopefully provide the necessary background for exploring such models.

1 Consumer's Choice

1.1 Preferences

There are two goods X, Y. A consumption bundle is a pair (x, y), where x is the quantity of good X and y is the quantity of good Y. We will denote bundles explicitly, by the quantities of goods (x_1, y_1) or (x_2, y_2), or with capital letters $A, B, C, ...$ Consumer's preferences are described by the **weak preference relation** $\succsim$, so that $A \succsim B$ means that bundle A is at "least as good as" bundle B. We assume that the weak preference relation $\succsim$ satisfies the following axioms:

(A1) **Completeness** - for any two bundles A and B, we have either $A \succsim B$ or $B \succsim A$ or both. Completeness means that people are able to rank any two bundles.

(A2) **Transitivity** - for any bundles A, B, C, if $A \succsim B$ and $B \succsim C$ then $A \succsim C$.

A preference relation $\succsim$ is called **rational** if it satisfies these two axioms (completeness and transitivity). In the appendix we will show that if a consumer has preferences that violate transitivity, then he can be deprived of all his wealth.

Using the weak preference relation, we can define two more relations:

1. **Strict preference relation** $\succ$, so that $A \succ B$ means that bundle A is "strictly better than" bundle B, and we define $A \succ B$ as $A \succsim B$ but not $B \succsim A$.

2. **Equivalence relation** $\sim$, so that $A \sim B$ means that bundle A is "equivalent to" bundle B, and we define $A \sim B$ as $A \succsim B$ and also $B \succsim A$.

If preferences satisfy the completeness and transitivity, and also another technical assumption (continuity), then it is possible to represent such preferences with continuous utility function $U(x, y)$. A utility function assigns a number to any bundle, and higher number means better bundle. Formally, we say that utility function $U : \mathbb{R}^2 \to \mathbb{R}$ represents the weak preference relation $\succsim$ if for any two bundles $(x_1, y_1) \succsim (x_2, y_2) \iff U(x_1, y_1) \geq U(x_2, y_2)$ for any two bundles (x_1, y_1) and (x_2, y_2).

In this course, we will make two more assumptions about preferences:

1. **Monotonicity** or "more-is-better": for all $\varepsilon > 0$ we have

$$\begin{aligned} (x + \varepsilon, y) &\succ (x, y) \\ (x, y + \varepsilon) &\succ (x, y) \end{aligned}$$

   Equivalently, using utility function, monotonicity is defined as for all $\varepsilon > 0$, we have

$$\begin{aligned} U(x + \varepsilon, y) &> U(x, y) \\ U(x, y + \varepsilon) &> U(x, y) \end{aligned}$$

2. **Convexity**, or average is preferred to extremes. We will discuss this assumption later.

1.2 Budget Constraint

We assume that consumers are price takers, i.e. the prices of the goods p_x, p_y and consumer's income is I are taken as given. The budget constraint is given by

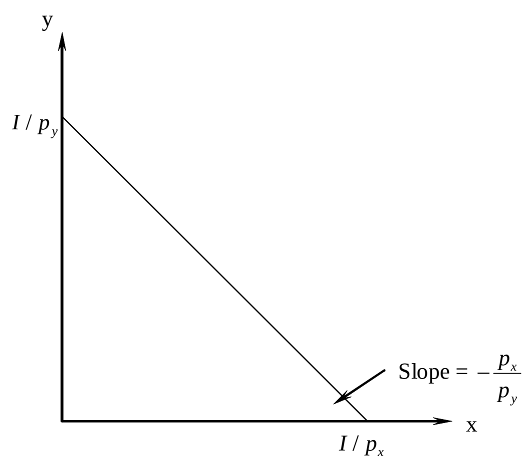
$$p_x x + p_y y = I$$

This means that the spending on x and the spending on y must add up to the income[1] I. Solving for y from the budget constraint gives

$$y = \frac{I}{p_y} - \frac{p_x}{p_y} x$$

and it is illustrated in figure 1

Figure 1: Budget constraint



Points on or below the budget frontier are feasible, while points outside the budget constraint are not feasible. Notice that changes in income will not affect the slope of the budget constraint. When income goes up, the budget constraint will shift outward while remaining parallel to the original one. Changes in prices will be reflected in the slope of the budget constraint. An increase in the price of x for example will decrease the maximal quantity of x that the consumer can purchase, so the x-intercept will move to the left. If both prices change by the same percent however, the slope of the budget constraint will not change.

[1] The more proper way of writing the budget constraint is $p_x x + p_y y \leq I$, which means that the spending on the two goods cannot exceed the income, but some income can be left over. We will always make the assumption of monotonicity though, that will guarantee that the consumer will always exhust his income.

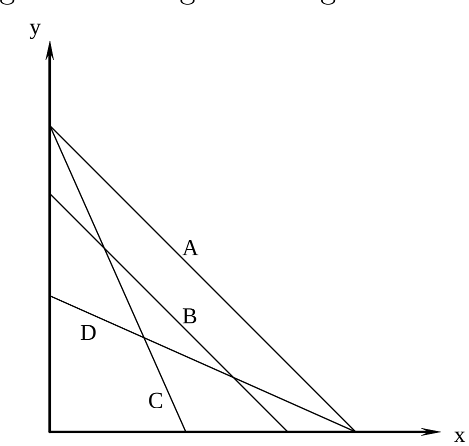
Review question.

