Finding Roots of Polynomials

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1 Introduction

Polynomials appear all over the maths and sciences, and are extremely important to a number of fields, despite their simple appeance.

2 General Formulas

If a polynomial is of degree 4 or less, the roots can easily (relatively) be found with explicit formulas. Galois showed us that there is not an explicit formula for 5th degree polynomials and above.

2.1 Quadratic Polynomials

For polynomials of the form $ax^2 + bx + c$, the roots are of the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{1}$$

as many people know from seventh or eighth grade.

2.2 Cubic Polynomials

For a polynomial of the form $ax^3 + bx^2 + cx + d$, the formula for the roots is slightly more complex than that of quadratic polynomials.[1]

$$x_k = -\frac{1}{3a}\left(b + u_k C + \frac{\Delta_0}{u_k C}\right),\ k \in 1, 2, 3, \tag{2}$$

where

$$u_1 = 1,\ u_2 = \frac{-1 + i\sqrt{3}}{2},\ u_3 = \frac{-1 - i\sqrt{3}}{2}$$

are the cubic roots of unity and

$$C = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}},$$

with

$$\Delta_0 = b^2 - 3ac$$
$$\Delta_1 = 2b^3 - 9abc + 27a^2 d.$$

Also,

$$\Delta_1^2 - 4\Delta_0^3 = 27a^2\Delta,$$

where Δ is the discriminant, $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$.

This formula was probably not taught to most middle schoolers.

3 Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

4 Graeffe's Method

Graeffe's Method is also known as Graeffe's Root Squaring Method. For a proof on why Graeffe's Method works, see [2].

Let p be a polynomial of degree n, where

$$p(x) = (x - \zeta_1)(x - \zeta_2)\dots(x - \zeta_n).$$

Then,

$$p(-x) = (-1)^n(x + \zeta_1)(x + \zeta_2)\dots(x + \zeta_n).$$

Let $q(x)$ be the polynomial with roots $\zeta_1^2, \zeta_2^2, \dots, \zeta_n^2$,

$$q(x) = (x - \zeta_1^2)(x - \zeta_2^2)\dots(x - \zeta_n^2).$$

As shown in [3],

$$q(x^2) = (-1)^n p(x)p(-x).$$

$q(x)$ can be computed with operations on the coefficients of $p(x)$. Let

$$p(x) = x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$
$$q(x) = x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n.$$

Then,

$$b_k = (-1)^k a_k^2 + 2\sum_{j=0}^{k-1}(-1)^j a_j a_{2k-j},\ a_0 = b_0 = 1.$$

Repeating k times gives a polynomial

$$q^k(y) = y^n + a_1^k y^{n-1} + \dots + a_{n-1}^k y + a_n^k$$

with roots

$$\gamma_1 = \zeta_1^{2^k}, \gamma_2 = \zeta_2^{2^k}, \dots, \gamma_n = \zeta_n^{2^k}.$$

Then, Vieta's relations[4] are used:

$$\begin{aligned}
a_1^k =& -(\gamma_1 + \gamma_2 + \cdots + \gamma_n) \\
a_2^k =& \gamma_1\gamma_2 + \gamma_1\gamma_3 + \cdots + \gamma_{n-1}\gamma_n \\
&\vdots \\
a_n^k =& (-1)^n(\gamma_1\gamma_2 \ldots \gamma_n).
\end{aligned}$$

Since the roots are seperated, the first term is larger than the rest.[5] Thus we have

$$\begin{aligned}
a_1^k \approx& -\gamma_1 \\
a_2^k \approx& \gamma_1\gamma_2 \\
&\vdots \\
a_n^k \approx& (-1)^n(\gamma_1\gamma_2 \ldots \gamma_n).
\end{aligned}$$

4.1 An alternate approach

In his graduate thesis[6], Wankere Mekwi details a similar variant of Graeffe's Method.

$$p_0(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \tag{4}$$

where $a_n = 1$. Equation 4 is replaced with a polynomial still of degree n whose roots are the squares of the roots of Equation 4.

