HOMEWORK 4

Section 4.6: 12, 13, 21[1] 22, 33, 34, 35, 41, 47, 50

1. Given a nonnegative random variable X we can define the *probability generating function of* X for each $0 \le t \le 1$ as

$$f(t) = Et^X = \sum_{k=0}^{\infty} P(X = k)t^k.$$

For example, $Et^{\text{Ber}(p)} = (1-p)t^0 + pt^1 = (1-p) + pt$. Answer these questions for arbitrary X.

(a) What is $f(0)$?

(b) Explain why $f'(1) = EX$.

(c) Explain why $f''(1) = EX(X-1)$.

(d) If $X = \text{Poi}(\lambda)$ show that $f(t) = e^{-\lambda(1-t)}$.

(e) Use the formulas from (b) and (d) to show that $\text{var}(\text{Poi}(\lambda)) = \lambda$.

2. You flip a fair coin and win \$1 with probability $p < 1/2$ and lose \$1 with probability $1 - p$. You start with \$0. Let T be the time it takes for your net winnings to be \$1.

(a) Let T' and T'' be iid copies of T and $Y = \text{Ber}(p)$. Explain why T has the same distribution as

$$Y + (1 - Y)(T' + T'' + 1).$$

(b) Let $f(t) = Et^T$. Explain why the previous equation implies that

$$f(t) = pt + (1-p)tf(t)^2.$$

(c) The resulting quadratic equation $0 = (1-p)tf(t)^2 - f(t) + pt$ has solutions

$$f(t) = \frac{1 \pm \sqrt{1 - 4(1-p)pt^2}}{2(1-p)t}.$$

Use the fact that $f(0) = P(T = 0) = 0$ to decide which root to use.

(d) Show that $f(1) = p/(1-p)$.

(e) Notice that by definition

$$f(1) = \sum_{k=0}^{\infty} P(T = k) = P(T < \infty).$$

Use our assumption that $p < 1/2$ and (d) to explain why $P(T = \infty) = \frac{1-2p}{1-p} > 0$.

(f) Interpret what (e) says about this gambling game.

[1] This was incorrectly #20 for awhile, you can turn in either, but 21 is better