1. Draw a budget constraint that would result after a decline in income.
2. Draw a budget constraint that would result after a proportional increase in both prices, p_x and p_y.
3. Draw a budget constraint that would result after an increase in p_x.
4. Draw a budget constraint that would result after an increase in p_y.
5. Draw a budget constraint that would result after an increase of 50% in income and both prices.

Answer. Figure 2 shows 4 budget constraints labeled A, B, C, and D. The original budget constraint, before any changes took place, is labeled A. The changes and the corresponding budget constraints are listed below.

1. B
2. B
3. C
4. D
5. A

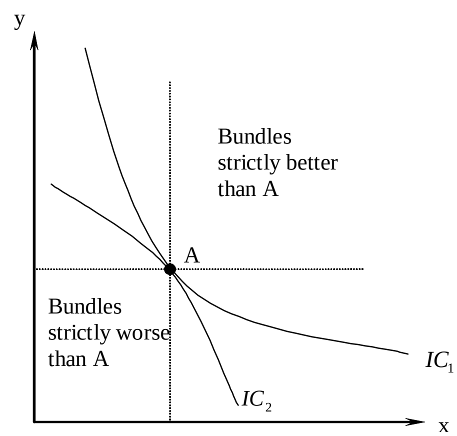
Figure 2: Changes in budget constraints



1.3 Indifference Curves

We can represent preferences graphically with indifference curves. An indifference curve is a collection of all the bundles that are equivalent. Based on our assumptions about consumer's preferences, what would the indifference curves look like? Choose an arbitrary bundle, such as bundle A in figure 3. The figure shows the regions of bundles that are strictly better than A and strictly worse than A. recall that the monotonicity assumption implies that all the bundles that have more of at least one good are strictly better, and all the bundles that have less of at least one good are strictly worse. Thus, the bundles that are equivalent to A cannot be in those regions. Thus, the indifference curve that goes through the point A, i.e. the curve that contains all the bundles that are equivalent to A, must look like the curve labeled IC_1 or IC_2 in figure 3. Any indifference curve must be decreasing, again by the assumption of monotonicity. In this course we will also assume that the indifference curves have a shape of IC_1, i.e. that the indifference curves are *convex*.

Figure 3: Indifference curves



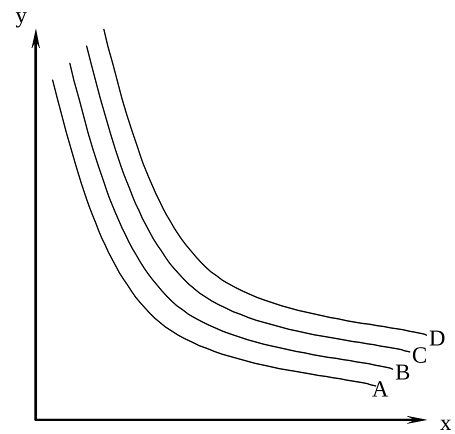
We could have chosen any bundle in the non-negative quadrant, and draw an indifference curve through it. Thus, preferences can be described graphically by an indifference map, as shown in the figure 4. You should be able to prove that higher indifference curve represents bundles that are strictly better than bundles on a lower indifference curve. In figure 4 for example, any bundle on indifference curve labeled B is strictly better that any bundle on A. Thus, the consumer would like to choose a bundle on his budget constraint that attains the highest indifference curve. Another property of indifference curves that is very important is that two distinct indifference curves cannot intersect. You should be able to prove this property as well.

To summarize, there are three important properties of indifference curves that you should be able to prove:

1. Indifference curves are downward slopping.

2. Higher indifference curve contains bundles that are strictly better than those on lower indifference curves.

Figure 4: Indifference map

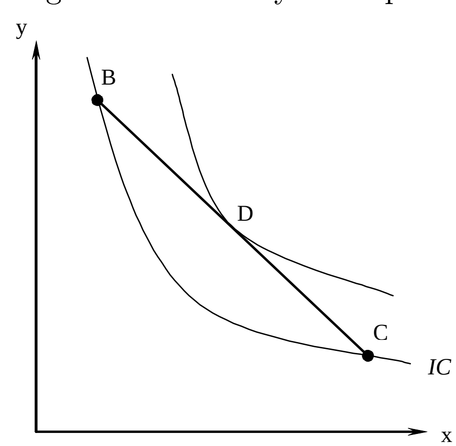


3. Two distinct indifference curves cannot intersect.

1.3.1 Convexity assumption

We assume that the indifference curves are convex. This assumption means that the consumer prefers the average over extremes. Figure 5 illustrates this point. Consider bundles B and C in figure 5. These are extreme bundles; B contains very few X and a lot of Y and C contains a lot of X and very few Y. The picture shows that this consumer would prefer a bundle like D which is a convex combination of B and C. In other words, D is an average of B and C. This assumption makes sense. Suppose that you throw a party and you need to buy drinks (X) and food (Y). Bundles B corresponds to having too little drinks and lots of food, while bundle C corresponds to having too little food and lots of drinks. Observe that

Figure 5: Convexity assumption

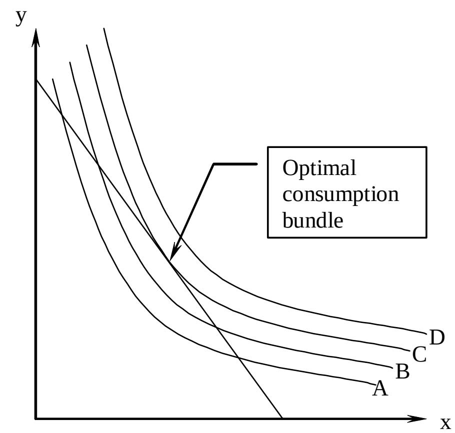


bundle D, which is an average of B and C is better than either of the extreme bundles.

