

Midterm Exam 2

October 29, 2019

Time: 1 hour, 30 minutes

Name: _____

Instructions:

1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (5 points). Find the limit

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$$

This is a limit of the form $\frac{0}{0}$. Using L'Hôpital's rule,

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{1} = 1$$

2. (10 points). Prove that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r$$

Remark: This is an important result in finance. When periodic interest rate is r , and the interest is compounded n times per period, the amount after one period is: $(1 + r/n)^n$. For example, r is usually annual percentage rate (APR), and $n = 12$ if the interest compounds monthly, and $n = 360$ if interest compounds daily, etc. If interest compounds every instant, the amount can be approximated by e^r - continuously compounded formula.

We can rewrite the required function in a way that allows the use of substitution rule and L'Hôpital's rule:

$$\left(1 + \frac{r}{n}\right)^n = \exp\left(n \ln\left(1 + \frac{r}{n}\right)\right) = \exp\left(r \frac{\ln\left(1 + \frac{r}{n}\right)}{r/n}\right) = \exp\left(r \frac{\ln(1+x)}{x}\right)$$

where $x = r/n$. Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} x &= 0 \\ \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} &= 1 \quad [\text{We proved previously, using L'Hôpital's rule}] \end{aligned}$$

Thus, the required limit can be written as

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = \lim_{x \rightarrow 0} \exp\left(r \frac{\ln(1+x)}{x}\right) = \lim_{y \rightarrow 1} \exp(ry) = e^r$$

3. (10 points). Given the utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0$$

(a) Prove that

$$\lim_{\sigma \rightarrow 1} u(c) = \ln(c)$$

Since the above limit has $\frac{0}{0}$ form (i.e. the numerator and the denominator have limit of 0 as $\sigma \rightarrow 1$), we can apply L'Hôpital's rule:

$$\lim_{\sigma \rightarrow 1} \frac{c^{1-\sigma} - 1}{1 - \sigma} = \lim_{\sigma \rightarrow 1} \frac{-\ln(c) c^{1-\sigma}}{-1} = \ln(c)$$

Here we used the rule of derivatives $(c^x)' = \ln(c) c^x$, together with the chain rule.

(b) Calculate the Arrow-Pratt measure of Relative Risk Aversion (RRA).

$$RRA = -\frac{u''(c)}{u'(c)}c = -\frac{-\sigma c^{-\sigma-1}}{c^{-\sigma}}c = \sigma$$

4. (10 points). Suppose $u(x, y)$ is a utility function, and $v(x, y) = f(u(x, y))$ is another utility function generated by composition $f \circ u$, where f is differentiable function. Using the implicit function theorem, prove that the indifference curves of u and v have the same slope at every point.

Indifference curves of u and v , in the form of implicit function:

$$\begin{aligned}u(x, y) &= \bar{u} \\v(x, y) &= f(u(x, y)) = \bar{v}\end{aligned}$$

Using the Implicit Function Theorem, the slopes of indifference curves are:

$$\begin{aligned}[IC_u] &: \frac{dy}{dx} = -\frac{u_x(x, y)}{u_y(x, y)} \\[IC_v] &: \frac{dy}{dx} = -\frac{v_x(x, y)}{v_y(x, y)} = -\frac{f'(u(x, y)) u_x(x, y)}{f'(u(x, y)) u_y(x, y)} = -\frac{u_x(x, y)}{u_y(x, y)}\end{aligned}$$

The last step is valid as long as $f'(u)$ is not zero. Thus, the slope of indifference curves of u is the same as the slope of indifference curves of v , as long as $f'(u) \neq 0$.

Remark. The above is an important result in consumer choice theory. If f is also *monotone increasing* function, then consumer with utility function $v(x, y) = f(u(x, y))$ will have the same demand as consumer with utility function $u(x, y)$. This result is known as *invariance of utility functions under monotone increasing transformation*. In other words, the same preferences can be represented with infinitely many utility functions, as long as they are monotone increasing transformations of each other.