$$p_1(z^2) = (-1)^n p_0(z) p_0(-z), \tag{5}$$

where p_1 is the polynomial whose zeros are the squares of the zeros of p_0. This can be repeated such that

$$p_{r+1}(z^2) = (-1)^n p_r(z) p_r(-z) \tag{6}$$

is the polynomial has roots which are the squares of the roots of p_r. Each iteration r is referred to as a Graeffe Iteration.[7] If the coefficients of $p_r(z)$ are $a_j^{(r)}, j = 0, 1, \ldots, n$, then

$$a_j^{(r+1)} = (-1)^{n-j}\left[(a_j^{(r)})^2 + 2\sum_{k=1}^{\min(n-j,n)} (-1)^k a_{j-k}^{(r)} a_{j+k}^{(r)}\right]. \tag{7}$$

The coeffecients of p_r satisfy

$$\begin{aligned}
a_0^{(r)} =&\sigma_0 = 1 \\
&\vdots \\
a_j^{(r)} =&(-1)^{n-j}\sigma_{n-j}\big(\zeta_1^{2^r}, \zeta_2^{2^r}, \ldots, \zeta_n^{2^r}\big) \\
&\vdots \\
a_{n-1}^{(r)} =&(-1)^{n-1}\sigma_{n-1}\big(\zeta_1^{2^r}, \zeta_2^{2^r}, \ldots, \zeta_n^{2^r}\big) \\
a_n^{(r)} =&(-1)^{n}\sigma_{n}\big(\zeta_1^{2^r}, \zeta_2^{2^r}, \ldots, \zeta_n^{2^r}\big),
\end{aligned} \tag{8}$$

where σ_j is the j-th Elementary Symetric Polynomial[8], and the ζ_j are the roots of p_0. If $|\zeta_1| < |\zeta_2| < \cdots < |\zeta_n|$, then we can make an approximation:

$$\begin{aligned}
a_0^{(r)} =&\sigma_0 = 1 \\
&\vdots \\
a_j^{(r)} \approx&(-1)^{n-j}\zeta_{j+1}^{2^r} \cdots \zeta_n^{2^r} \\
&\vdots \\
a_{n-1}^{(r)} \approx&(-1)^{n-1}\zeta_2^{2^r} \ldots \zeta_n^{2^r} \\
a_n^{(r)} =&(-1)^{n}\zeta_1^{2^r}\zeta_2^{2^r} \ldots \zeta_n^{2^r}.
\end{aligned} \tag{9}$$

Thus we have

$$\zeta_j^{2^r} \approx -\frac{a_j^{(r)}}{a_{j+1}^{(r)}}. \tag{10}$$

4.2 Effectiveness

The run-time of Graeffe's Method is $\mathcal{O}(n^2)$ for each iteration, where n is the degree of the polynomial. Thus r Graeffe Repitions is $\mathcal{O}(rn^2)$.

The biggest weakness of Graeffe's Method is that the coefficients of p_r often become too large for most computers, overflowing the floating-point system.

5 Analytic Functions

5.1 Argument Principle

According to [9], if C is a closed curve in the complex plane which does not pass through a zero of $f(z)$, and R is in the interior of C, then

$$s_n = \frac{1}{2\pi i}\oint_C z^N \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{v} z_j^N, \tag{11}$$

where $z_j, j \in \{1, 2, \ldots, v\}$ are the zeros of f which lie in R. A multiple zero is counted according to its multiplicity. When $N = 0$, equation 11 gives the number of zeros in C. This is a result of the argument principle.[10]

Unfortunately, calculating s_n using the Residue Theorem usually involves finding the roots anyways.

5.2 Numerically Finding Roots

[9] gives a method, which I will refer to as the "Delves-Lyness Method", for numerically locating the zeros of an analytic function (hence the title), which has four sections:

1. Evauluate the number of roots, $s_0 = v$, in the region. If the number is manageble, calculate $s_1, s_2, \ldots, s_v$, and carry on to step (3). Otherwise, continue.
2. Subdivide the region into smaller subregions and repeat step (1).
3. Given a region and $s_1, s_2, \ldots, s_v$, construct and solve the equivalent polynomial $p(z)$.
4. Optionally, take the roots of $p(z)$ as approximations to the roots of $f(z)$, and refine these using an iterative method on $f(z)$.

The paper considers two shapes for R, squares and circles, and gives three methods for determining s_N in the case of circles.

References

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