1.4 Optimal Choice: graphical illustration

The consumer wants to choose the best bundle that is feasible. Figure 6 illustrates the optimal choice for the consumer. Notice that the consumer would like to consume a bundle on indifference curve D, but these bundles are not affordable. The highest indifference curve that can be attained within the budget constraint is C.

Figure 6: Optimal Choice



Under the assumption that indifference curves are convex, the optimal bundle is at the point where the highest possible indifference curve is tangent to the budget constraint. This means that at the optimal point, the slope of the budget constraint is equal to the slope of indifference curves.

Condition for optimality of consumption bundle:

Slope of indifference curve = Slope of the budget constraint

1.5 Optimal Choice: mathematical treatment

The consumer's problem is

$$\max_{x,y} U(x,y)$$
$$\text{s.t.}$$
$$p_x x + p_y y = I$$

This is a standard form of a constrained optimization problem. After the "max" follows the *objective function,* which is what we attempt to maximize. In the consumer optimization problem the objective function is the utility. Under the max we write the *choice variables*, these are the variables that we choose. In the consumer optimization problem the choice variables are the quantities of the goods purchased. The abbreviation "s.t." means subject to or such that. The next component of the constrained optimization problem is the constraint, which is the budget constraint in the consumer's problem. Thus, in any constrained optimization problem we maximize the objective function subject to some constraints. In the consumer's problem we maximize the utility subject to the budget constraint[2]. We will show two methods of solving the above problem.

1.5.1 Method 1: using the optimality condition

In this method we apply directly the optimality condition derived above (Slope of indifference curve = Slope of the budget constraint). The slope of the budget constraint is

$$-\frac{p_x}{p_y}$$

What is the slope of the indifference curves? Mathematically, an indifference curve is defined as follows:

$$U(x,y) = \bar{U}$$

where $\bar{U}$ is some constant utility level, say 17. Thus, an indifference curve consists of all the bundles (x,y) that give the same utility of $\bar{U}$. $U(x,y)$ is "implicit function", as opposed to "explicit", where y is described as a function of x. By the implicit function theorem, the slope of the indifference curves is given by[3]

$$\frac{dy}{dx} = -\frac{U_x(x,y)}{U_y(x,y)}$$

where $U_x(x,y) = \partial U(x,y)/\partial x$ and $U_y(x,y) = \partial U(x,y)/\partial y$ are partial derivatives of U. These partial derivatives are called "marginal utility" in economics. Thus, $U_x(x,y)$ is the marginal utility from x and $U_y(x,y)$ is the marginal utility from y. The marginal utility from x, $U_x(x,y)$, measures the change in total utility as x changes a little. Recall that mathematically, the partial derivative of U with respect to x is defined as

$$\lim_{\Delta x \to 0} \frac{U(x+\Delta x, y) - U(x,y)}{\Delta x}$$

which measures the rate at which U changes when x changes by a small unit. Similarly, $U_y(x,y)$ measures the change in U as we change y by a small unit.

Thus, mathematically, the condition of optimality of consumption bundle is given by

$$-\frac{U_x(x,y)}{U_y(x,y)} = -\frac{p_x}{p_y}$$

or

$$\frac{U_x(x,y)}{U_y(x,y)} = \frac{p_x}{p_y}$$

The term on the left hand side is called the Marginal Rate of Substitution between x and y and denoted by $MRS_{x,y}$. It tells us how many units of y is the consumer willing to trade for one unit of x. Suppose that $U_x(x,y) = 3$ and $U_y(x,y) = 4$. This means that an extra unit of

[2] For more on constrained optimization see section 5.3 in the Math Review notes.

[3] The proof is in the appendix

x increases the utility by 3 and extra unit of y increases the utility by 4. The $MRS_{x,y} = 3/4$ which means that the consumer is willing to trade 1 unit of x for $3/4$ units of y. This makes sense because $3/4$ units of y generate the same utility as 1 unit of x.

To solve the consumer's problem, we need to combine the optimality condition with the budget constraint and solve for x and y:

$$\begin{aligned} 1.\ \frac{U_x(x,y)}{U_y(x,y)} &= \frac{p_x}{p_y} \\ 2.\ p_x x + p_y y &= I \end{aligned}$$

This is a system of two equations with two unknowns (x and y), and we can solve for the optimal x and y as a function of the *exogenous parameters* p_x, p_y, and I. The solution to the consumer optimization problem is called *demand*:

$$\begin{aligned} \text{Demand for } x &: \quad x(p_x, p_y, I) \\ \text{Demand for } y &: \quad y(p_x, p_y, I) \end{aligned}$$

1.5.2 Method 2: Lagrange method

This method allows us to get rid of the constraint at the cost of introducing additional variable into the optimization problem. The consumer's problem is

$$\max_{x,y} U(x,y)$$
$$\text{s.t.}$$
$$p_x x + p_y y = I$$

The corresponding Lagrange function (Lagrangian) is

$$\mathcal{L} = U(x,y) - \lambda [p_x x + p_y y - I]$$

The first term in the Lagrangian is the objective function, λ is the Lagrange multiplier[4], and it multiplies the difference between the left hand side and the right hand side of the budget constraint. Now we have a new function $\mathcal{L}$ to maximize. This function has 3 variables (x, y, λ), but the advantage is that it does not have constraints. The first order conditions are (differentiating with respect to all 3 variables and equating to zero):

$$\begin{aligned} (1)\ \mathcal{L}_x &= U_x(x,y) - \lambda p_x = 0 \\ (2)\ \mathcal{L}_y &= U_y(x,y) - \lambda p_y = 0 \\ (3)\ \mathcal{L}_\lambda &= p_x x + p_y y - I = 0 \end{aligned}$$