5. (15 points). Suppose that the demand function for some good is given by $x = Ap_x^{-1.2}p_y^{0.2}I$, where p_x is the price of x , p_y is the price of some other good y , and I is buyers' income.

- (a) Based on the given information, x is a substitute/complement for y . Circle the correct answer and provide a brief explanation.

The demand for x is increasing when the price of y goes up.

- (b) Calculate the price elasticity of demand for x , the cross price elasticity of demand for x with respect to p_y , and income elasticity of demand for x . Denote these elasticities with η_{x,p_x} , η_{x,p_y} and $\eta_{x,I}$.

Taking logs of the demand function

$$\ln x = \ln A - 1.2 \ln p_x + 0.2 \ln p_y + \ln I$$

The required elasticities are:

$$\eta_{x,p_x} = \frac{\partial \ln x}{\partial \ln p_x} = -1.2$$

$$\eta_{x,p_y} = \frac{\partial \ln x}{\partial \ln p_y} = 0.2$$

$$\eta_{x,I} = \frac{\partial \ln x}{\partial \ln I} = 1$$

- (c) Calculate the approximate % change in the demand for x as a result of a 1% simultaneous increase in p_x, p_y and I .

$$\begin{aligned} \% \Delta x &= \eta_{x,p_x} \cdot \% \Delta p_x + \eta_{x,p_y} \cdot \% \Delta p_y + \eta_{x,I} \cdot \% \Delta I \\ &= -1.2 \cdot 1\% + 0.2 \cdot 1\% + 1 \cdot 1\% = 0 \end{aligned}$$

Remark. This has to be the case with any demand function. When prices and income increase by the same proportion, the budget constraint does not change, and the demand does not change as well. We therefore say that demand (for any good) is always *homogeneous of degree zero*.

6. (10 points). Consider a firm with Constant Returns to Scale production function $y = f(x_1, x_2)$, where y is output and x_1, x_2 are inputs. Prove that if the firm pays the inputs their marginal product, then the profit of the firm is zero, i.e.

$$\pi = f(x_1, x_2) - f_1(x_1, x_2)x_1 - f_2(x_1, x_2)x_2 = 0$$

In your proof must be direct, and cannot rely on the Euler theorem. That is, you must prove the Euler theorem here.

The definition of CRS (homogeneous of degree 1):

$$f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2) \text{ for any } \lambda > 0$$

Differentiate with respect to λ :

$$f_1(\lambda x_1, \lambda x_2)x_1 + f_2(\lambda x_1, \lambda x_2)x_2 = f(x_1, x_2) \tag{1}$$

Differentiating the definition of CRS with respect to x_1 and x_2 , proves that partial derivatives of h.d.1. function are h.d.0.:

$$\begin{aligned} f_1(\lambda x_1, \lambda x_2)\lambda &= \lambda f_1(x_1, x_2) \Rightarrow f_1(\lambda x_1, \lambda x_2) = f_1(x_1, x_2) \\ f_2(\lambda x_1, \lambda x_2)\lambda &= \lambda f_2(x_1, x_2) \Rightarrow f_2(\lambda x_1, \lambda x_2) = f_2(x_1, x_2) \end{aligned}$$

Plugging into (1), gives the result:

$$\pi = f(x_1, x_2) - f_1(x_1, x_2)x_1 - f_2(x_1, x_2)x_2 = 0$$

7. (15 points). Consider the following utility function from consumption in two periods:

$$u(c_1, c_2) = \frac{c_1^{1-\sigma} - 1}{1 - \sigma} + \beta \frac{c_2^{1-\sigma} - 1}{1 - \sigma}, \quad \beta, \sigma > 0$$

(a) Write the equation of a generic indifference curve with utility level \bar{u} .