The last partial derivative is simply the budget constraint. Typically, people don't bother to write it. The above system of equations has 3 equations and 3 unknowns (x, y, λ). Lets solve it. Rewriting (1) and (2) gives

$$\begin{aligned} (1)\ U_x(x,y) &= \lambda p_x \\ (2)\ U_y(x,y) &= \lambda p_y \end{aligned}$$

[4] See section 5.3 in the Math Review notes.

Dividing (1)/(2) gives

$$\frac{U_x(x,y)}{U_y(x,y)} = \frac{p_x}{p_y}$$

Which is the same condition for optimality of consumption bundle as we derived above graphically and mathematically, and it says that at the optimal consumption bundle the marginal rate of substitution between x and y is equal to the absolute value of the slope of the budget constraint.

Now we need to combine the optimality condition with the budget constraint and solve for x and y:

$$\begin{aligned} 1.\ \frac{U_x(x,y)}{U_y(x,y)} &= \frac{p_x}{p_y} \\ 2.\ p_x x + p_y y &= I \end{aligned}$$

This is a system of two equations with two unknowns (x and y), and we can solve for the optimal x and y as a function of the *exogenous parameters* p_x, p_y, and I. The solution to the consumer optimization problem is called *demand*:

$$\begin{aligned} \text{Demand for } x &: \quad x(p_x, p_y, I) \\ \text{Demand for } y &: \quad y(p_x, p_y, I) \end{aligned}$$

We can solve for λ as well from the first order conditions:

$$\lambda = \frac{U_x(x,y)}{p_x} = \frac{U_y(x,y)}{p_y}$$

The lagrange multiplier has economic meaning: at the optimum it is equal to the marginal utility from extra \$1 spent on x or on y. It can be shown that the value of λ also gives the increase in the maximal utility as we relax the budget constraint by \$1. Suppose that initially the income is \$40,000 and we solve the consumer's problem and find the value of utility achieved. Now suppose that I want to know by how much will my utility change if my income becomes \$40,001. The answer is that the maximal utility will go up by approximately λ. See the Math Review notes for additional illustration of this point.

1.5.3 The intuition behind the optimality condition

The condition for optimality of consumption bundle is

$$\frac{U_x(x,y)}{U_y(x,y)} = \frac{p_x}{p_y}$$

This condition can be rewritten as

$$\frac{U_x(x,y)}{p_x} = \frac{U_y(x,y)}{p_y}$$

The left hand side is the marginal utility from extra \$1 spent on x and the right hand side is the marginal utility from extra \$1 spent on y. To see this, observe that \$1 buys $1/p_x$

units of x or $1/p_y$ units of y. Thus the utility generated by extra \$1 spent on x is equal to the marginal utility from x times how many units of x that \$1 can buy. Suppose that the consumer purchased a bundle (x, y) for which the above condition does not hold, e.g.,

$$\frac{U_x(x,y)}{p_x} > \frac{U_y(x,y)}{p_y}$$

This means that by increasing the spending on x and decreasing the spending on y the consumer is able to increase his utility and therefore the bundle is not optimal. Similarly if

$$\frac{U_x(x,y)}{p_x} < \frac{U_y(x,y)}{p_y}$$

the consumer can increase his utility by spending more on y and reducing the spending on x.

Notice that in the discussion of the Lagrange method, the Lagrange multiplier is equal at the optimum to the marginal utility from extra \$1 spent on x or on y.

1.6 Examples

1. Suppose that consumer's preferences are represented by $U(x,y) = x^{\alpha}y^{\beta}$, $\alpha, \beta > 0$. This is called Cobb-Douglas utility.

   (a) Derive the consumer's demand for x and y.
   The Lagrangian

   $$\mathcal{L} = x^{\alpha}y^{\beta} - \lambda[p_x x + p_y y - I]$$

   The first order conditions

   $$\begin{aligned} (1)\ \ \mathcal{L}_x &= \alpha x^{\alpha-1}y^{\beta} - \lambda p_x = 0 \\ (2)\ \ \mathcal{L}_y &= \beta x^{\alpha}y^{\beta-1} - \lambda p_y = 0 \end{aligned}$$

   From (1) and (2) we obtain the condition for optimality of consumption bundle

   $$\begin{aligned} (1)\ \ \alpha x^{\alpha-1}y^{\beta} &= \lambda p_x \\ (2)\ \ \beta x^{\alpha}y^{\beta-1} &= \lambda p_y \end{aligned}$$

   $$(1)/(2)\ \ \frac{\alpha x^{\alpha-1}y^{\beta}}{\beta x^{\alpha}y^{\beta-1}} = \frac{p_x}{p_y}$$

   Thus, the condition for optimality is

   $$\frac{\alpha y}{\beta x} = \frac{p_x}{p_y}$$

   Here the LHS is $MRS_{x,y}$ and RHS is the absolute value of the slope of B.C. Solving for $y$gives

   $$y = \frac{\beta}{\alpha}\frac{p_x}{p_y}x \tag{1}$$

Substituting into the budget constraint

$$\begin{aligned}
p_x x + p_y \left(\frac{\beta}{\alpha} \frac{p_x}{p_y} x \right) &= I \\
p_x x + \frac{\beta}{\alpha} p_x x &= I \\
p_x x \left(1 + \frac{\beta}{\alpha} \right) &= I \\
p_x x \left(\frac{\alpha + \beta}{\alpha} \right) &= I
\end{aligned}$$

This gives the demand for x

$$x = \left(\frac{\alpha}{\alpha + \beta} \right) \frac{I}{p_x}$$

Plug this into equation (1) to find the demand for y

$$\begin{aligned}
y &= \frac{\beta}{\alpha} \frac{p_x}{p_y} \left(\frac{\alpha}{\alpha + \beta} \frac{I}{p_x} \right) \\
y &= \left(\frac{\beta}{\alpha + \beta} \right) \frac{I}{p_y}
\end{aligned}$$

Thus the demand for x and y is given by

$$x = \left(\frac{\alpha}{\alpha + \beta} \right) \frac{I}{p_x}, \quad y = \left(\frac{\beta}{\alpha + \beta} \right) \frac{I}{p_y}$$

(b) Find the optimal consumption bundle when $\alpha = 3$, $\beta = 7$, $p_x = 2$, $p_y = 4$, and $I = 1000$.

$$\begin{aligned}
x &= \left(\frac{3}{3 + 7} \right) \frac{1000}{2} = 150 \\
y &= \left(\frac{7}{3 + 7} \right) \frac{1000}{4} = 175
\end{aligned}$$

Optimal consumption bundle is $(150, 175)$.

(c) Based on the demand functions, what happens to the quantity demand for x and y if I goes up?

The quantity demanded of both goods goes up.

(d) Based on the demand functions, what happens to the quantity demand for x and y if p_x goes up?

The quantity demanded of x goes down, but nothing happens to the quantity demanded of y.

(e) Based on the demand functions, what happens to the quantity demand for x and y if p_y goes up?

The quantity demanded of y goes down, but nothing happens to the quantity demanded of x.

(f) In this example x and y

i. substitutes
ii. complements
iii. not related as substitutes or complement
They are not related (iii).

(g) What happens to the fraction of income spent on each good as the prices or income change?

With Cobb-Douglas preferences, the fraction of income spent on each good is fixed and does not depend on prices or income. The demand is

$$x = \left(\frac{\alpha}{\alpha+\beta}\right)\frac{I}{p_x}, \quad y = \left(\frac{\beta}{\alpha+\beta}\right)\frac{I}{p_y}$$

The fraction of income spent on x is

$$\frac{p_x x}{I} = \frac{\alpha}{\alpha+\beta}$$

and the fraction of income spent on y is

$$\frac{p_y y}{I} = \frac{\beta}{\alpha+\beta}$$

These fractions depend only on the parameters of the utility function, α and β, but not on prices or income. In this example, $\alpha = 3$, $\beta = 7$, so the consumer will always spend 30% of his income on x and the rest 70% on y, regardless what the prices and income are.

1.7 Invariance of utility functions

An important and very useful property of utility functions is invariance with respect to monotone increasing transformations. What this means is that if $U(\cdot,\cdot)$ is a utility function that represents some preferences, and f is some monotone increasing function, then

$$V = f(U)$$

represents the same preferences. In other words, consumer with utility function V will make the same choices as consumer with utility function U.

1.7.1 Examples

1. Suppose that $U(x,y) = x^a y^b$, $a,b > 0$. Let $f(U) = U^{\frac{1}{a+b}}$, thus f is a monotone increasing function. Then

$$V(x,y) = f(U(x,y)) = \left(x^a y^b\right)^{\frac{1}{a+b}} = x^{\frac{a}{a+b}} y^{\frac{b}{a+b}}$$

   represents the same preferences. Denoting $\alpha = \frac{a}{a+b}$ and thus $(1-\alpha) = \frac{b}{a+b}$ allows us to represent the Cobb-Douglas preferences with

$$V(x,y) = x^{\alpha} y^{1-\alpha}$$

   that is, we can normalize the exponents to sum up to 1. The demand for x and for y in the previous example becomes much simpler

$$x = \alpha \frac{I}{p_x}, \;\; y = (1-\alpha)\frac{I}{p_y}$$

2. Suppose that $U(x,y) = x^a y^b$, $a,b > 0$. Let $f(U) = \ln(U)$. Then these same preferences can be represented by

$$V(x,y) = \alpha \ln x + (1-\alpha)\ln y$$

   If you solve the consumer's problem with U or V, the solution will be exactly the same, and the demand will be

$$x = \alpha \frac{I}{p_x}, \;\; y = (1-\alpha)\frac{I}{p_y}$$

3. Solve for the consumer's demand with Cobb-Douglas preferences, when there are arbitrary number of goods. For the solution see the appendix.

1.8 Income and substitution effects

What do we expect to happen to the quantity demanded of a good when the price of a good increases? We might think at first that the quantity demanded should necessarily go down, that is the demand curve is downward slopping. Economic theory however does not preclude the possibility of an upward slopping demand curve. There is no such thing as "Law of Demand" which guarantees that the demand curve must be decreasing in price, but rather some people make this as an assumption.