Indifference curve:

$$\frac{c_1^{1-\sigma} - 1}{1 - \sigma} + \beta \frac{c_2^{1-\sigma} - 1}{1 - \sigma} = \bar{u}$$

(b) Find the marginal rate of substitution between c_1 and c_2 (MRS_{c_1, c_2}).

Note, the indifference curve is implicit function. Using the implicit function theorem:

$$\frac{dc_2}{dc_1} u(c_1, c_2) = -\frac{u_1(c_1, c_2)}{u_2(c_1, c_2)} = -\frac{c_1^{-\sigma}}{\beta c_2^{-\sigma}} = -\frac{c_2^\sigma}{\beta c_1^\sigma}$$

The marginal rate of substitution between c_1 and c_2 (MRS_{c_1, c_2}) is the absolute value of the slope of indifference curves, i.e.

$$MRS_{c_1, c_2} = \frac{1}{\beta} (c_2/c_1)^\sigma$$

(c) Find the elasticity of substitution between goods c_1 and c_2 (ES_{c_1, c_2}).

$$\begin{aligned}\frac{1}{ES_{c_1, c_2}} &= \frac{\partial MRS_{c_1, c_2}}{\partial (c_2/c_1)} \cdot \frac{c_2/c_1}{MRS_{c_1, c_2}} \\ &= \frac{\partial \frac{1}{\beta} (c_2/c_1)^\sigma}{\partial (c_2/c_1)} \cdot \frac{c_2/c_1}{\frac{1}{\beta} (c_2/c_1)^\sigma} \\ &= \frac{1}{\beta} \sigma (c_2/c_1)^{\sigma-1} \frac{c_2/c_1}{\frac{1}{\beta} (c_2/c_1)^\sigma} = \sigma\end{aligned}$$

Thus,

$$ES_{c_1, c_2} = \frac{1}{\sigma}$$

8. (15 points). Consider the following system of nonlinear equations:

$$\begin{aligned}xy - \frac{4}{x} + y - 1 &= 0 \\x^2 - y + a &= 0\end{aligned}$$

The endogenous variables are x and y , while a is an exogenous variable (or parameter).

- (a) Suppose that initially, $(x, y, a) = (2, 1, -3)$. Verify that this point is a solution to the given system of equations.

$$\begin{aligned}2 \cdot 1 - \frac{4}{2} + 1 - 1 &= 0 \\2^2 - 1 - 3 &= 0\end{aligned}$$

- (b) Derive the Jacobian matrix with respect to endogenous variables, and calculate the Jacobian determinant at the initial point.

The given system can be written as:

$$\begin{aligned}F_1(x, y, a) &= 0 \\F_2(x, y, a) &= 0\end{aligned}$$

The Jacobian with respect to the endogenous variables (x, y) :

$$J_{xy} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} y + \frac{4}{x^2} & 1 + x \\ 2x & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

Jacobian determinant:

$$|J_{xy}| = 2 \cdot (-1) - 4 \cdot 3 = -14 \neq 0$$

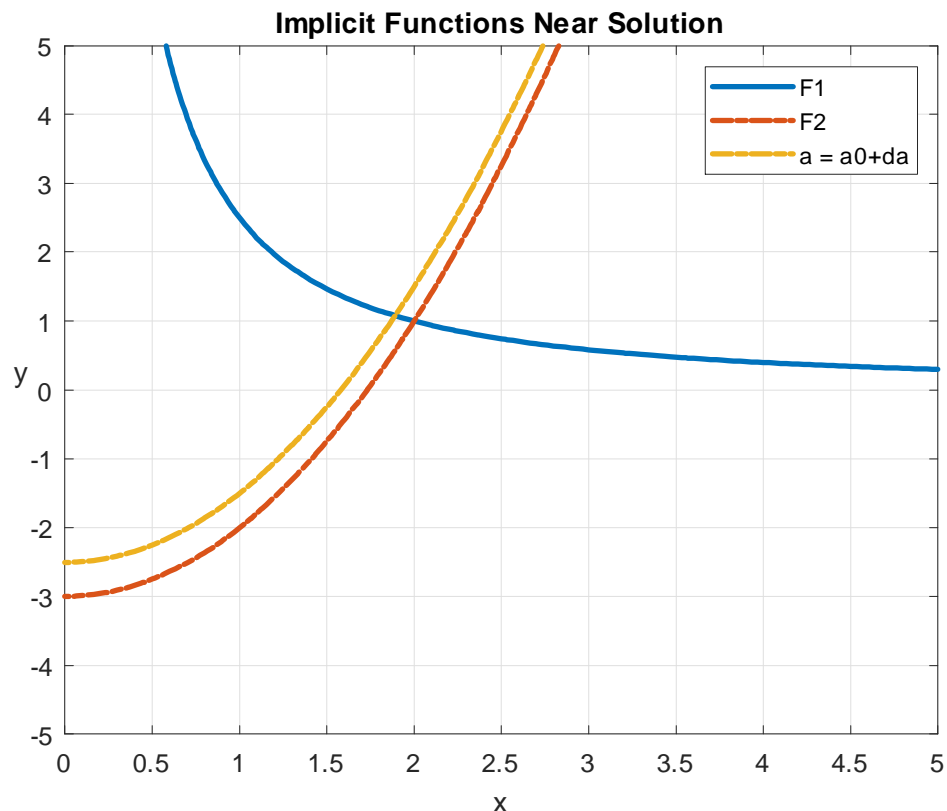
- (c) Suppose the exogenous parameter changes: $da = 0.5$. Calculate the approximate changes in endogenous variables: dx , dy .

$$\begin{aligned} \begin{bmatrix} dx \\ dy \end{bmatrix} &= -J_{xy}^{-1} \begin{bmatrix} \frac{\partial F_1}{\partial a} \\ \frac{\partial F_2}{\partial a} \end{bmatrix} da = - \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} da \\ &= - \frac{\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}}{-14} \begin{bmatrix} 0 \\ 1 \end{bmatrix} da = \frac{1}{14} \begin{bmatrix} -3 \\ 2 \end{bmatrix} \cdot 0.5 = \frac{1}{14} \begin{bmatrix} -1.5 \\ 1 \end{bmatrix} \\ &\approx \begin{bmatrix} -0.107142857142857 \\ 0.0714285714285714 \end{bmatrix} \end{aligned}$$

Remark: when solving the system numerically with $a = a_0 + da$ (using Matlab), the precise changes in endogenous variables are

$$\begin{aligned} \Delta x &= -0.10862598991465484157214652630658... \\ \Delta y &= 0.077295646026319329066832836530986... \end{aligned}$$

demonstrating that the approximation above is not too bad, even for such large increment $da = 0.5$. The next figure shows the plot of the two implicit functions at the original solution, which suggests that the two functions can be reasonably well linearly approximated near the initial solution.



9. (10 points). Briefly explain what the following Matlab commands are doing.

(a) `z = linspace(-5,5,101)`

Creating evenly spaced grid (named `z`) of 101 points between -5 and 5.

(b) `syms x y`

Declaring symbolic variables `x` and `y`.

(c) `limit(x^x,x,0)`

Finding the limit

$$\lim_{x \rightarrow 0} x^x$$

(d) `diff(x^a, a)`

Calculating the partial derivative

$$\frac{\partial}{\partial a} x^a$$

(e) `J = jacobian([a*x^2+x*y + b; a*y^2 - x/y + b],[x,y])`

Calculating the Jacobian (matrix of second derivatives) of the system of implicit functions:

$$\begin{aligned} F_1(x, y, a, b) &= ax^2 + xy + b \\ F_2(x, y, a, b) &= ay^2 - \frac{x}{y} + b \end{aligned}$$

That is,

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$