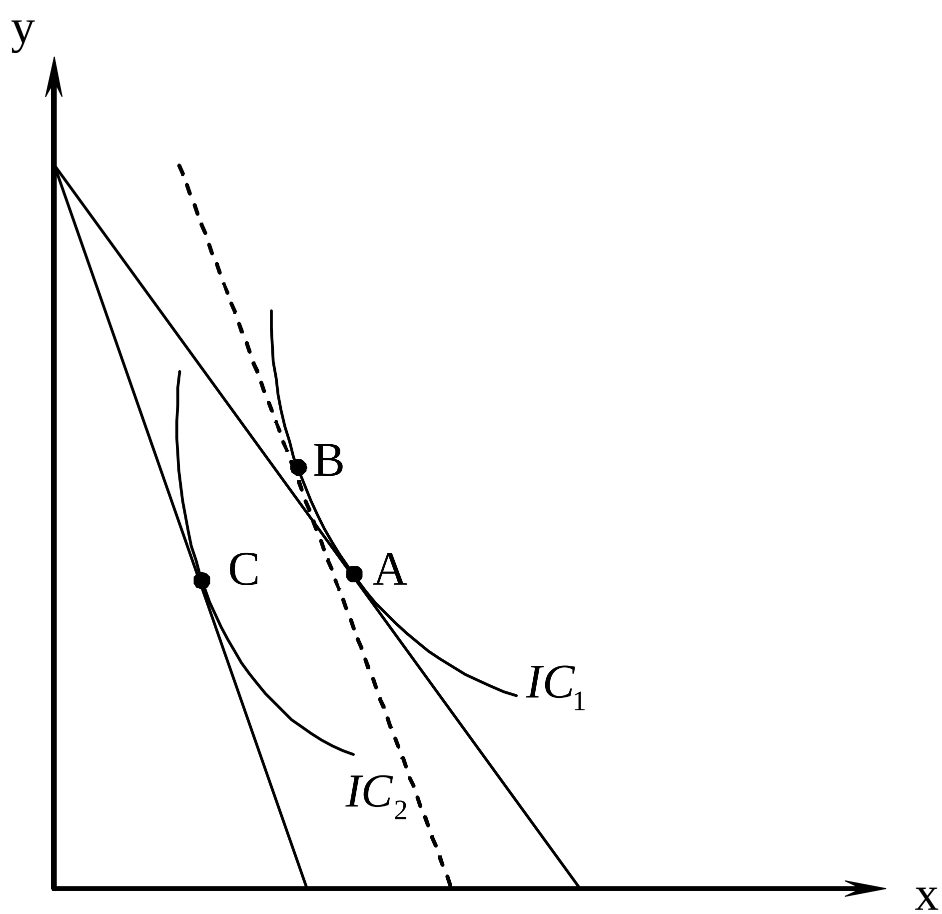
Economists recognize that when the price of a good goes up, there are two effects on the quantity demanded:

1. **Substitution effect**, the change in the quantity demanded due to the change in relative prices, keeping the purchasing power constant.

2. **Income effect**, the change in the quantity demanded due to changes in the purchasing power.

Intuitively, when the price of some good goes up, this has a negative income effect because with the same income we can no longer buy the same bundle as before the price increase and we cannot achieve the level of utility as we had before the price change. Now, suppose that somebody compensated us for the loss of purchasing power by giving us enough income to attain the same level of utility with the new prices as we had with the old prices. Then, this compensation would have neutralized the negative income effect and the only change in consumption bundle would be due to the substitution effect. Figure 7 illustrates the income and substitution effects.

Figure 7: Income and substitution effects



The initial consumption bundle is A. Now suppose that $p_x \uparrow$, which is reflected in the increase in the slope of the budget constraint. The new budget constraint is below the old one, and the new optimal consumption bundle is C. Thus, the total effect of an increase in the price of x is the move from bundle A to bundle C. Now we want to break this change into the income and substitution effects. Suppose that we compensate the consumer for the price increase and give him enough income so that with the new prices he can attain the same utility level as before. This compensation would have resulted in the dashed budget constraint which is tangent to the initial indifference curve (IC_1) and has the same slope as the new price ratio. Thus, the compensated budget constraint neutralized the income effect, so that the only remaining effect is the substitution effect. With the compensated budget constraint the consumer would have chosen the bundle B. Thus the change $A \to B$ is the substitution effect. Therefore, the change $B \to C$ is the income effect. To summarize, we have decomposed the total effect of a price change $A \to C$ into two effects: (1) substitution effect $A \to B$ and (2) income effect $B \to C$.

Now we are ready to answer the question "what happens to x when $p_x \uparrow$? The substitution effect is always negative for a price increase, because when $p_x \uparrow$, the substitution effect is reflected in a movement up along the indifference curve. We know that indifference curves must be downward slopping, so the substitution effect always causes the consumer to substitute away from the good which became more expensive. In figure 7, as a result of a substitution effect the consumer reduces x and increases y. What about the income effect? From principles of microeconomics, we know that for a **normal** good the quantity demanded changes in the same direction as income, while for *inferior* goods the quantity demanded changes in the opposite direction from income. Thus, when $p_x \uparrow$, this is a negative income effect, so if x is normal the income effect is to reduce x while if x is inferior, the income effect is to increase x. The following table summarizes these results

$p_x \uparrow$

	x is normal	x is inferior
Substitution effect	$x \downarrow$	$x \downarrow$
Income effect	$x \downarrow$	$x \uparrow$
Total effect	$x \downarrow$	$x?$

Economic theory therefore has the following result: if the price of a *normal* good increases, then the quantity demanded of the good will go down for sure. If however the good is *inferior*, the quantity demanded will go down only if the substitution effect is stronger than the income effect.

2 Producer's Choice

We represented the consumer's preferences with a utility function. In a similar fashion we represent the production technology with a production function. We assume that there are two inputs, capital (K) and labor (L).

Definition 1 *A* ***production function*** $F(K, L)$ *gives the maximal possible output that can be produced when using K units of capital and L units of labor.*

Example. A widely used production function in economics is the Cobb-Douglas production function

$$Y = AK^{\theta}L^{1-\theta}, \quad 0 < \theta < 1$$

where Y is the output, A is productivity parameter, K is the capital, L is labor, and θ is called the capital share in output. We will discuss this parameter later. Suppose that $A = 10,\ K = 7,\ L = 20,\ \theta = 0.35$. What is the maximal output that can be produced with this technology and these inputs? Answer:

$$Y = 10 \cdot 7^{0.35} \cdot 20^{1-0.35} \approx 138.5$$

Definition 2 *A production $F(K, L)$ exhibits* ***constant returns to scale*** *if*

$$F(\lambda K, \lambda L) = \lambda F(K, L), \quad \forall \lambda > 0$$

This means if a function exhibits constant returns to scale, then when we double all the inputs, the output is also doubled. To see this let $\lambda = 2$ in the above definition. Then

$$F(2K, 2L) = 2F(K, L)$$

Example. The Cobb-Douglas production function exhibits constant returns to scale.

$$A(\lambda K)^{\theta}(\lambda L)^{1-\theta} = \lambda^{\theta}\lambda^{1-\theta}AK^{\theta}L^{1-\theta} = \lambda AK^{\theta}L^{1-\theta}$$

Definition 3 *The marginal product of capital is $F_K(K, L)$ and the marginal product of labor is $F_L(K, L)$.*

Thus, the marginal product of each input is the partial derivative of the production function with respect to the input. In words, the marginal product of capital is the change in total output that results from a small change in capital input. The marginal product of labor is the change in the total output that results from a small change in the labor input. The marginal product is completely analogous to the marginal utility in the consumer theory.

2.1 Firm's profit maximization problem

We assume that the firm is competitive in both the output market and the input markets. That is, the firm takes the prices of output and inputs as given. Let P be the price of the output, and W and R be the wage and the rental rate of capital respectively. The firm's maximization problem is

$$\begin{aligned} & \max_{Y,K,L} P \cdot Y - RK - WL \\ & s.t. \\ Y &= F(K,L) \end{aligned}$$

Hence the firm chooses the output level and how much inputs to employ, and it maximizes profit subject to the technology constraint. The term $P \cdot Y$ is the revenue and therefore $P \cdot Y - RK - WL$ is the profit (revenue - cost). In applications to Macro, we will often express all the magnitudes in real terms. Thus we will divide the profit by the price level P and denote the real prices of inputs by

$$r = \frac{R}{P}, \quad w = \frac{W}{P}$$

The profit maximization problem then becomes

$$\begin{aligned} & \max_{Y,K,L} Y - rK - wL \\ & s.t. \\ Y &= F(K,L) \end{aligned}$$

The easiest way to solve the profit maximization problem is to substitute the constraint into the objective

$$\max_{K,L} F(K,L) - rK - wL$$

This is an unconstrained optimization problem, and the only choice of the firm is the quantities of inputs K and L. The first order conditions for optimal input mix:

$$\begin{aligned} F_K(K,L) - r &= 0 \\ F_L(K,L) - w &= 0 \end{aligned}$$

or

$$\begin{aligned} F_K(K,L) &= r \\ F_L(K,L) &= w \end{aligned}$$

Thus, a competitive firm pays each input its marginal product.

Example. Write the profit maximization problem for a competitive firm with Cobb-Douglas technology and derive the first order conditions for optimal input mix.

Answer:

$$\text{Profit maximization problem:} \quad \max_{K,L} AK^{\theta}L^{1-\theta} - rK - wL$$

First order conditions for optimal input mix:

$$\begin{aligned}\theta A K^{\theta-1} L^{1-\theta} &= r \\ (1-\theta) A K^{\theta} L^{-\theta} &= w\end{aligned}$$

Thus, in competition the rental rate of capital is equal to the marginal product of capital and the wage is equal to the marginal product of labor.

2.2 Factor shares

Suppose that the aggregate output in the economy (real GDP) can be modeled as Cobb-Douglas production function:

$$Y = AK^{\theta} L^{1-\theta}, \quad 0 < \theta < 1$$

And suppose that each input is paid its marginal product. Then a fraction θ of the total output is paid to capital and a fraction $(1-\theta)$ of the total output is paid to labor. To see this notice that the payment to capital is

$$\begin{aligned}rK &= \theta A K^{\theta-1} L^{1-\theta} \cdot K = \theta A K^{\theta} L^{1-\theta} = \theta Y \\ \text{thus } \frac{rK}{Y} &= \theta\end{aligned}$$

and the payment to labor is

$$\begin{aligned}wL &= (1-\theta) A K^{\theta} L^{-\theta} \cdot L = (1-\theta) A K^{\theta} L^{1-\theta} = (1-\theta) Y \\ \text{thus } \frac{wL}{Y} &= 1-\theta\end{aligned}$$

3 Appendix

3.1 Transitivity assumption

Suppose that for some consumer we have $A \succsim B$ and $B \succsim C$ but $C \succ A$. Suppose he owns bundle A. We offer him an exchange of C for A and since he strictly prefers C to A he will be willing to pay 1 penny for the exchange. Then we offer him bundle B in exchange for C and he will accept because $B \succsim C$. Next we offer him to exchange B for A and again he will accept since $A \succsim B$. Now he has the bundle A, the one he had in the beginning, but we took 1 penny from him. By repeating the scheme as many times as we want, he will loose all his money. Thus, transitivity assumption is an assumption about consistency, and it seems reasonable. For example, if you prefer BMW to Infinity and you prefer Infinity to Nissan, then the transitivity assumption implies that you will also prefer BMW to Nissan.

3.2 The slope of indifference curves

The indifference curve is given by

$$U(x, y) = \bar{U}$$

Fully differentiating both sides gives

$$U_x(x, y)\, dx + U_y(x, y)\, dy = 0$$

The full differential gives the total change in the function U as we change x by dx and y by dy. This change corresponds to moving slightly along the indifference curve, so the total change in utility is 0. Rearranging the above gives the slope of the indifference curves (and also proves the part of the implicit function theorem that we used in the notes):

$$\frac{dy}{dx} = -\frac{U_x(x, y)}{U_y(x, y)}$$

3.3 Demand with Cobb-Douglas Preferences and n goods

In this appendix we solve the consumer's problem with Cobb-Douglas preferences and n goods. The consumer's utility over the n goods is given by the utility function $U(x_1, x_2, ..., x_n) = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot ... \cdot x_n^{\alpha_n}$, $\alpha_i > 0\ \forall i$. It is convenient however to work with the logarithmic transformation of this utility function:

$$\tilde{U}(x_1, x_2, ..., x_n) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2 + ... + \alpha_n \ln x_n = \sum_{i=1}^{n} \alpha_i \ln x_i$$

At this point it is important to recall that utility is invariant under monotone increasing transformations. Thus, the consumer's demand will be the same whether his preferences are represented by U or by $\tilde{U} = \ln(U)$.

The consumer's problem that we solve now is therefore

$$\max_{\{x_i\}_{i=1}^n} \sum_{i=1}^n \alpha_i \ln x_i$$
$$s.t.$$
$$\sum_{i=1}^n p_i x_i = I$$

The Lagrangian is

$$\mathcal{L} = \sum_{i=1}^n \alpha_i \ln x_i - \lambda \left[\sum_{i=1}^n p_i x_i - I \right]$$

and the first order conditions are

$$\begin{aligned} \mathcal{L}_{x_i} &= \frac{\alpha_i}{x_i} - \lambda p_i = 0, \ i = 1, ..., n \\ \mathcal{L}_\lambda &= \sum_{i=1}^n p_i x_i - I = 0 \end{aligned}$$

We now find the demand for an arbitrary good i. Take the first order conditions for two goods, i and j, where the good j is can be any good other than i.

$$\begin{aligned} \frac{\alpha_i}{x_i} &= \lambda p_i \\ \frac{\alpha_j}{x_j} &= \lambda p_j \end{aligned}$$

Dividing the first one by the second one gives

$$\frac{\alpha_i}{\alpha_j} \frac{x_j}{x_i} = \frac{p_i}{p_j}$$

thus

$$x_j = \frac{\alpha_j}{\alpha_i} \frac{p_i}{p_j} x_i$$

Now we can substitute the above in the budget constraint

$$\begin{aligned} p_i x_i + \sum_{j \neq i} p_j x_j &= I \\ p_i x_i + \sum_{j \neq i} p_j \left(\frac{\alpha_j}{\alpha_i} \frac{p_i}{p_j} x_i \right) &= I \end{aligned}$$

and solving for x_i

$$\begin{aligned} p_i x_i + \frac{p_i x_i}{\alpha_i} \sum_{j \neq i} \alpha_j &= I \\ p_i x_i \left(1 + \frac{1}{\alpha_i} \sum_{j \neq i} \alpha_j \right) &= I \\ p_i x_i \left(\frac{\alpha_i + \sum_{j \neq i} \alpha_j}{\alpha_i} \right) &= I \\ p_i x_i \left(\frac{\sum_{i=1}^{n} \alpha_i}{\alpha_i} \right) &= I \end{aligned}$$

Thus, the demand for good i is given by

$$x_i = \left(\frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} \right) \frac{I}{p_i}$$

The interpretation is intuitive. The parameter α_i is the weight on good i in the utility function, and $\sum_{i=1}^{n} \alpha_i$ is the sum of all the weights. Thus, the *relative* weight on good i is $(\alpha_i / \sum_{i=1}^{n} \alpha_i)$, and the consumer spends this fraction of his income on good i.

The next example illustrates the results further. Suppose that the consumers solves

$$\max_{x_1, x_2, x_3, x_4} \; 0.2 \ln x_1 + 0.4 \ln x_2 + 0.6 \ln x_3 + 0.8 \ln x_4$$
$$s.t.$$
$$p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 = I$$

The weights on the four goods are $0.2, 0.4, 0.6, 0.8$. The sum of the weights is $0.2 + 0.4 + 0.6 + 0.8 = 2$. Thus, the consumer will spend a fraction $0.2/2$ of his income on the first good, a fraction $0.4/2$ of his income on the second good, a fraction $0.6/2$ of his income on the third good and a fraction $0.8/2$ of his income on the fourth good. The demand is then given by

$$x_1 = \frac{0.1I}{P_1}, \; x_2 = \frac{0.2I}{P_2}, \; x_3 = \frac{0.3I}{P_3}, \; x_4 = \frac{0.4I}{P_4}$$